Dynamics of Machines

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Module No. #12

Lecture No. #4

Free Vibration of Multiple Degrees of Freedom Systems:

Flexibility Matrix and Influence Coefficients

We mentioned in the previous lecture that, when model consist of a large number of degrees of freedom, it is algebraically and computationally difficult task to directly solve the characteristic equation and to determine the natural frequencies and mode. Instead, I think there are some simple iteration procedure which can lead to reasonably good values of the frequencies and the corresponding modes. We will discuss these methods which are very useful for the designers and also during the designing it is not possible all the time to switch over to a computational program. Therefore, such techniques can be used and reasonably good answers can be obtained with the help of a simple calculator and such techniques can be quite often very useful.

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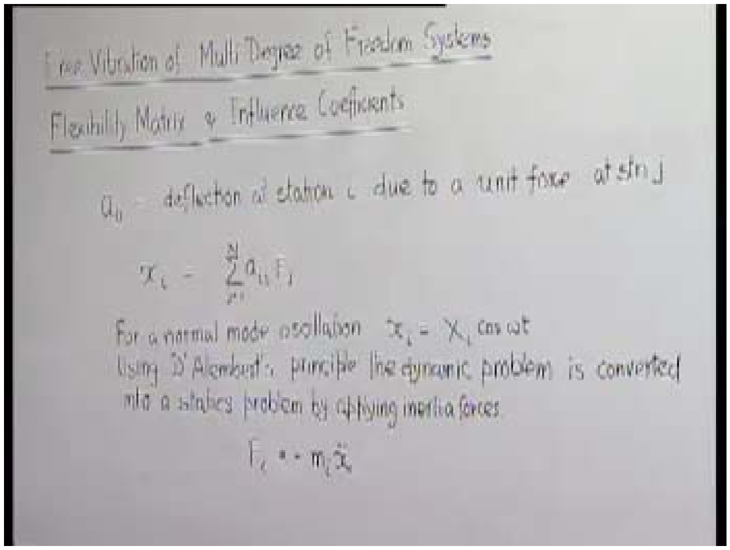

Before we go directly, we discuss one more concept that is flexibility and influence coefficient. Before one attempts to start the solution process, it is essential to write down the matrix equation of motion. In some cases that itself may take some time, but with the help of one new concept like influence coefficients and flexibility matrix, the equation of motion in matrix form can be found out in a reasonable time. We will define a quantity a_{ij} has the deflection at station i due to a unit force at station j. For an n degree freedom system the displacement at any instant at the ith station will be a_{ij}, force which is acting at the jth station summed over j equal to 1 to N. For a normal mode oscillation x_i is equal to capital X_i cosine omega t.

Now, the question is, when such a vibration takes place? What will be the forces acting? How do you convert? As a static problem, you can see it is very simple, but as a dynamic problem we have to convert this static problem by applying D' Alembert's law or D' Alembert's principle that we have done earlier. At every station say for ith station it is having an amplitude capital X_i with a frequency omega and that station carries a block of mass mi. What we can do is that we apply or introduce the inertia forces at the each station. At the ith station, inertia force will be minus m_i into omega square rather what we should write is minus $m_i x_i$ two dot.

When you consider that means, at any instant we can use the acceleration of that station multiplied by the mass of station and negative of that is the inertia force. We assume that it is being applied at the ith station and the combined effect of all such inertia forces in the absence of any other force because it is a free vibration and the displacement at this, we can find here.

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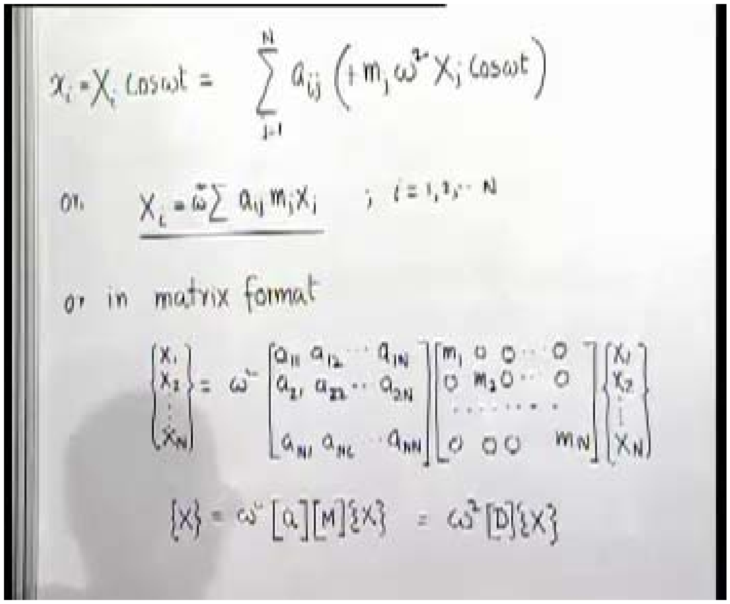

Thus, displacement which is X_i cosine omega t which is x_i will be sum total of a_{ij}. x_j will be minus m_j into x_j two dot. How much is x_j two dot? x_j two dot will be minus another minus omega square X_j cosine omega t summed over all such j is from 1 to N or we get X_i is equal to omega square summation of a_{ij} into m_jX_j. This we can write in matrix form i equal to 1, 2 up to N. We can write in matrix form, all the equations can be represented and will be X_1 X_2 upto X_N equal to omega square a_{11} a_{12} a_{1N} (Refer Slide Time: 10:04). This is the expanded form; in short form we can write it as X equal to omega square a matrix m matrix. Now, this we call the flexibility matrix because a_{ij} represents the deflection or unit load at another station, so it is nothing but a measure of flexibility which is just opposite of stiffness, because we can very easily find that according to this definition X will be flexibility into force.

If you pre-multiply by a inverse this will be a inverse into a unit matrix this is F, but we know f matrix can be written as k matrix into x matrix, so k matrix is nothing but inverse of a or a matrix is nothing but inverse of k, this is the stiffness matrix and its inverse is the flexibility matrix.

We write this as omega square D, where D is another form of dynamic matrix but it is nothing a into m this equation we have to solve by iteration technique. Iteration technique means that we assume some value of X or some model matrix X anything we multiply this and then by normalizing, normalizing means maybe one of the terms like the first term we always keep one, so that way when we do it let us see whether we get back the same matrix or not, normally we will not get, then we use the second matrix that the result what we get that matrix again we put it here and do the operation again and again (Refer Slide Time: 13:26). I will explain this with the help of an example.

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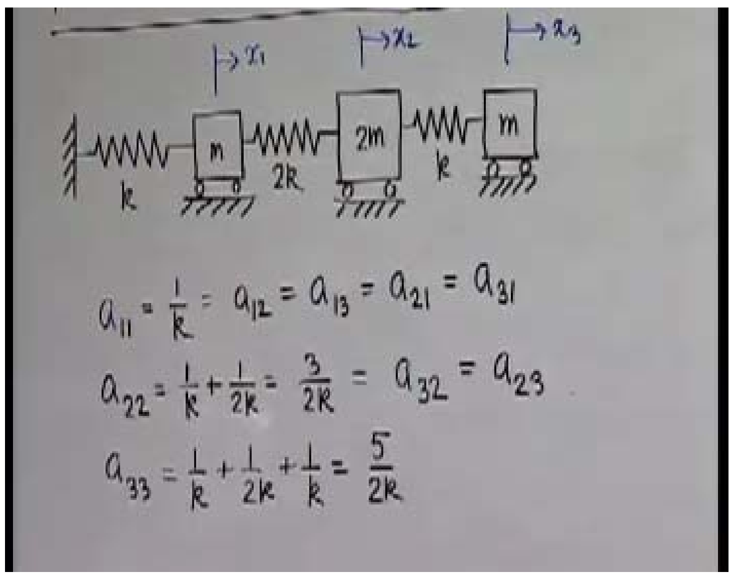

Let us consider the three degree freedom system. This is simple three degree freedom system and our objective is to find out the natural frequencies and the natural mode. First, we have to determine the D matrix, m matrix is quite straight forward but D matrix we have to find out the flexibility matrix. Let us see, what is a_{11} that is, if you apply a unit load at station one what is the deflection of station one? Obviously, it is only the space of string and that is nothing but 1 by k.

Now, what will be the deflection of this position if you give a unit load here? Again it is nothing but the stretch of this due to the unit load which is also passing through this (Refer Slide Time: 15:36) this will be simply as a_{12}. Similarly, if you apply a unit load at this location that is third station again the stretch of this string will be same and that reflection of station one. On the other hand, if I apply a unit load at one, only this string stretches this string just display as a rigid body without any distortion. The displacement at station two due to a unit load applied at station one will be same as 1 by k and that is a_{21}.

Similarly, if I apply unit load here, the stretch of this will be 1 by k and this is moving just like a rigid body; so the displacement of station three will be also equal to 1 by k. Next a_{22}; a_{22} means that if I apply a unit load here what will be the displacement of this (Refer Slide Time: 16:51) . It is obviously the stretch of the string due to unit force. It is 1 by k, 1 by 2k that is the influence coefficient a_{22} and obviously, if you apply unit load here the displacement of this will be same as the displacement of this, because this moves just like a rigid body this string is not, this is obviously a_{32}.

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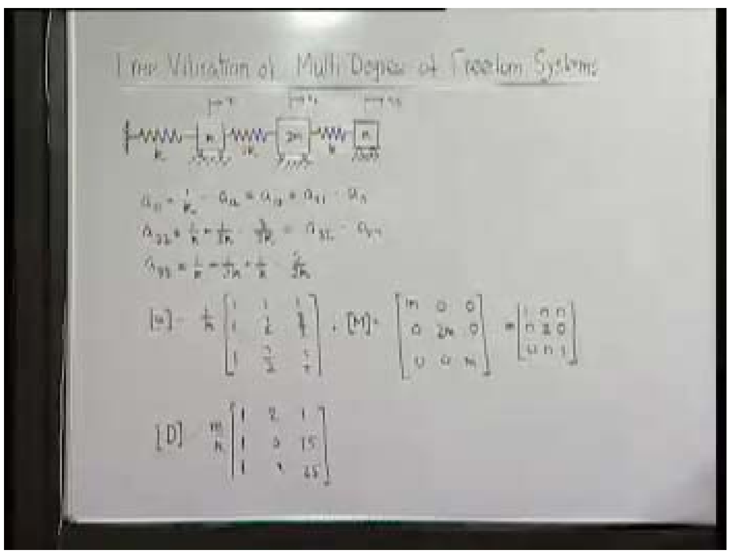

At the same time, if I apply unit load here, what will be the displacement of this? It will again be the stretch of this string and this string, which is this. (Refer Slide Time: 17:38) So this is a_{23} also. The only one which is still not determined is a_{33}; that is if I apply unit load at station three what will be the displacement of this station three it will be the stretch of these three strings. Obviously, it will be 1 by k plus 1 by 2 k plus 1 by k equal to 5 by 2k, a matrix can be written as 1 by k. a_{11} is 1, a_{12} is 1, a_{13} is 1, a_{21} is 1, a_{22} is 3 by 2 and a_{23} is 3 by 2, a_{31} is 1, a_{32} is 3 by 2 and a_{33} is 5 by 2.This is the flexibility matrix you also know the mass matrix will be simply first mass is m, second mass is 2m, Third mass is m and these terms are all zero because mass matrix is a diagonal matrix. This is nothing but m into 1, 2, 1, 0, 0, 0, 0, 0, 0. The D matrix is equal to m by k into this into this. The result which I give is a straight forward.

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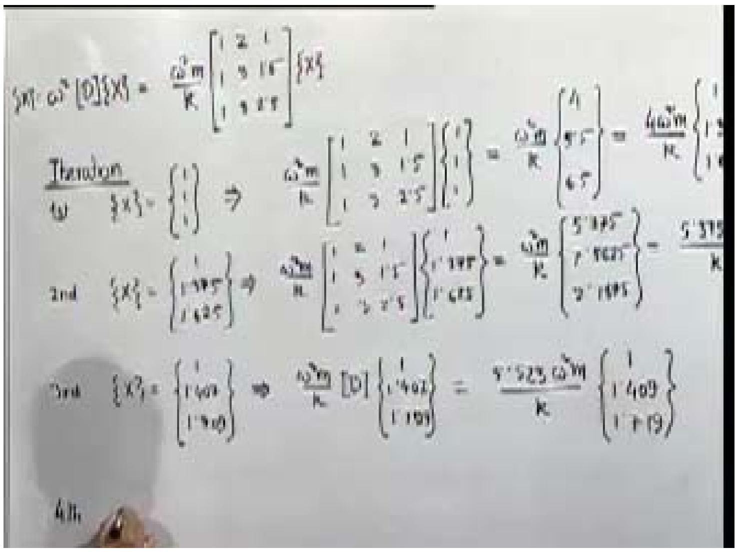

Thus we are now ready for our iteration part we will write this equation as omega square D X is omega square m by k into 1, 2, 1. First state we can start with any assumption on it, Iteration process let us start first assumption X is simply let us take as 1, 1, 1. This if we substitute here, we will get omega square m by k one particular case I am writing 1, 1, 1. This will be omega square m by k into 4, 5.5, 6.5. Now to normalize let us always make the first one always one because the ratio of the various terms are actually important not the values so this will be 4 omega square m by k into 1, 1.375, 1.625 this is the value we get now.

In the second stage we assume X is equal to 1, 1.375, 1.625. If we apply this and pre-multiply this by D matrix(Refer Slide Time: 23:58). This will be now as shown above, then to normalize the first term it has to be kept one, we write this as follows. If we apply now, I will not write the in-between thing so the third step we have X equal to 1, 1.407, 1.709. You can see we started with this and we got back this. Therefore, it is still not satisfying this equation that means, if we multiply this we will get back X that was suppose to get that we are not getting. Now, if we do this now what we will get this as 5.523 omega square m by k into 1, 1.409, 1.719.

Now, one can easily see that; we have gradually closing that; I started with this(Refer Slide Time: 26:33) and came back to something which is not very far from this; maybe in one more step we will bring this to the final result, say let us take a fourth step, if I want to make more accurate depending on the amount of accuracy, one desire; If we use this, we will get; now we consider that we have these tags. The equation what we have is now fulfilled with this particular moment. Now the equation to be fulfilled that means, if this as to be equal to 1, 1.409, 1.720 then what does it mean? That is to be equal to 1. If this quantity is equal to 1 then only this equation is satisfied that means, your corresponding omega will be how much? omega will be correspondingly 0.425, the normal mode is equal to 1, 1.14, 1.72, this is the mode and this is the corresponding frequency.

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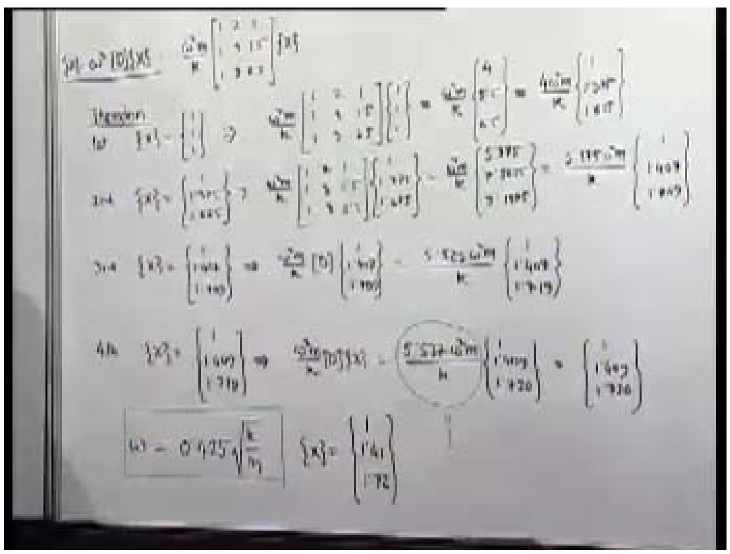

Whatever may be the starting point 1, 1, 1; we can take 1, 0, 1; we can take 3, 2, 1; we can take 7.5 minus 3; whatever maybe the starting point, you will find that we will always convert to this mode and this is the corresponding sequence, it may look like a miracle, but I will show you the reason why it is happening. The question is that which mode is this? Now it can be proved, as I will be doing now, this is nothing but the first mode or the fundamental natural frequency and this is the first mode. Why it is converting all this to the first mode? I think, if that is the question we have to understand; then we will diagnose from the equation procedure for a short while.

Now, we have seen that omega square D X say for a particular mode i, the particular frequency i is equal to. (Refer Slide Time: 30:50) Here, I have not used i, but always remember that this is valid only for a natural mode of oscillation and let this be ith mode or we can write D which is nothing but 1 by omega i square, for all the natural mode i is equal to 1,2,3.

When we assume something, we assume mode shape; the assumed mode shape can be always expressed in terms of the first mode, a combination of C_1 or if we still further normalize, we can always make this divide the whole thing by C_1. I can make it like this, C_2 by C_1 is one quantity, we call them as prime, now I call it as C_2, it is nothing but C_2 prime by C_1 prime.

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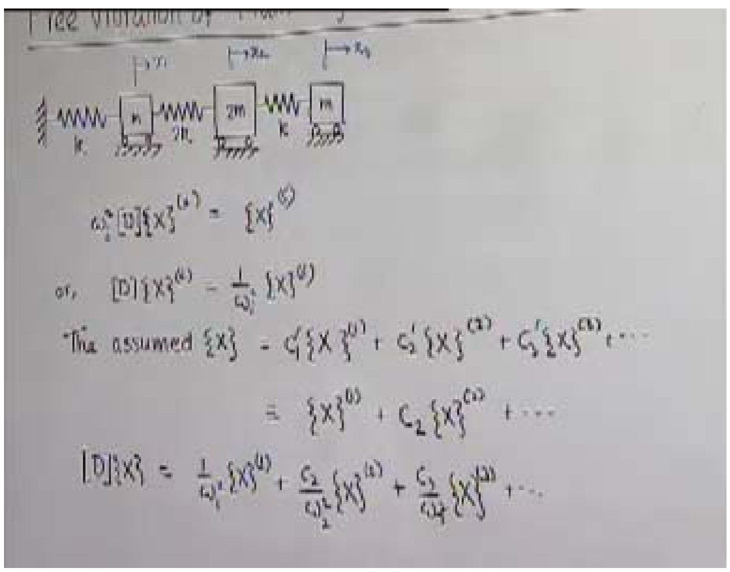

It is always possible whatever may be this sum value, it can be expressed in terms of degrees of all the natural modes. Now, if I apply this D into X that means, when I pre-multiply X by D, what does it mean, I am pre-multiplying each one by D. Now, since this is a natural mode, when I pre-multiply this by D,(Refer Slide Time: 33:34) I will get 1 by omega_1 square X one; according to this, plus C_2 omega_2 square X two; like that C_3 by omega_3 square.

If I normalize the result X, I will get more, it will be somewhat different from the previous one, because the coefficients are different from coefficients earlier. If I again pre-multiplying this by D what will happen? Again when I pre-multiply X one by D, this term will become (Refer Slide Time: 34:45). After n such iterations of this, we will get 1 by $omega_1$ to the power n and so on.

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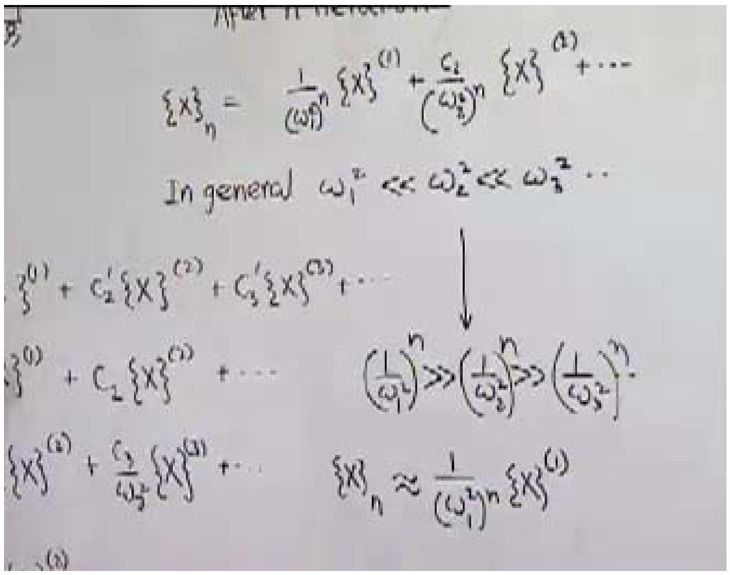

We should notice that in general $omega_1$ square is much less than $Omega_2$ square is much less than $Omega_3$ square and so on. What will happen that omega ,$omega_1$ square is more than 1,2 square. When you raise these things to power when n is sufficient till that, then you will find almost approximately equal to the this (Refer Slide Time: 36:35)

Therefore, what is happening the most important thing that since, I think we will do it this way therefore, if we make it n to the power this is whole this is aggregated. Only these things are far smaller compared to this term so the pre-dominant term what we get after n number of iterations is approximately equal to (Refer Slide Time: 37:45). What does it mean forget about this for if mode is concerned, look at the ratios of the various term it really approaches the first and the corresponding frequency becomes the first. That is why we are whatever may be the initial value if it is closer to first mode then number of iterations will be less if it is very different from the actual first mode it will take a longer time to convert to the first mode, but it will convert to the first mode.

Therefore, this is quick way of getting an idea about the natural frequency and the first natural mode not solving any matrix equation because it involves only simple matrix multiplication it is much easier and quick time we can do it. Next the question comes that if we require to find out the hire and hire natural frequency, that means here there will be three natural frequencies and three modes we would have found only the fundamental quantities.

How to find out the higher modes so to determine the higher modes? We will take this approach, first let us keep in mind the very important conditions which we derive (Refer Slide Time: 38:56). These are the orthogonality of normal modes this condition we have to keep in mind. Now, any mode we assume can be expressed as C_1 into the first mode shape as I mentioned just now C_2 into the second mode shape and so on. Now, we see that when now we pre-multiply this by D matrix again and again and again gradually this term tends to become prominent because Omega$_1$ by omega$_1$ square by two which it is being multiplied again and again for each pre-multiplication because of this. We can express it in this way so every time if we multiply by D, one such term comes and therefore, as omega is the smallest this is the largest and repeated multiplication causes this to be the prominent.

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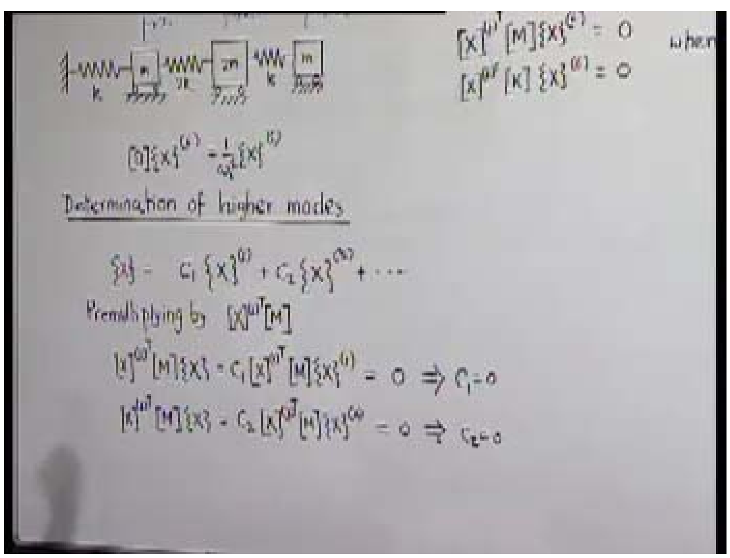

If we could somehow make C_1 zero then our assumed shape would have started from here and if we repeatedly multiply or pre-multiply this by D then the whole thing will convert to the second mode and we will get the second natural frequency. Similarly, if we could make both C_1 and C_2 zero by some means then the whole thing the next term it will convert to that which is nothing but the third term. In this particular case there will be only three it will convert to the third frequency and the mode shape we will get by the same process.

Our objective is now to find out ways of making a thing or making X such C_1 is 0. Now, one thing if we pre-multiply by this what we will get all other term will be 0 because of the orthogonality condition. This is the only term when this is 1 and this is 1 later this will be 1 and the other will be 2,0 and so on. And therefore, I will again take this example, this example will give us x_1 for mode one ,x_2 for mode one, x_3 for mode one, m_1 0, 0, m_2 ,0 for three degree freedom system and when I am not putting any things superscripts it means it is not actually a mode. Whenever it is a mode it is there this is equal to 0 that what we I am saying that if I make this equal to 0. Since this is not 0, we have said that when both are there this is nothing but the generalized mass which is not 0 it can be 0 only if C_1 is 0 therefore, putting this equal to 0 means C_1 is becoming 0.

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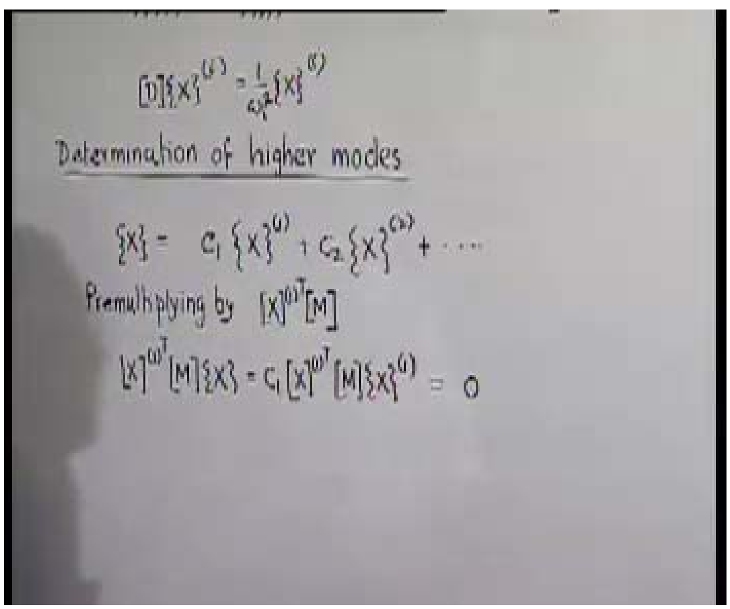

This implies C_1 equal to 0 therefore, if I this is nothing but this and when I am putting this equal to 0 it means C_1 has become 0 automatically. If you expand this we will get $m_1x_1$one x_1 or you can write x_1 is equal to minus (Refer Slide Time: 45:23).Now, we have got this equal to zero means $x_{1,}x_2$,x_3 cannot be independent because x_1 can be the in terms of x_2 and $x_3$using this equation x_2 of course, is equal to x_2 x_3 equal to $x_{3.}$

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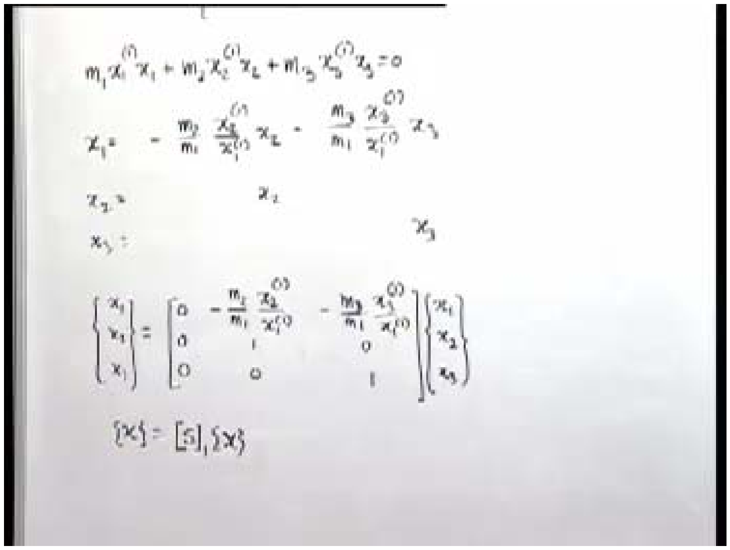

This whole thing can be written in matrix form x_1,x_2,x_3 is equal to or symbolically we call it as S_1 small x or capital X does not matter cosine omega t get cancelled. Now what is happened now we have already solved the first mode that is always be remember so these quantities are known x_1 one x_2 one these are all known quantities m_2 m_1 m they are also known quantities. This is a known matrix and when this matrix pre-multiplies this write like this. Then, it means that the first mode as been stripped out as C_1as been made 0 that’s why we call it as stripping matrix.

Similarly, C_2 can be made 0 by pre-multiplying this x_2. when both this and this equations are satisfied then both $C_1$and C_2 as been stripped out only C_3 is left. Then if we do the iteration we go to the third mode before we do that, I think so this example if we do so in this case that means both are striped out that means two equations will be there one will give raise to this equation and this will give raise to another equation. So, therefore, the second equation will lead to another stripping matrix so I will write the final answer x_1,x_2, x_3 equal to 0, 0, 0 ,0, 0, 0 and x_{31},x_{22} minus. So, if we use this it means both the boards have been stripped out. So to solve the problem same problem because already first mode as been solved for which we know that your first mode $omega_1$ is 0.425 square root of k by m and x_1 is 1,1.41, 1.71 or something.

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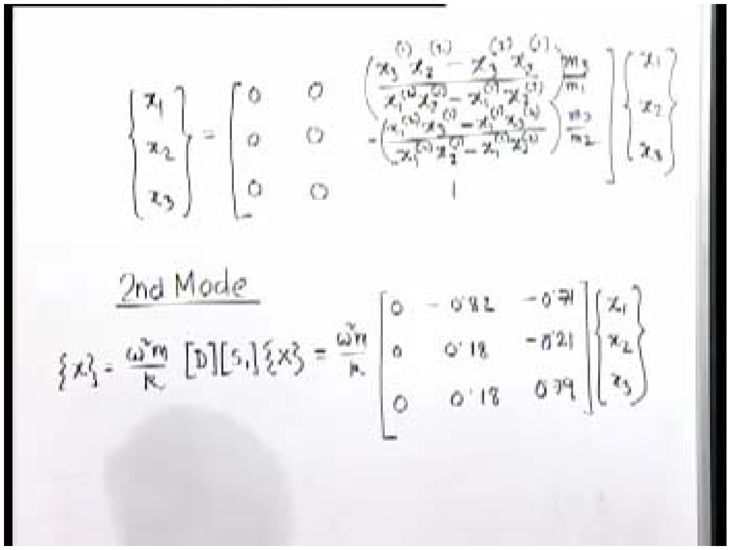

This is solved therefore, now we should be able to figure out how much is going to be the second mode. Now it involves too much of calculation, you do not want to do it what we will show just this that x is equal to omega square m by k same kind of formulation here is the D matrix and stripping matrix for first mode into X and this becomes omega square m by k in this case 0 ,0 ,0, minus 0.82,0.18,0.18 minus 0.71, minus 0.21,1.79, this is the final. Now, we have to assume something here and repeatedly pre-multiply by this which is nothing but D into S_1 therefore, this is going to strip out the first mode and it will now convert to second mode. If we do it if we start with 1, 1, 1 it does not matter what we will be converging to $omega_2$ or $omega_{n2}$ will be 1.18 square root of k by m and second mode will be 1, 0.7 minus 1.8. This matrix actually this is stripping matrix when you multiply D matrix. By this, we get if we just do the same thing that means stripping matrix if we have used both stripping matrix that means that C_1C_2 zero if you multiply that by D the equation will get. For the third mode that is the stripping matrix to pick out both first and second mode your third mode equation becomes this matrix becomes ok, directly we can write.

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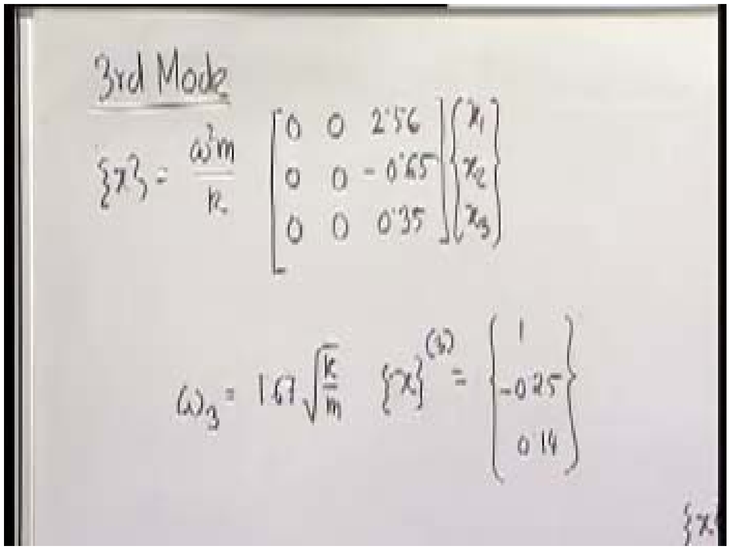

This is D; we again assume any value of x and pre-multiply by this again and again we will convert to omega$_3$ and third mode will be 1 minus 0.245. Now, we have to remember that we have solved first mode and second mode. That is why we are now able to calculate all this terms because here you need both the first mode term and second mode term. Thus we get all the higher modes by using this stripping matrix technique. If you solve directly a matrix equation of course, then we get all the modes and all the frequencies or not necessary but, it is very computer intensity process. I think what now you can see that it is very even if it is simple but then it takes some time.

The question is, are there means or methods by which we can quickly estimate the fundamental frequency, which is generally of maximum importance?. I think in the next session we will discuss techniques by which a quick estimate of the fundamental frequency or first natural frequency can be found without going through such detailed calculation procedure.