Dynamics of Machines

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Module 12 Lecture 3

Design of Vibration Absorbers: Free Vibration of Multiple Degrees of Freedom Systems; Orthogonality of Normal Modes

We have seen that it is possible to utilize principle of 2 degree of freedom system in neutralizing the vibration of an absorber. To demonstrate how, this kind of tuned vibration absorbers are designed, we will check up with specific examples.

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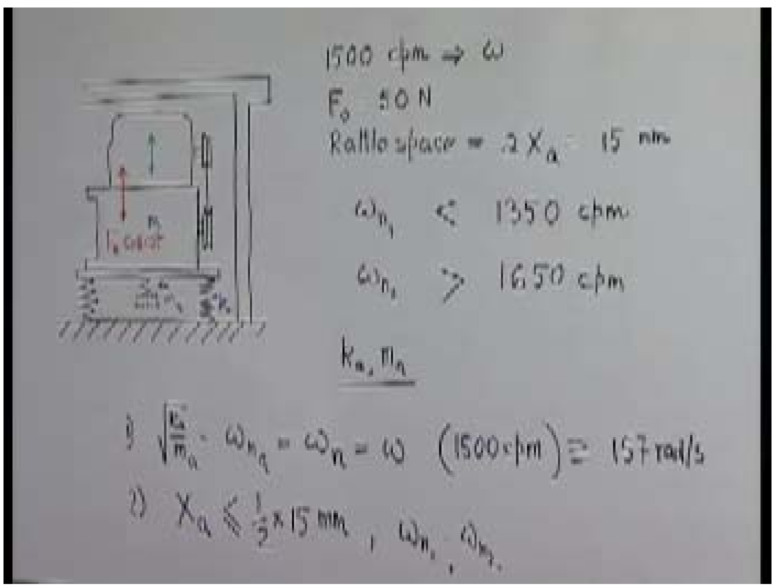

This is a vacuum pump; we need not go into the details of the story behind it. Main thing is that the machine which is being driven by a motor is suspended or supported on two strings. It is capable of making vertical movements whose frequency will be same as the unbalanced force which is generated in the machine, may be a vacuum pump of cylinder type. So, you have to design a vibration absorber. Vibration absorber means that another spring mass system is to be connected here or here (Refer Slide Time: 01:57 min). So, what is given in this case is that the machines run with nominal speed at fifteen hundred cycles per minute. We can easily find out the corresponding omega and obviously, the frequency of the force generated will be same; that means this related to omega.

The magnitude of the force at this speed is 50 Newton. The rattle space is nothing but twice the amplitude of the absorber mass wherever it is, if it is here or if it is here (Refer Slide Time: 03:02 min); I have kept here, as larger space is here. So, the maximum amount of displacement m_a can have on both sides of its equilibrium position that is called rattle space; that is given as 15 millimeter.

Now, it is also very important to see that after adding the absorber mass or absorber system, the natural frequencies of the resulting 2 degree of freedom system should be wide apart. In this case, it has been told that omega $_{n1}$ should be below 1350 cpm and omega $_{n2,}$ the larger one should be above 1650 cpm. We can easily find out radians per second and we can convert. So, that means the two natural frequencies should correspond to values. They should lie outside this region where the nominal operational speed of the system is this. Obviously, we know that this will be also the resonating frequency of the primary system. So, we have to find out k_a and m_a.

So, the three conditions will be: one is that square root of k_a by m_a is the natural frequency of the absorber system, should be equal to the natural frequency of the original system and which must also match with the operating frequency which is 1500 cpm; that is equivalent to 157 radians per second - that is one condition. Second condition; x_a must be less than equal to half into rattle space that is 15 millimeter. Third thing is that omega$_{n1}$ and omega$_{n2}$ must lie outside this.

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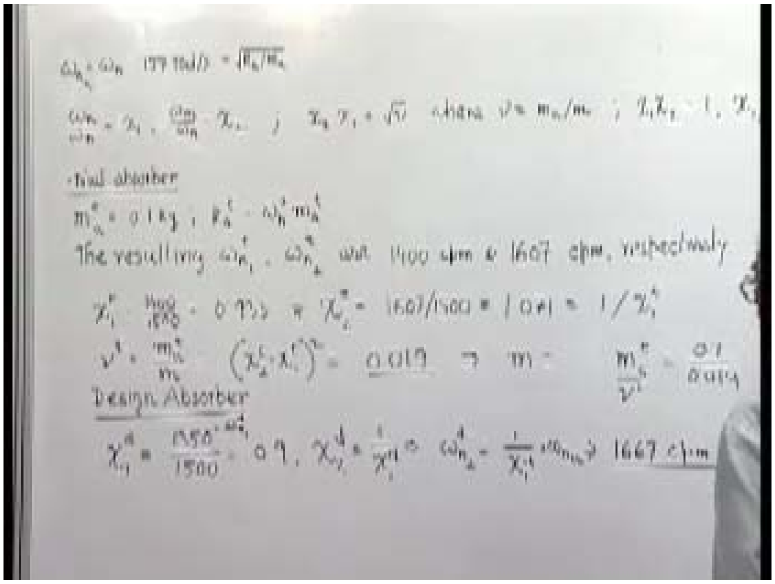

How do you proceed? First thing will be we have found out and we know that omega_a equal to omega_n is equal to 157 radians per second which is same as square root of R_a by m_a, which is actually omega_{na}. Next, we also know that omega_{n1} by omega_n equal to chi_1, omega_{n2} by omega_n we have found out chi_2 and we also know (Refer Slide Time: 07:15) Next, therefore what is done generally in such cases that we first take a trial auxiliary system.

So, we first attach a trial absorber and when we attach the trial absorber, say of mass m_a trial is 0.1 kg. Whenever we say that absorber mass, it has to be tuned. Though it is not mentioned, it is obvious that k_a t should be such that it should be omega $_n$ square into m_a t, where omega_n is the operational speed that is 157 radians per second and trial mass is here; so this can be found. We also know that once we attach this trial absorber, the resulting omega_{n1} t and omega_{n2} t are 1400 cpm and 1607 cpm respectively. Since the two resulting natural frequencies are not outside the region, obviously this trial absorber is not satisfactory. But with this experiment we can check the result; that means, omega_{n1} t is this much; therefore, trial value of chi_1 t will be 1400 by 1500; so that is equal to 0.933 and chi_2 t we can find out either as 1607 by 1500 which will be 1.071. We also know that this is nothing but 1 by chi_1, because chi_1 into chi_2 must be always equal to 1; that we have derived. We should also mention that chi_{12} can be written as half square root of 4 plus v, plus or minus root v. So, now we know the trial values of this and from this we can easily find out now trial mass, because we know that the trial value of m is nothing but trial value of the absorber mass by m, will be nothing but from this relation we get chi_2 trial minus chi_1 trial square, which is 0.019.

So, we can definitely find out from this, what is the value of your original mass. This can give us the value, because, the m of original system is nu trial into m_a trial. If necessary, you may want to find out, we are not required to do it, but we can easily find out how much it is going to be. Anyhow, we will not do that; rather, let us find out the design absorber, final one if you want to do.

Now, we know there are many constraints; let us start satisfying one constraint exactly. What we may say, let us take this constraint to be satisfied, that means, with design absorber chi_1 d will be 1350 which is just the limit by 1500, which is equal to 0.9. Obviously, with this we have to find out that chi_2 d will be then simply 1 by chi_1 d, that relation and that will lead to $omega_{n2}$ in the design stage will be equal to (Refer Slide Time: 14:12 to 14:57)) here actually this was design here, so this will be 1 by chi_1 d into $omega_{na}$, which is nothing but 157 radians per second or which is equivalent to 1500 cpm. So, this will give us 1667. So, when we put the trial mass here or a design absorber here, in such a way that when the two frequencies are splitting up, the lower frequency is just equal to this.

Then the upper frequency which we will get will be above this; this means the frequency condition is being satisfied. But there is another condition which you have to now define. So, two conditions we are satisfying; that is, natural frequency is same as the exciting frequency and also which is equal to the natural frequency of the original system. We have also put a design absorber in such a way that if you put a design absorber to make the lower natural frequency exactly equal to the limiting value of the natural frequency, we avoid any large oscillation when the exciting frequency varies, till the upper frequency is above the other thing; that means we are satisfying that condition.

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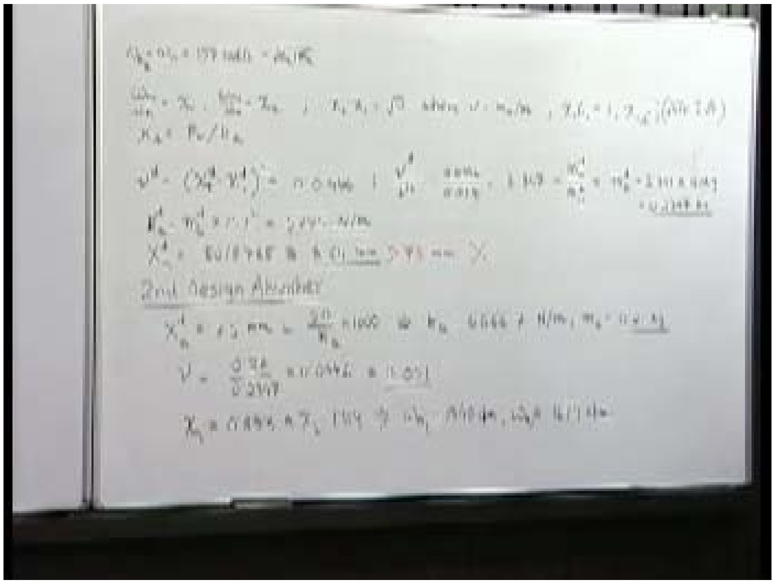

So, the other condition, the rattle space, we have to find out. To find out the rattler space we have to find X_a and X_a means the amplitude of the absorber mass is simply F_0 by k_a; that we have derived in the previous lecture. That is x_a is nothing but (Refer Slide Time: 16:27). So, what is F_0 is given. So, we have to find out k_a. If we can find out k_a, we can find out x_a. Now, how to do that? You know design value of this nu is simply design value of chi_2 minus design value of chi_1 square and that is equal to 0.0446. Since chi_1 is this; chi_2 is 1 by 0.9; so, we can easily find out.

Design value of nu by trial value of nu becomes trial value of nu which is equal to 2.347. Now, since nu is nothing but m_a by m; so, nu d by nu t will be m_a d by m_a t (Refer Slide Time: 18:02). This will give us the design value of the absorber mass will be equal to 2.347 into trial value of the absorber mass which was 0.1 kg; so the whole thing is 0.2347. So, we have found out the mass of the design absorber, not the trial mass.

So, once we find out that, we know the natural frequency. Since we know the natural frequency, we can easily find out k_a d will be equal to simply 157 square and that will be equal to 5785 Newton per meter; that is the stiffness of the absorber. Now, we know x_a is going to be with the design absorber 50 Newton divided by this (Refer Slide Time: 19:32), which is equivalent to 8.64 millimeter; we know that our limiting value is 7.5. So, this is more than 7.5 millimeter; so, this condition is not satisfied; so, we cannot use this; we have to now do the design again. This is a straining design, a mechanical design is not a straightforward problem applying formally and finding out values. It is always trying to fit in the constraints along with the desired conditions to be satisfied.

So here, we have found that starting from this it will not be possible. Now, one thing, should we start satisfying this condition (Refer Slide Time: 20:49 to 20:56). Now without going through the calculation I can tell you and one can verify, that if we satisfy this then the lower value of $omega_n$ which will be resulting here, the way we have found out will be more than 1350; that is, the frequency condition will not be satisfied. Therefore, let us not start from that side; rather now, let us try to see if we satisfy the rattle space what happens. So, this is the second design mass or second design attempt.

So, our next attempt is this. Now, let us start designing the other work starting from the rattle space condition; that means, let x_a d be exactly equal to 7.5 millimeter. Now, with a 15 Newton force, so this is nothing but equivalent to 50 by k_a into 1000, where k_a is in Newton per millimeter. So, this will give us the value of the stiffness of the absorber springs the final design which we are trying and this will become something like 6666.7 Newton per millimeter. So, once we know the stiffness, we know the frequency has to always match, because, it is 157 radians; we get absorber mass also will be equal to 0.27 kilograms. So, k_a by m_a is equal to square of this. We can easily calculate and find out. Now, once we get this, we can find out the value of nu. What will be the mass of the system? So, with 0.1 kg nu as this much (Refer Slide Time: 23:50), so with so much kg what will be the value of nu? Obviously, 0.27 by 0.2347 into 0.046 and this becomes 0.051.

We have been able to find out nu. Once we know nu, we should be able to find out chi_1, how? We just use this relation (Refer Slide Time: 24:33 to 25:50 min) putting the value of nu here we get chi; when it is minus sign it is $chi_{1;}$ when it is plus sign it is chi_2. So, using this calculation gives us chi_2 will be 1.119. Since chi_1 is nothing but $omega_{n1}$ by $omega_n$ and chi_2 is nothing but $omega_{n2}$ by this. So, directly they will give us the value of $omega_{n1}$ which is equal to or equivalent to 1340 cpm and which will be equivalent to

1679 cpm. Thus, we find that we start by satisfying the rattle space condition, like this, find out the stiffness of the absorber satisfying this and the resulting m_a because it has to have the same natural frequency as this. Once you know the mass of the absorber and we know for a given mass of the absorber what the nu was, we can find out what should be the nu in this case.

For example, if we had nu here (Refer Slide Time: 26:05 to 27:05 min) in this, the mass of the system, I think we can easily find out from this without any problem. Now, that means, for example if 0.1 kg mass of the absorber nu found out trial mass any how we do not require that. So, it was this much 0.019. So, if nu is this much what will be the value of the trial mass with this? I want to know nu directly. This relation gives us the chi values and the chi values are nothing but the natural frequencies divided by the natural frequency of the primary frequency or the exciting frequency or the absorber frequency, because, they are all kept same. So, this now we find, the lower one is below 1350 and the upper one above. So, it is satisfying this frequency condition also. So, this is just a simple example of indicating that how an absorber is actually designed in the real life situation. It is always a trial and error back and forth satisfying the equation.

Now, if a system consists more than 2 degrees of freedom, we do not give a separate 3 degree of freedom or 4 degree of freedom, what we now consider in general cases of multiple degrees of freedom. 2 degrees of freedom was a special case of multiple degrees of freedom system; nevertheless, all the important basic concepts like natural mode, natural frequencies that we have already discussed in 2 degrees of freedom system and they will be all applicable in case of multiple degrees of freedom system.

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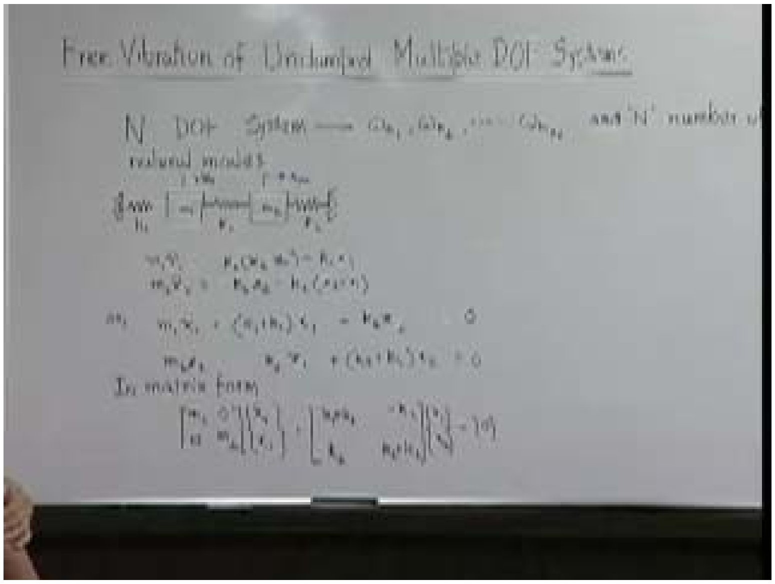

So, therefore in N degree of freedom system, we will have N number of natural frequencies and N number of modes. So, now, it will be necessary to know new concepts. But to present or to handle the whole analysis or discussion in a very compact manner, it will be necessary to utilize metric representation. So, to indicate that how matrix equations are written, I will take a simple example of a 2 degree freedom system.

This is a 2 degree of freedom system. We have already seen this; it is familiar to us. What will be the equations of motion? Equations of motion will be m_1 x_1 2 dot will be equal to k_2 into x_2 minus x_1 minus k_1 x_1. We can reorganize and we can finally write, or same thing can be written as m_1 x_1 2 dot plus k_1 plus k_2; we have done this before. For the second mass, we can write in this manner (Refer Slide Time: 31:31 to 32:17 min). This is simple reorganization and obviously, we are discussing the problem of free vibration. Now, how we can represent this set of equations in matrix form? The same equation you will find can be written as this. So, we use second bracket for column matrixes and third bracket for square or rectangular matrixes.

So, you know matrix representation. Therefore, there is no need to explain in detail, but you will find this is the matrix (Refer Slide Time: 33:20 min to 33:32) representing the mass of the whole system. This square matrix represents the stiffness of the whole system; this column matrix represents the mode shape. You can say when it is oscillating in a natural mode, this will represent the column of the natural mode oscillation and this is the acceleration direction.

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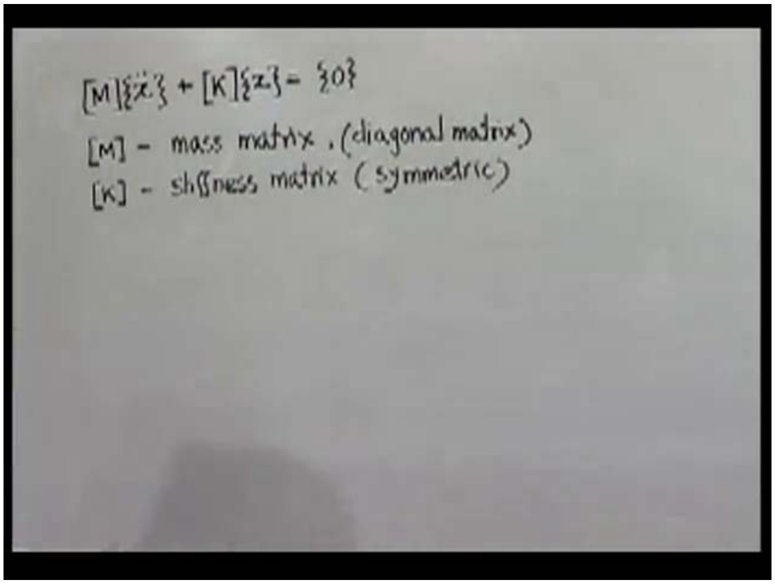

Therefore, in compact form we write that mass matrix into acceleration matrix plus stiffness matrix into displacement matrix is equal to column normal, where m is the mass matrix. If we take the coordinates from equilibrium position, it will be always a diagonal matrix and k is the stiffness matrix. This will always be a symmetric matrix; diagonal matrix also, of course is a symmetric matrix, because, all diagonal terms are equal to 0. Here (Refer Slide Time: 35:15 min) the diagonal term with this and this; that means, one half of the matrix that is above the diagonal term will be same as the terms in the locations below the diagonal. So, this is a general characteristic of all multiple degree freedom system.

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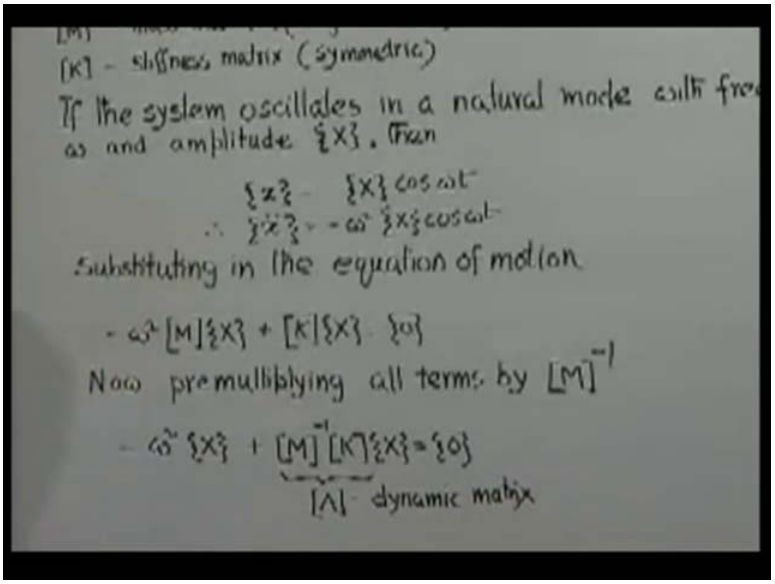

Now, if this system oscillates in a natural mode with frequency omega and amplitude by a column matrix X like capital X_1, capital X_2 and capital X_3. These are amplitudes of the mass of the coordinates 1, 2, 3 and so on; frequency is same for all.

So, therefore then displacement of the system can be written as (Refer Slide Time: 36:46 to 39:00 min) and obviously, acceleration will be this. If you substitute in this equation, what we get? We get this term into minus omega square, cosine omega t has been cancelled because it was common to both terms. Now, pre-multiplying both sides, all terms, by M inverse, what happens? When you do it, M inverse becomes unit matrix, so it is 1. So, unit matrix into X symmetric plus M inverse into k stiffness matrix into column matrix X is equal to 0. Now, this matrix we can give a name say A matrix which you call it as dynamic matrix.

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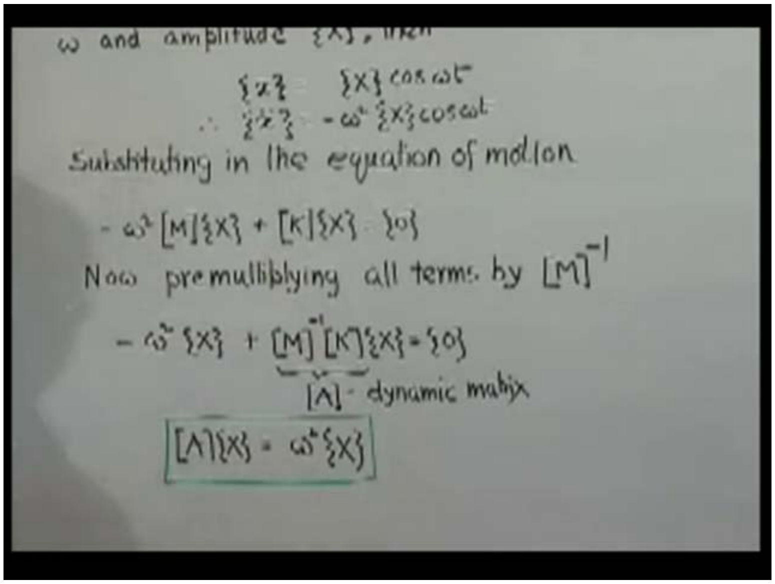

So, we can write the equation in the form A X is equal to omega square X for a particular natural mode. Remember, if it is the I th mode, this frequency will be omega$_I$ and mode shape will be X_I. Now, this is a typical equation which perhaps familiar to you. (Refer Slide Time: 39:46) Now, this matrix equation is satisfied only for specific sets of Xs, what I mean to say that when there is a column matrix, you multiply the column matrix by square matrix, you get back the column matrix only with a constant equation. This is possible only for some specific column matrices and for some resulting values of some specific coefficient of this one. This is called the Eigen values problem and the value which satisfies this equation here with omega are called Eigen values and these columns are also called vectors.

So, these vectors are called Eigen vectors which are nothing but the normal mode and the Eigen values are nothing but the squares of the natural mode. Of course, these can be written in equation form also.

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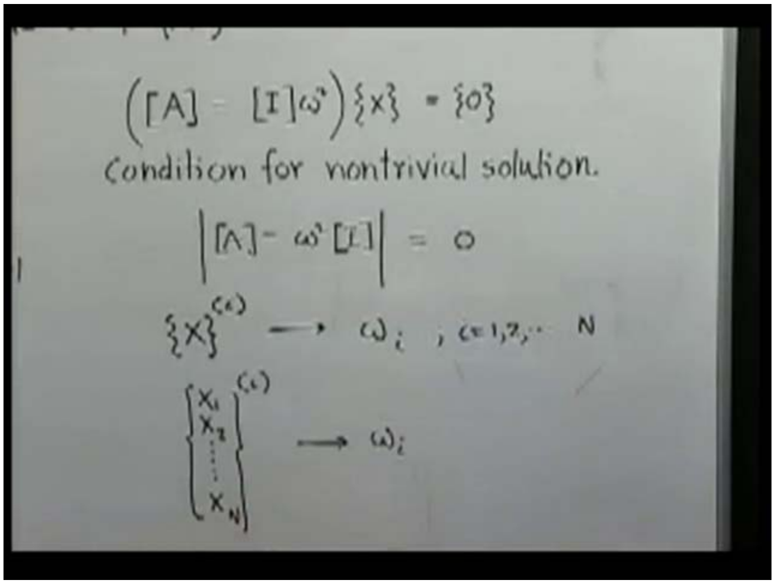

We get a state of homogeneous equation just like previous stages and the condition for non-trivial solution will be the determinant of this matrix (Refer Slide Time: 41 22). I is the unit matrix, which is again nothing but the characteristic equation solution which will be nothing but the values of omegas of the natural frequency. That we have already seen. So, for each natural mode, that means, if natural mode is identified by this I, corresponds to a natural frequency omega_{ni} or we can write it in omega_i. We do not want to write all the time $\text{omega}_{n,}$ because now we will be writing this too often, so we may write this simply as omega_i. This is nothing but the Ith matrix. So, that n subscript we can remove from all the part. So, if there are N number of degrees of freedom then i will vary from 1, 2 up to N; N sets of values that means x_1 x_2 up to x_N. So, there are N coordinates and for the ith matrix, this is the natural mode of the ith matrix and this corresponds to omega_i.

Now, we can see these natural modes are really nothing but Eigen vectors of this equation. Now, another very important thing; we will find that natural mode cannot be just arbitrarily anything. We will now discover some extremely important properties of this natural mode which will be extremely useful and let us see this.

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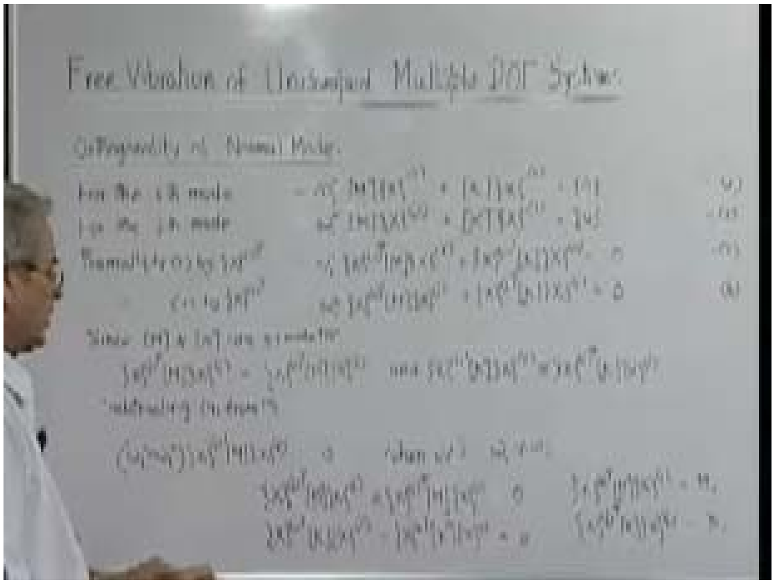

So, let us consider the ith mode. For ith mode, what will be the equation of motion? You can see, it will be minus omega I square, let us call this as equation 1. For the jth mode equation will be this (Refer Slide Time: 45:05 to 46:40 min) and we call this as equation 2. Now pre-multiply 1 by x of j transpose; I am quite sure you understand transpose; automatically just exchanging the columns and rows. So, if we pre-multiply this equation by this, what we will get? We get this. Let us call it as equation 3. Now, pre-multiply 2 by x of i transpose. So, that equation will be this.

Now, it has been already mentioned that m and k are both symmetric matrix; of course, m is a diagonal matrix. So, it is obviously [47:16], but this is a symmetric matrix. Therefore, since m and k are symmetric matrices, they satisfy this condition, where x of j transpose m x i will be same as x of i transpose m into j and this (Refer Slide Time: 48:05 to 50:04 min). j transpose into k is a property of symmetric matrix. When a symmetric matrix is pre-multiplied by the transpose of a column matrix and post-multiplied by another column matrix that is equal to the transpose of the column matrix post-multiplied by m and here it is like this, both for m and k will be like this. So, now subtract equation 4 from equation 3; if you subtract this from this, what we will get? Omega$_j$ square minus omega$_i$ square because this is equal to this. So, we can write any one of them. So, we will write this j transpose m X_i transpose and this equal to this. So, subtracting it will be 0.

When i is not equal to j, in general, $omega_i$ is not equal to $omega_j$. So, the only way this condition can be satisfied is like this (Refer Slide Time: 50:45). Now if this is 0, (Refer Slide Time: 51:19) this term is 0, then this has to be also 0; So, it follows. This is a very special and important property; that means that two different modes when post-multiplied and pre-multiplied, then both the mass matrix and pre matrix, it becomes 0. They are not 0 only when i and j are equal and they are called generalized mass and generalized stiffness; that means, if I put X_i transpose m X_i this will be a quantity which we may call it as m_i and we can give the name generalized mass.

Similarly, xi transpose k X_i will be again 0 quantity and we call it generalized stiffness. This property is called Orthogonality of Normal Modes. These are two very important properties and at later stage, you will find they will be extremely useful to us. Now, I think our task is determination of the natural mode and natural frequency. If we directly solve these and try to get the values, of course, it is fine and we will get the values as we have already done in one or two cases. However, when a system is large, quite often what happens? A regular normal actual system is modeled in the form of a parameter system, but to make the model very realistic resulting in acceptable values of the answer, so the number of degrees of freedoms is very large.

So, for example, a machine tool or car, vehicle all such kinds of actual systems, the presentation will be meaningful only when the numbers of degrees of freedom are large. So, when capital N, that is degrees of freedom is very large then solving this equation directly, it becomes serious one and that is why many methods are being revised. It can be applied to simple engineering design problems and this method which now we will take up rather than solving the direct equations and one type of method called as iteration method; we will iterate the matrix and we will get the value. That is of course, a repetitive process, but it converges to the answer after few steps. Only thing that computationally it is far less intensive than solving, say for example, a 10 degree of freedom system directly solving 10 degree polynomials.

We will next take up the methods of solving this multiple degree system. First, we will discuss matrix iteration method, and subsequently, we will just discuss some approximate methods for resulting in very quick values giving an idea about the order of magnitude of the fundamental natural frequency that is the lowest natural frequency.