Dynamics of Machines

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Module - 12 Lecture - 2

Forced Vibration of Two Degrees of Freedom Systems; Tuned Vibration Absorber

So we continue our discussion on free vibration of a two degree of freedom systems of course without damping. Now, I hope you remember that we mentioned that under certain very specific initial conditions a system may vibrate in a particular natural mode, where each and every member executes simple harmonic motions with same frequency. However, if the initial conditions are not chosen in that particular specific manner then the general free vibrations what results from an arbitrary initial disturbance can be expressed in terms of the natural mode of oscillations in the following way.

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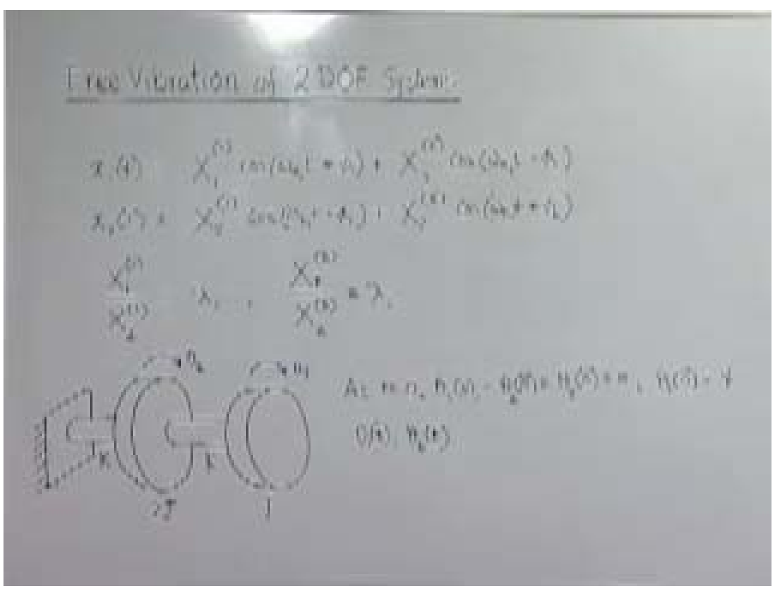

x_1 (Refer Slide Time: 01: 22) We also know that the ratio of the amplitude in a particular mode we have determined from free vibration analysis (Refer Slide Time: 03: 01). So, what actually remains to be done is in the four unknown quantities it is actually not like as it appears to be, because I can always expect the terms of x_2 as $lambda_1$ into $X_2 1$ and this should be $lambda_2$ into $X_2$2. So, there will be actually only four unknown quantities that will be $X_2$1, $X_2$2, phi_1, and phi_2. So, these four quantities can be determined from the initial condition.

Let us solve a problem to demonstrate this. (Refer Slide Time: 03:49) Let us take a particular case of torsional oscillation (Refer Slide Time: 03:54). We call this preferably J1, the moment of inertia of this is J, (Refer Slide Time: 04:42) this is 2J and the torsional stiffness of this part of the shaft is capital K; same is the case with this one if they are two identical shafts. The displacements are denoted by two angular rotations, (Refer Slide Time: 05:05) one for this which is called $theta_1$, and one for this which is called $theta_2$. Now, the initial condition is started or the oscillation is started at t is equal to0, $theta_1$ dot 0 equal to $theta_2$0, $theta_2$0 equal to 0, they are all 0. Only thing what we do is, we keep everything as it is only and rotate this (Refer Slide Time: 05:46). That means, $theta_1$ at 0 is say some angle phi.

So, how we have started the vibration? We have provided no velocity and we have kept this where in its original position, $theta_2$ equal to 0; only twisted by an amount phi and then, release the whole thing to execute free oscillation. Therefore, we have to find out $theta_1$t, $theta_2$t and we know in general that when it is done arbitrarily, they will be composed of both the modes as shown here (Refer Slide Time: 06:38). Now, before we solve a few vibration problems, it is obviously essential to find out the natural mode oscillation.

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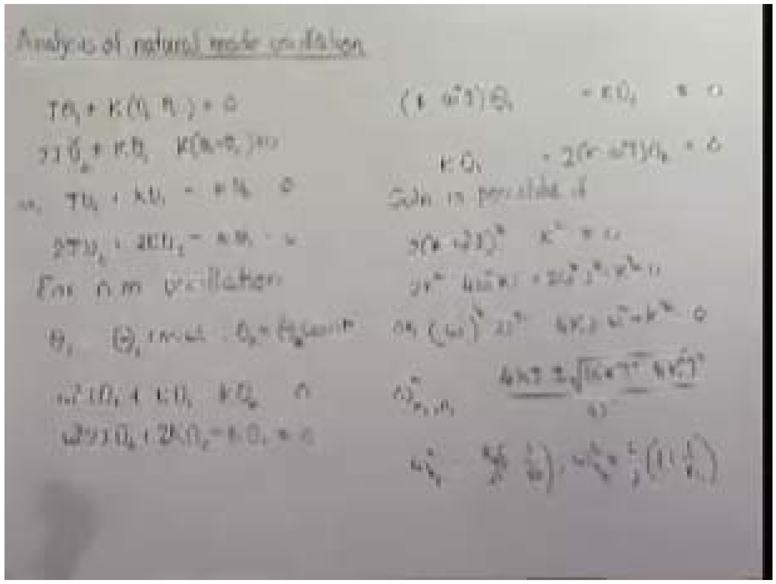

First, let us do that. Now, what will be the equations of motion? However, the twist of this shaft is always theta$_1$ minus theta$_2$. Therefore, for the first mass, if we denote the mass of the shaft with J into theta$_1$two dot plus K into theta$_1$ minus theta$_2$ equal to 0; that is the equation of motion of the first disc. The equation of motion of second disc is 2J theta$_2$ two dot plus K theta$_2$ minus K into theta$_1$ minus theta$_2$.

You can write it like this J theta$_1$two dot plus K theta$_1$ minus K theta$_2$ equal to 0. This will be 2J theta$_2$ two dot plus 2K theta$_2$ minus K theta$_1$ equal to 0. So, these are the equations of motion. Since it is natural mode oscillation, we can assume that both are simple harmonic functions of time and theta$_1$ equal to theta$_1$ cosine omega t and theta$_2$equal to theta$_2$ cosine omega t. Now, if we substitute these two in these equations, what we will get? We get minus omega square J theta$_1$ plus K theta$_1$ minus K theta$_2$ equal to 0. This second equation will give me minus omega square 2J theta$_2$ plus 2K theta$_2$ minus K theta$_1$ equal to 0. When we write these equations properly, what we will get? We will get K minus omega square J into theta$_1$ minus K theta$_2$ equal to 0. This will be the second equation (Refer Slide Time: 10:53). Now, obviously, solution will be possible for theta$_1$ and theta$_2$ of the state of homogeneous equations if the determinant is 0. So, the characteristic solution is possible if 2 into K minus omega square J whole square minus K square equal to 0 (Refer Slide Time: 11:35).

This solution will give us the natural frequencies. What are those natural frequencies for the system? It will be 2K square minus 4 omega square KJ plus 2 omega square J square minus K square equal to 0. Otherwise, it can be written as omega square whole square into 2J square minus 4 KJ omega square plus K square equal to 0. So, omega_{n1} and omega_{n2} square will be 4 KJ plus square root of 16 K square J square minus 8K square J square divided by 4J square. Therefore, the two roots will become, omega_{n1} square will be K by J 1minus 1 by root 2 and omega_{n2} square will be K by J into 1 plus 1 by root 2. So, the two natural frequencies are determined. Now, let us move onto the mode state. I have told you how to find out the mode state, use any of the equations and substitute a frequency there and find out what happens. Say, for example, let us takes this (Refer Slide Time: 14:30).

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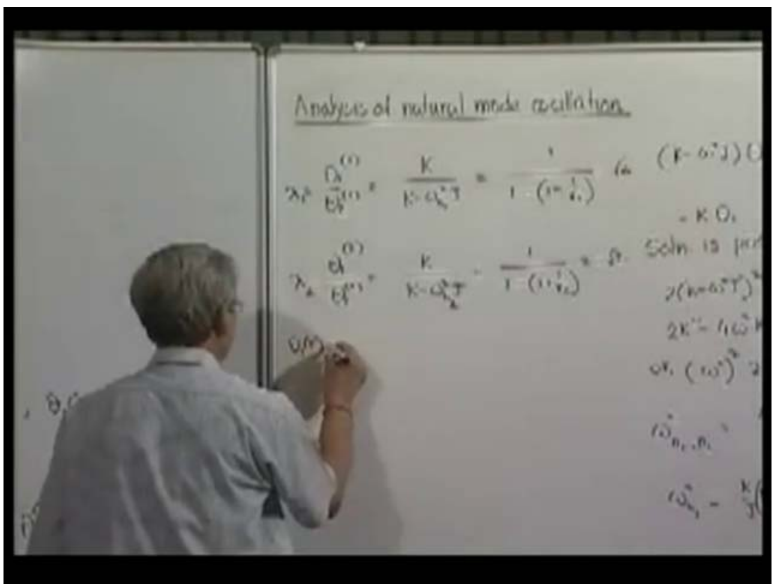

Now, if we take that one, we get theta_1 by theta_2 of first mode equal to K by K minus omega_{n1} square J and for the second mode (Refer Slide Time:15:12), which we call as lambda_1 (Refer Slide Time: 15:17) lambda_2 it will be K by K minus omega_{n2} square J. Then, how much is this? We can find out, 1 by 1 minus (Refer Slide Time:15:37) omega_n square into J by K will be 1 minus 1 by root 2 and this will be equal to root 2. In this case, it will be 1 by 1 minus omega_n square into J by K will be 1 plus 1 by root 2, that will be again equal to minus root 2. So, the mode states are also determined. Once the mode states and the natural frequencies are determined, we can now use this expressions (Refer Slide Time: 16:21) or this type of representation for our ultimate final result.

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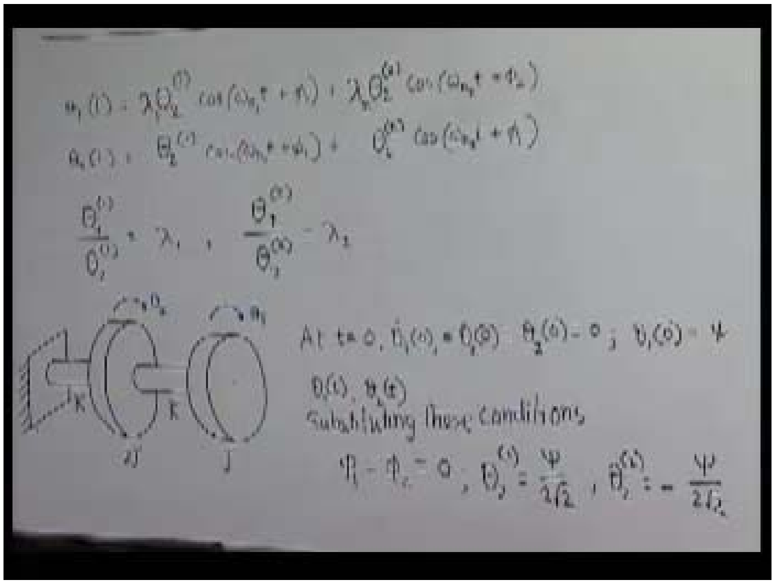

So, let us replace x by theta. Now, it is the actual vibration, not normal mode oscillation. So, this theta$_1$ we should not confuse with the solution of theta$_1$what you are doing and the amplitudes are theta$_1$and theta$_2$. Similarly, here also we have used this (Refer Slide Time: 17:41). What can we do? Now, we can write this as theta$_1$1, which is nothing but lambda$_1$ into theta$_2$1 and theta$_1$2 is nothing but lambda$_2$ into (Refer Slide Time: 17:55) theta$_2$1. Now, both these lambdas and omegas are all known.

Now, what we do? We use this initial condition and we find out theta$_1$ dot and theta$_2$ dot and t is equal to 0. We substitute both as 0 then theta$_2$ also at t is equal to 0 and theta$_1$ at 0 is equal to phi, using all these we get phi$_1$ equal to phi$_2$ equal to 0. theta$_2$1 is equal to psi by 2 root 2 and theta$_2$2 equal to minus psi by 2 root 2. Using these expressions, we will get theta$_1$ t. This is the complete solution (Refer Slide Time: 21:12). Therefore, we can solve the free vibrations problems of a 2 degree of freedom system for an arbitrary initial condition.

The procedure is: first, we solved for the natural mode vibration, determined the mode state and the corresponding natural frequencies; then, used the general expression in terms of the two natural modes. Then, using the initial conditions, we will find out the final one.

It is important for us to solve the force vibration problem of a 2 degree of freedom system. Why we are doing it? Because it is not only for the sake of completeness of the analysis of 2 degree freedom system, but also for the force vibration of 2 degree freedom system has some very important practical applications in engineering. In force vibration of 2 degree freedom systems, we now take up and we solve a particular system, though there is no problem in handling a case, but we will involve ourselves primarily with this system as it has considerable amount of practical applications.

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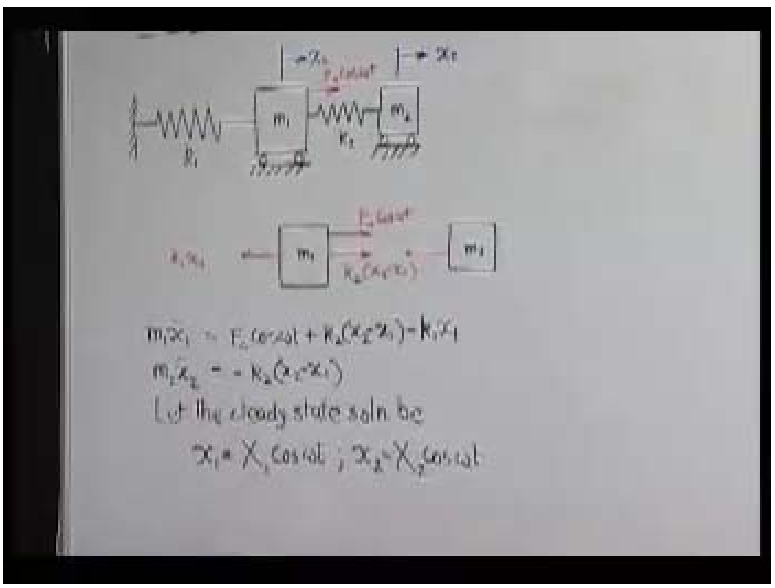

So, our system is quite simple. It is k_1m_1 and then, this is k_2m_2 and this system is again connected with m_1. There is a simple harmonic force applied to this mass m. What will be the solution? Now, one thing we should all the time remember that in the steady state solution of a force vibration problem, even though there is no damping associated but in nature everything ultimately gets damped. So, in the steady state, we will have solution of the system in a way that everything vibrates with the same frequency. Therefore, we can always assume the steady solution of a force vibration problem as again another simple harmonic function of time, with the same frequency.

The pivoted diagrams if you want to draw, in this direction (Refer Slide Time: 24:47) the state of the spring here is x_2 minus x_1. It will be k_2 into x_2 minus x_1 acting in this direction. This is externally applied force. Of course, the force is equal and opposite of this. So, the equation of motion in this direction is m_1x_12 dot and this must be equal to total force in this direction is F_0 cosine omega t plus k_2 into x_1 minus x_2 minus k_1x_1. There is one opposite force, which is minus k_1x_1. For mass, two total force in that direction is minus k_2 into x_2 minus x_1.

Now, let the steady state solution be x_1 is equal to X_1 cosine omega t; x_2 is equal to X_2 cosine omega t. That is the matter of the phase because we know when there is no separate damping; the phase difference between various bodies can be either in same phase or in opposite phase. So, that matter is taken care of by the signs of x_1 and x_2. If x_1 and x_2 are of the same sign, then they will be same. If they are of opposite signs, they will be (26:54). Now, substituting this in these two equations of motions, what do we get? Then if were organize this will become minus omega square capital X_1 and cosine omega theta everywhere will ultimately get cancelled.

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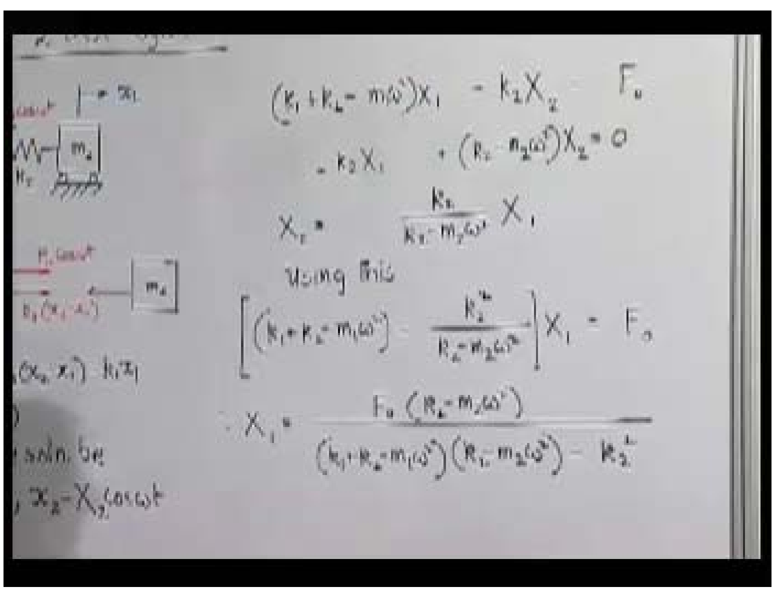

So, we can write directly as k_1 plus k_2 minus m_1 omega square X_1 minus k_2X_2 is equal to F_0. The second equation will become minus k_2X_1 plus k_2 minus n_2 omega square into X_2

equal to 0. Therefore, there are two equations and only 2 unknown X_1 and X_2, because, you know in all forced vibration problems, our objective is to find out the magnitude of vibration. Frequency is already known, which is equal to the force into frequency. In free vibration, our final objective is to find out the sequence and the amplitude depends on the initial disturbance given. To solve it, let us express X_2 in terms of X_1, so X_2 is k_2 by k_2 minus m_2 omega square into X_1. Substituting this in the equation (Refer Slide Time: 29:15) we get X_1 equal to F_0 into k_2 minus m_2 omega square divided by k_1 plus k_2 minus m_2 omega square into k_2 minus m_2 omega square minus k_2 square.

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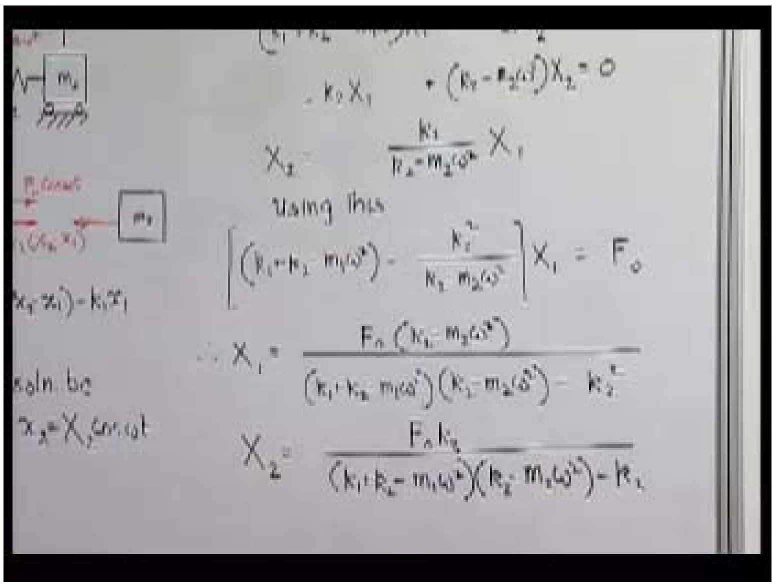

Using this, if we multiply this quantity by this, X_2 will become like this (Refer Slide Time: 30:45). Now, here one thing is very clear, both X_1 and X_2 become infinite when the denominator becomes 0.

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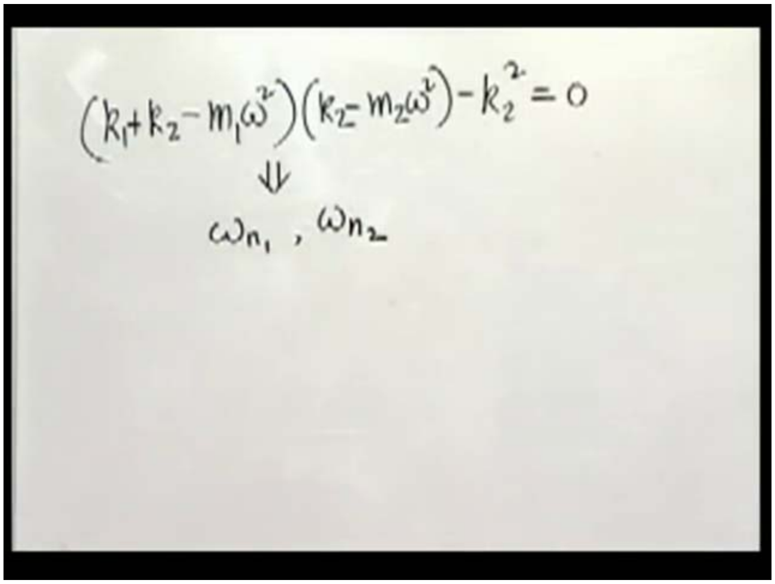

The denominator becomes 0, that is, the solution of this equation will lead to two values of omega. Now, remember this omega is not the total frequency. I am saying at which this becomes 0. So, obviously, this will give two frequencies. One will be the first natural frequency and the other will be second natural frequency. Therefore, this will be 0, when the exciting frequency omega matches one of the two natural frequencies of this system when it is allowed to vibrate freely. The solution to this $omega_{n1}$ and $omega_{n2}$ can be found out from this characteristic equation we can find that for the few vibration problems, F_0 is 0 and so, the two equations which we get here both are homogeneous and their solution will exist only when its determinant is 0, which is again nothing but this (Refer Slide Time: 32:32).

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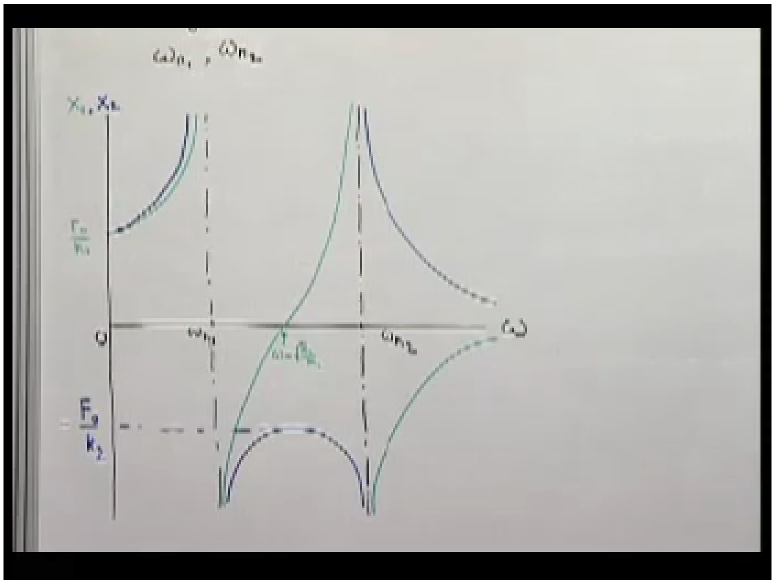

If we plot, let this be omega$_{n1}$, this be omega$_{n2}$ is equal, and this will be0. The vertical axis we will have two, one is X_1 and the other is X_2. Now, when omega is 0, X1 is equal to 0. So, omega is 0, top one will be F_0 into $k_2$and the bottom will be k_1 into k_2. $k_2$squarewill get cancelled; so, only k_1 into k_2. So, the total thing will be F_0 by k_1. It starts with F_0 by k_1. Then it increases as this omega tends to omega$_{n1}$ and this denominator tends to be 0. So, as omega tends to omega$_{n1}$, this tends to be like this (Refer Slide Time: 34:42). Then, when omega crosses omega$_{n1}$, then it starts and it becomes negative. Again, it becomes smaller and it becomes 0, when omega is equal to square root of k_2 by m, again it becomes positive. When omega crosses omega$_{n2}$, we can show that this is the nature of function X_1, when you plot against omega.

If you plot X_2, what we get? When omega is 0, obviously, this is 0, this is 0, $k_2$squaregets cancelled. So it starts from the same point but again it goes up, but when this crosses 0, then we will find that it can be shown like this (Refer Slide Time: 36:18).

So, this is the nature of variation of (36:20) (Refer Slide Time: 36:24). When at this position, omega square is equal to k_2 by M, this term is 0. So, X_2in that condition will be minus F_0 by k_2. So, and this much will be minus F_0 by k_2. Now, the important thing to note is that for a particular situation, when exciting sequence is equal to square root of k_2

by m_2, then the primary mass m_1 stops vibrating. When this stops vibrating (Refer Slide Time: 37:40), we can use these techniques for stopping vibration in system by attaching an auxiliary mass and this is called vibration absorber.

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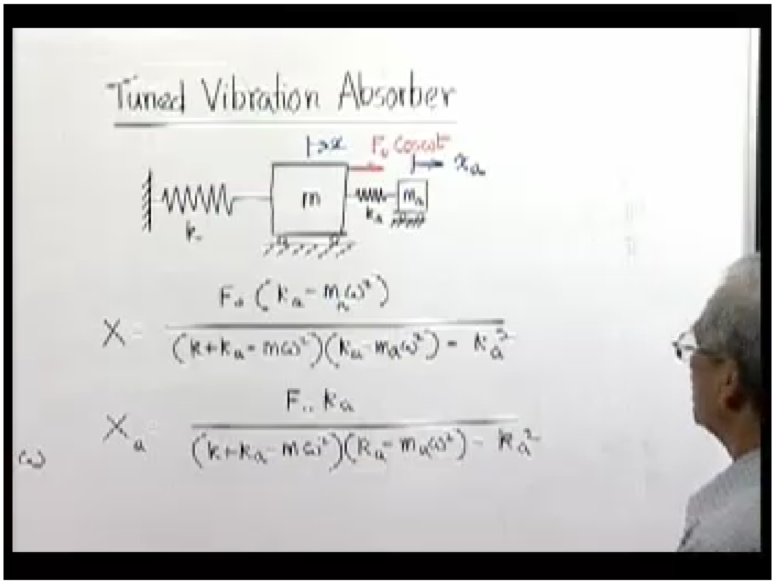

In vibration absorber, this is the primary system or system which is the real system and it is being subjected to simple harmonic excitation. Then we want to stop this vibration, because it is obviously going to vibrate, the technique is then to attach a separate system which we call the absorber system. Obviously, the absorber vibration is x_a. So, the same equations apply; our amplitude of vibration X under this condition will be F_0 into k_a minus m omega square by k plus k_a minus m omega square into k_a minus m_a omega square minus k_a square (Refer Slide Time: 40:22). So, this is the amplitude of the primary mass oscillation and this is the amplitude of the absorber mass oscillation (Refer Slide Time: 40:40). Since our objective is to absorb or stop the vibration, we should always tune it in such a manner, so far as the vibration absorber is concerned that the exciting frequency is equal to k_a by m_a from there, because under that condition, this will be 0.

The primary mass vibration stops. Also, generally since the vibration of a system is more problematic when it resonates, therefore, we have to be careful about those situations, where omega is equal to the natural frequency of the primary system.

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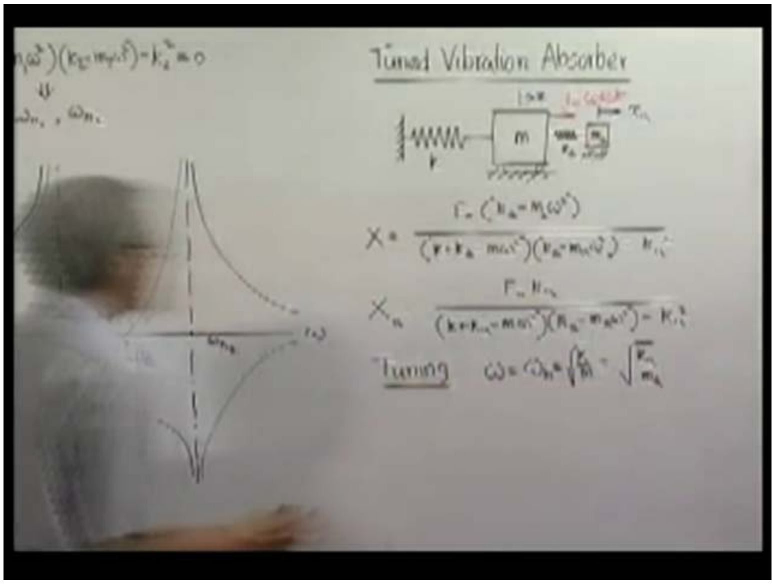

Therefore, generally the tuning is done in this manner. The natural frequency of the system is equal to natural frequency of the absorber, omega should also match this. Then, we will get the best results that this package vibration force is there and the system is trying to resonate, then we attach an absorber mass and the vibration stops. Now, why it stops can be easily seen. What happens when force is equal to or omega is equal to square root of K_a by m_a?

When omega is equal to square root of k_a by m_a, we find that X is equal to 0. How much is X_a now? k_a by m_a is equal to omega. So, this becomes 0; so this goes. Hence, X_a is nothing but F_0 by k_a with a minus sign.

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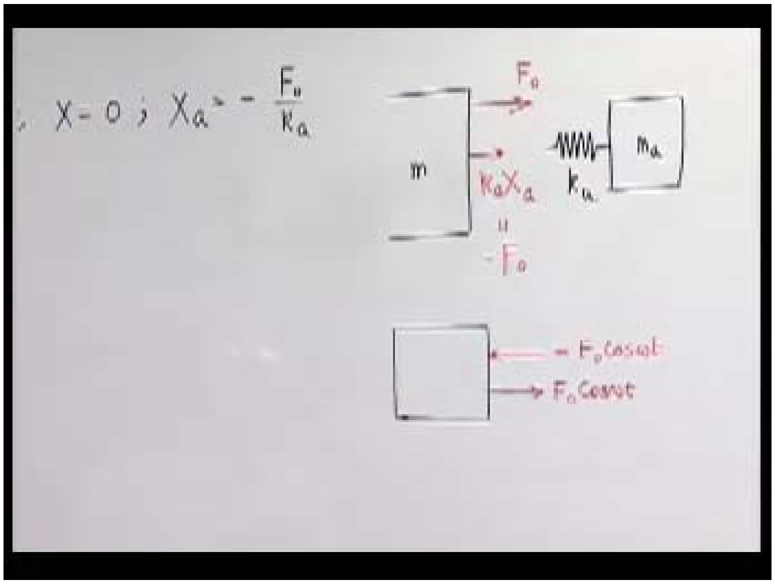

Now, what will be happening if this is the auxiliary system and this is the main system, then the force which is acting here, it will be simply k_a into X_a. Of course, there is another force here, which is F_0. Now, how much is this under this tuned condition? It is minus F_0 plus every instant of course when you multiply by cosine omega t in both the places, what happens?

When this primary mass is subjected to two forces, one is F_0 cosine omega t and the other is minus F_0 cosine omega t. Therefore, the sum total of the force acting on the primary mass is always pivoted at every instant. So, every time the force which is acting here is neutralized by the spring force attached due to vibration of the auxiliary system. Thus we find that the vibration of the primary mass stops at this equations, only because the vibration of the auxiliary mass produces a force on the primary mass at every instant which is equal and opposite to the externally applied shear. So, actually a better term would have been vibration neutralizer, because it really neutralizes the force in action. Since the vibration stops, people have generally always called it as vibration absorber, but there is no dissipation of energy as you can see. Therefore, designing vibration absorber is a very important thing.

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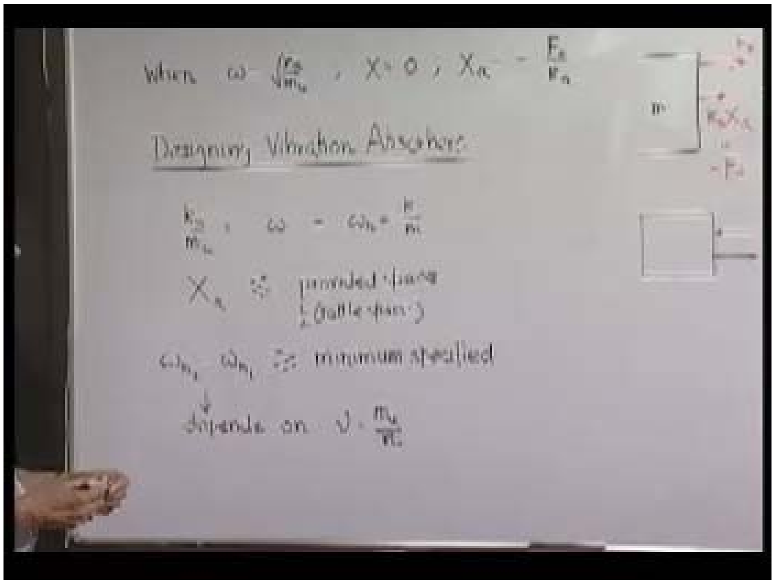

When you want to design a vibration absorber, its conditions will be primarily that k_a by m_a will be equal to omega, which is known and (Refer Slide Time: 46:25) this is generally omega$_n$ of this (Refer Slide Time: 46:20) Then, we have to also be careful about it must be less than or equal to provided space or called the rattle space. If everything vibrates, the total amount of space is occupied; it will be on both sides of the initial position by an amount X_a. So, the total amount of rattle space is always twice X_a. So, X_a should be less than equal to half the rattle space provided in the system; it cannot be anything. Therefore, these are the two very important conditions. Another important condition is that what we find here that if we operate exactly at that frequency, there is no problem and X_1 is 0.

But always you know in our real system, the exciting force may fluctuate in its frequency. If it happens that rather than that your operation is here (Refer Slide Time: 47:47),then, you find that your amplitude will be this point (Refer Slide Time: 47:55), which otherwise perhaps it would have been much less that even without the absorber, if the force acts, sometimes it may happen that the amplitude of the system is less than what you will have with the vibration absorber, because now this one natural frequency is split into two and you are going very near to another one. Therefore, it is important that it does not happen. You would like to have as much separation or spreading out of the natural frequencies as possible So that even if there is some fluctuation of the force in frequencies, the chances of it will be going to one of the natural frequencies after this absorber is attached [ ]. Therefore, the other conditions will be omega n_2 the two natural frequencies emerge only because we have attached the absorber system otherwise it was a single degree freedom system should be more than equal to some minimum specification point.

The points that we have to keep in mind while designing vibration absorber: one is that its natural frequency of the absorber systems will be exactly equal to the total frequency, which normally is near the natural frequency of the system. This actually tells you what can be the maximum value of the amplitude of the absorber mass and what minimum spreading is required between the two natural frequencies of the resultant system to avoid a large oscillation in case of fluctuation in the exciting frequency. These are the three conditions. Now, this depends on as you can show on a quantity which is called nu, which is the ratio between the two masses which determine the ratio. To find out the characteristic equation, it can be solved like this.

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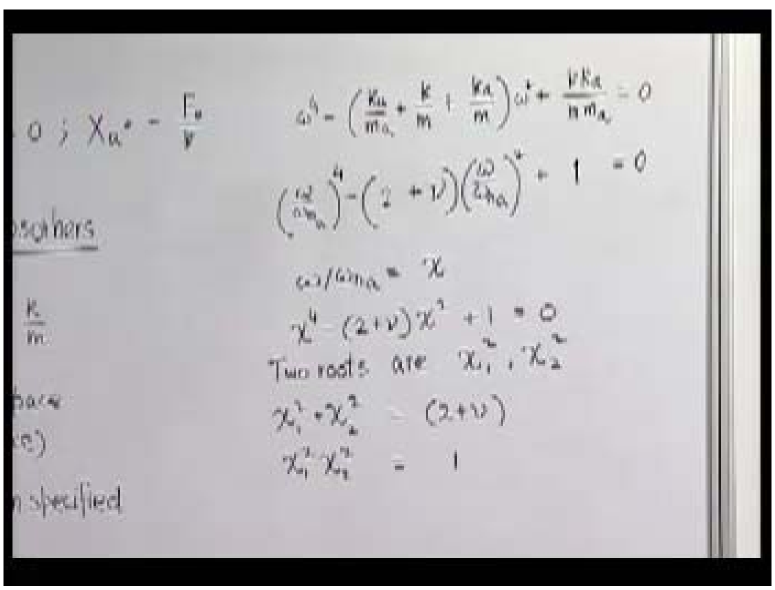

So, this equation becomes omega to the power 4 minus k_a by m_a plus k by m plus k_a by m into omega square plus kk_a by mm_a equal to 0. If we expand, this will be the equation.

Now, we can divide the whole thing by omega$_n$ to the power 4. Now, if you take k_a by m_a common, remember k_a by m_a is equal to k by m. So, there actually, it is nothing but 2 into k_a by m_a, take k_a by m_a common, which is omega$_{na}$ square and if you divide by omega$_{na}$ to the power 4, then below we get omega$_{na}$ square. Therefore, we will have this one and this 2 plus, if we divide this by k by m_a, what we will get is m_a by m, which is nothing but nu and this will be (Refer Slide Time: 52:02) k_a by m_a and k by m are both equal to omega$_{na}$ square. So, this whole thing is nothing but omega$_{na}$ to the power 4 and when you divide by omega$_{na}$ to the power 4 this is 1.

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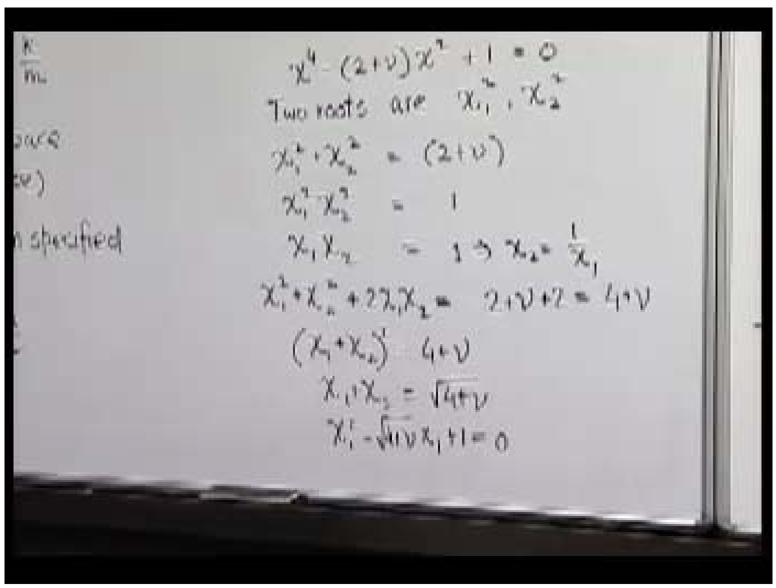

Let us call this to be this equation: x to the power 4 into 2 plus nu into x square plus 1 equal to 0 (Refer Slide Time: 52:30). So, it will lead to two roots, which are chi$_1$ square and chi$_2$ square. So, we know from that quadratic equation, the sum of the two roots equal to minus 2 plus nu and the product of the two rows are equal to 1. Therefore, you can write chi$_1$ into chi$_2$ is obviously 1; not minus 1, because chi$_1$ and chi$_2$ are both positive.

Now, if we add chi$_1$square and chi$_2$ square and we multiply this by 2, it becomes 2 chi$_1$ chi$_2$. So, what we can do? We can add 2 chi$_1$ plus chi$_2$, this is equal to 2 plus nu and this is equal to 1 plus 2 equal to 4 plus nu. This whole thing is nothing but again chi$_1$ plus chi$_2$

whole square. So, chi_1 plus chi_2 is equal to square root of 4 plus nu. This tells us that chi_2 is equal to 1 by chi_1. If we use this here we will get chi_1 square minus 4 plus nu chi_1 plus 1 equal to 0.

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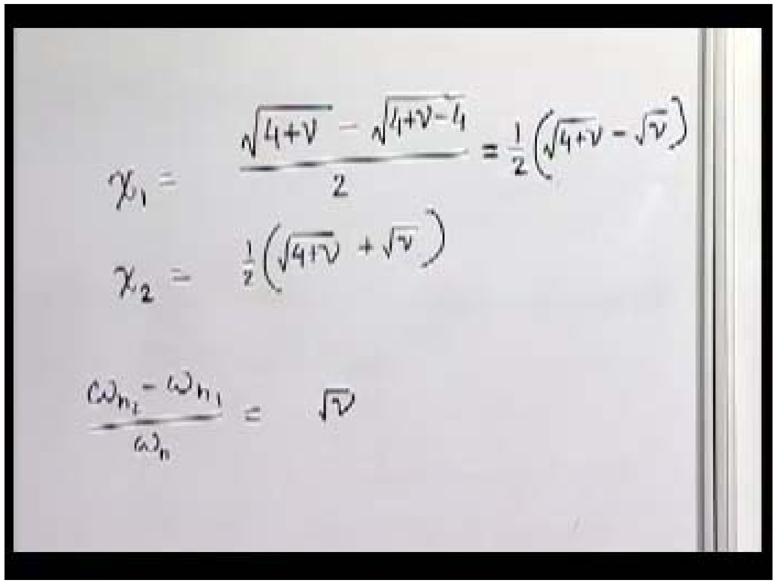

Solving this, we get chi_1 equal to half into square root of 4 plus nu minus root nu and chi_2 equal to half (Refer Slide Time: 55:47) into square root of 4 plus nu plus root nu. So,$omega_{n2}$ minus $omega_{n1}$ divided by $omega_n$ of the primary original system that is (56:20), so we will find this is equal to root nu. Therefore, (Refer Slide Time: 56:42) two natural frequencies between that and this is given by square root of k by m, this is a known quantity (Refer Slide Time: 56:56), which is same as omega equal to omega n_a. So, it depends on this mass ratio (Refer Slide Time: 57:06).

Therefore for a given primary system, what will be the mass of the auxiliary? What will be the thickness of the auxiliary? All these two things you have to find out satisfying this condition. May be solving one example (57:24) will try to solve one problem to demonstrate how this vibration absorber is designed.