Dynamics of Machines

Prof. Amitabha Ghosh

Indian Institute of Technology, Kanpur

Module-12 Lecture-1

Systems with 2 Degree of freedom; free Vibration

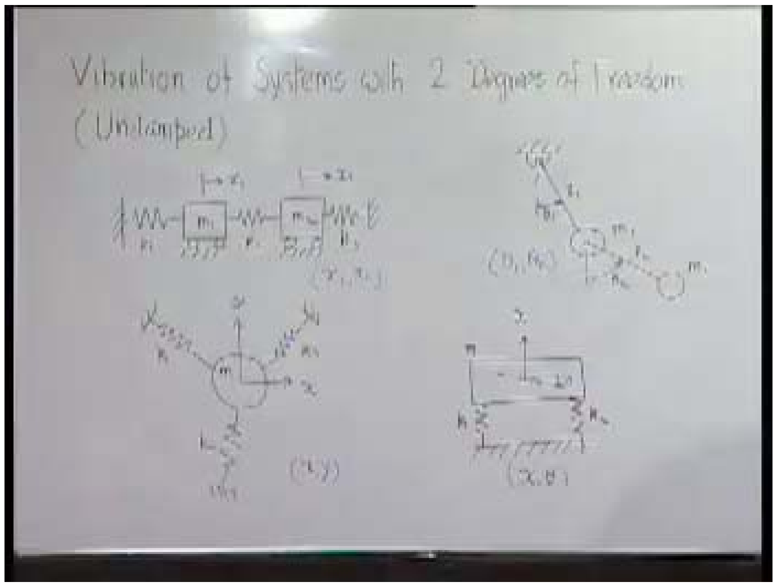
Till now we have been discussing the vibration problems involving systems which could be described with only one coordinate that means which possess single degree of freedom. However we encounter larger number of situations where the number of independent coordinates required describing the systems configuration is more than one. So therefore, it is necessary to discuss problems where systems have more than a single degree of freedom. In this direction we will encounter some new concepts where not present in single degree of freedom systems so before we take up discussions in detail. First let me present certain basic ideas about these new concepts which will be involved in place of vibration of systems with more than one degree of freedom.

So we start with the simplest next possible case that means vibration of systems with two degrees of freedom. Of course, we should also keep in mind that we are discussing undamped vibration. As just now mentioned that, this system will require two independent coordinates to describe the systems configuration, it can be in many ways. Maybe, it can be in the form of two blocks having vibration. It can be seen easily that this system requires two independent quantities x_1 and x_2, the displacement of the two masses forms their respective equilibrium position.

Similarly, in case of angular motion we can have double and in another case of two degree freedom as you can see the displacements from the equilibrium position. They are represented by two angular rotations of the 2 rods of length l 1 and l 2, the displacement coordinates are $theta_1$ and $theta_2$. Sometimes, a system with a two degree freedom system may not involve more than 1 lump of body.

For example, you take a system with a single body m. So, let us consider this body is supported by three strings. Mass of the body is m and let the body be confined to the plane of the o. So in that case, again a complete description of the location of this mass will require two coordinates x and y and therefore again the system will have two degrees of freedom. Similarly, in engineering we encounter situations where a body is resiliently supported.

(Refer Slide Time: 06:11)



So, in this case the body's motion again is confined to be black board. Maybe, body is the center of mass of the body can have only vertical motion, but again it can have a tilt. So x and theta will be the two coordinates here, x and y were the two coordinates. Here, $theta_1$ and $theta_2$ are the two coordinates required here x_1 and x_2 are the other two coordinates. So these are some examples of two degree of freedom systems. Now there are some fundamental points. Let us understand at the beginning, one thing that we are considering random system to begin with, and I will refer to situation when damping exists briefly, after we discuss this.

Now one thing we have seen a single degree freedom system, if we disturb the system from its equilibrium position, it executes simple harmonic motion invariably, without any problem. In case two degree freedom system suppose, this one I disturb somehow within the bodies execute simple harmonic motion. The answer is no that is a very fundamental difference with the single degree freedom of system that means the oscillation, free oscillation of this will not be necessarily a simple harmonic motion, if the displacement is given in an arbitrary manner.

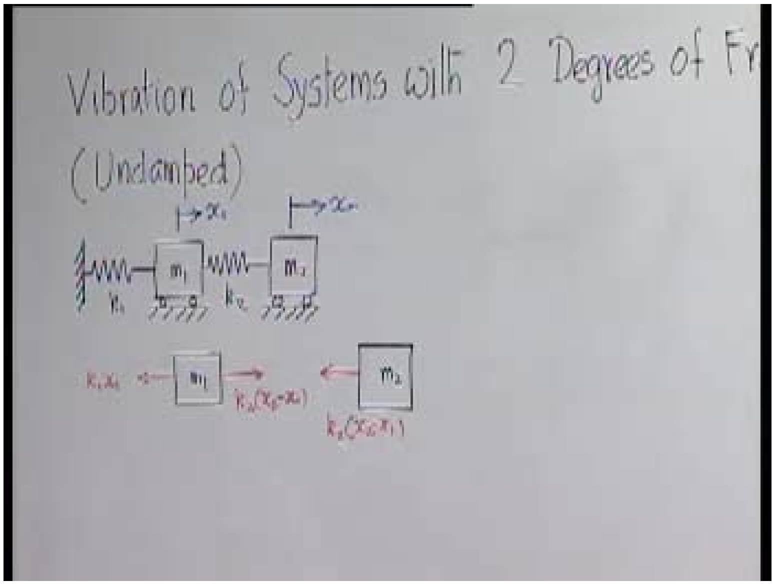
On the other hand, it is possible to give the initial condition to start the motion of a two degree freedom system, in such a way that both the bodies will execute simple harmonic motion and we will see in the case when one body is executing simple harmonic motion, both the bodies are executing simple harmonic motion the frequency of that oscillation for both the bodies are same. Phase difference can be there that means, they can be either in phase or out of phase that we will see later. But the frequency of oscillations will be same and both will execute simple harmonic motion only for some specific cases where the initial conditions are adjusted properly. So therefore, there is a special situation which exists in cases, where more than one degree of freedom exist that, the vibration is not necessarily simple harmonic if it is disturbed from the equilibrium position in an arbitrary manner.

On the other hand, just now as I mentioned if we disturb it in a particular way satisfying certain conditions then the system will execute simple harmonic motion both the bodies at the same frequency. Now this type of motion that means, where the system is executing simple harmonic motion with the same frequency all the parts of the body are called either natural mode of vibration or normal mode of vibration, both terms are used.

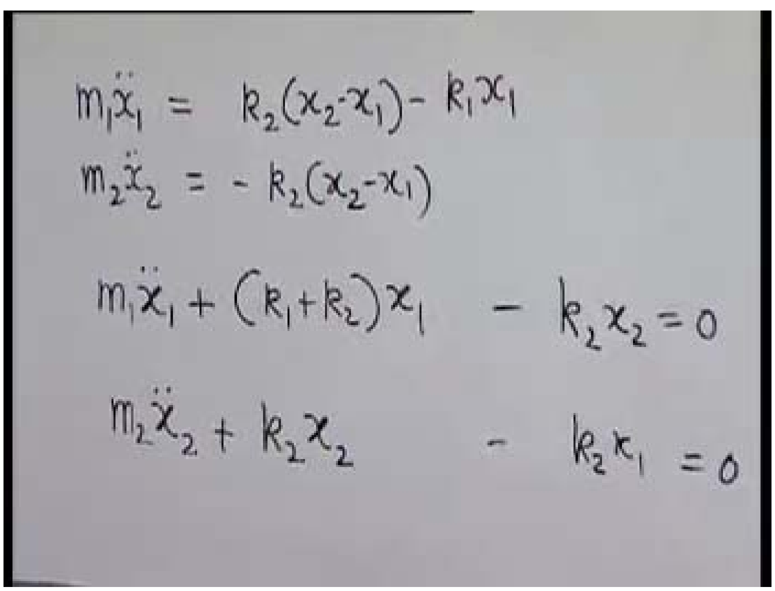
So this type of vibration where the whole system executes simple harmonic motion with the same frequency, that particular type of vibration is called natural mode vibration or normal mode vibration and the corresponding frequencies are called natural frequency. Next question will be that how many such natural mode vibration will exist and how many corresponding natural frequencies will exist. Now that will be exactly equal to number of degrees freedom. So natural frequency omega_n will have number of natural frequency which will be the number of degrees of freedom.

In this case, when you are considering two degrees of freedom system will have two natural frequencies and two natural mode of vibration. Now let us start investigating one, two degree freedom system of this lump parameter. We need not put the other this is also a two degree freedom system and really speaking we have not lost much in the generality of the case, if we want we can put, only it will complicate the expressions more we did all the time. This is a two degree of freedom system and the equations of motion, you can write down, this body in displaced condition will be here, this will be here, the forces on this, there will be only one force. Here there will be a force in this direction.

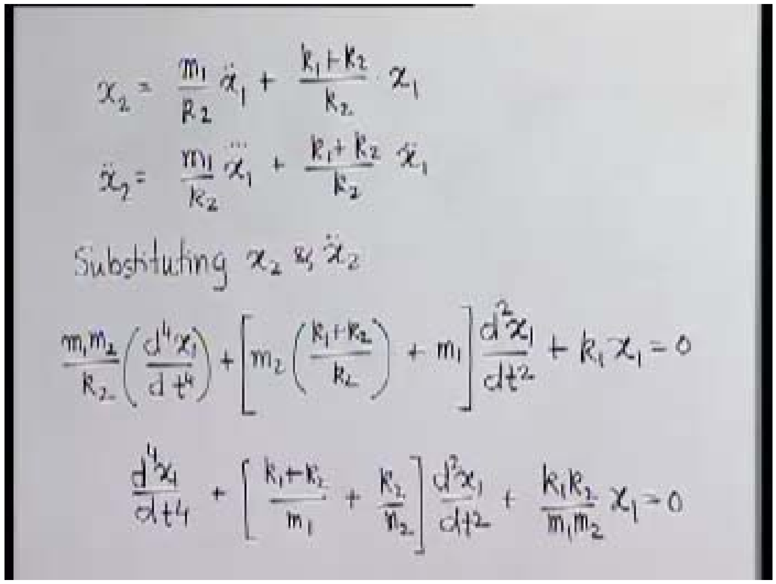
(Refer Slide Time: 08:44)



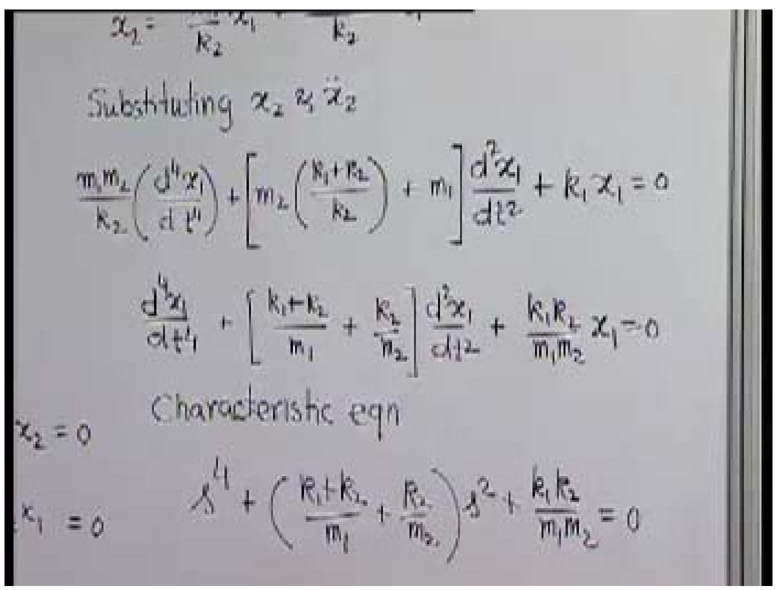
(Refer Slide Time: 13:33)



(Refer Slide Time: 20:00)



(Refer Slide Time: 28:32)



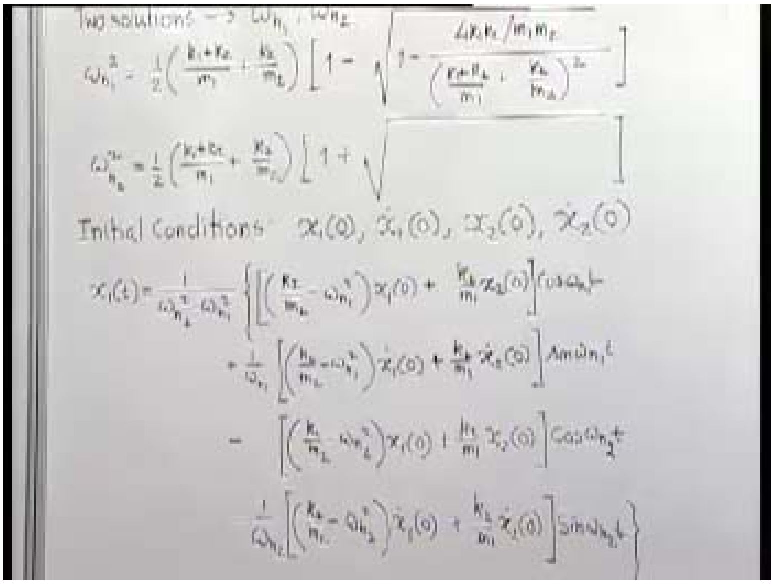
Same magnitude and these are the few diagrams and so the corresponding equations will be, and reorganizing it will be m_1. Now from this, we can express x_2 in terms of x_1 and x_1 2 dot. So let us write x_2 in terms of x_1 because this is case of simultaneous frequency equations two unknown variables and, let us put in a solvable manner by eliminating 1m $_1$ x_2 and x_2 2 dot will be the obviously. Now, if you substitute x_2 and x_2 dot in this equation say this equation so then, what we will get finally.

You can see now the actually this fourth order equation. Next one if we take the solution of form. The characteristic equation will become s to the power 4 plus, and the solution to this will provide the 2 omega squared values and those two will be the natural frequencies. So if we want to solve these, it can be solved. The two solutions, two natural frequencies, give the two natural frequencies omega$_{n1}$ square, omega n_2 square and expressions will be omega n_1 the whole square equal to and omega n_2 the whole square.

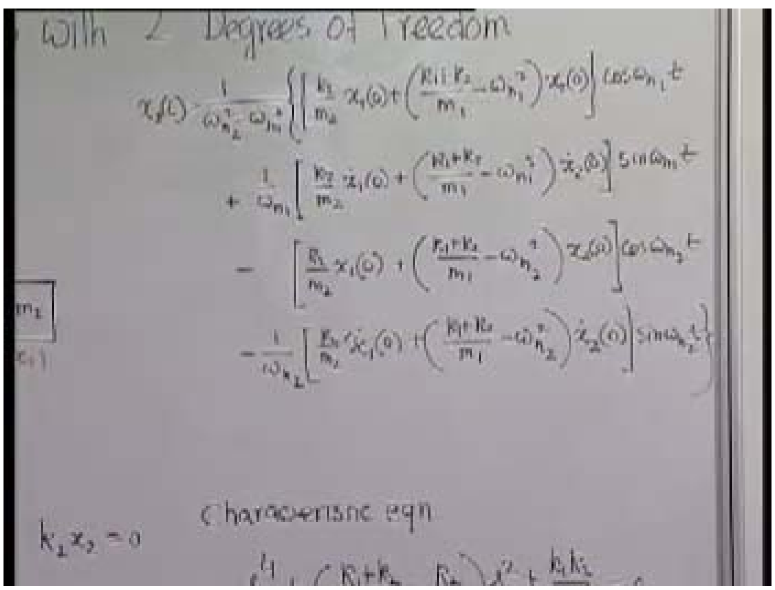
Now the solution will require the initial condition, the initial conditions what we can do we can provide just generalized equation that means, at t is equal to 0. x_1 is x_1 0 and the velocity is x_1 dot 0. x_2 is x_2 0 and the velocity x2 dot is 0. These are the four initial conditions then the complete solution can be written, I think it is more of an algebra so, what we will do, I will give the general equation in the final form. Now the reason that I am writing this long expression is to emphasize one point which will be clear at a later time. So this is the solution for x_1 and we will get a similar solution for x_2, it is required for us to write here, otherwise the point will not be clear. So, these are the general solution given for a simple case like two degree of freedom system undamped free vibration.

Now it is very clear that they are all mixed up that means two frequencies are involved omega n_1 omega n_2 and obviously x_1 or x_2. None of them are simple harmonic. However, we can see that it is possible to give initial conditions that is x_1 0 x_2 0, x_1 dot 0 x_2 dot 0 in such a manner that the whole system oscillates in a harmonic way.

(Refer Slide Time: 32:07)

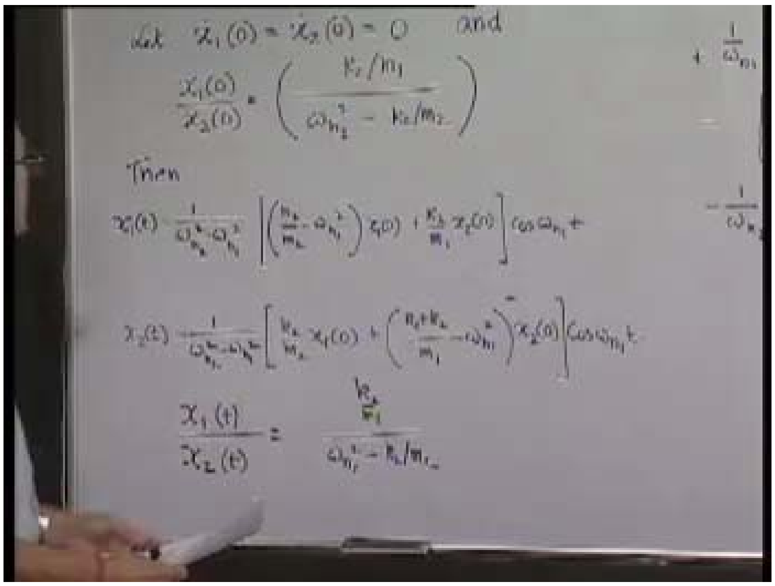


(Refer Slide Time: 33:38)



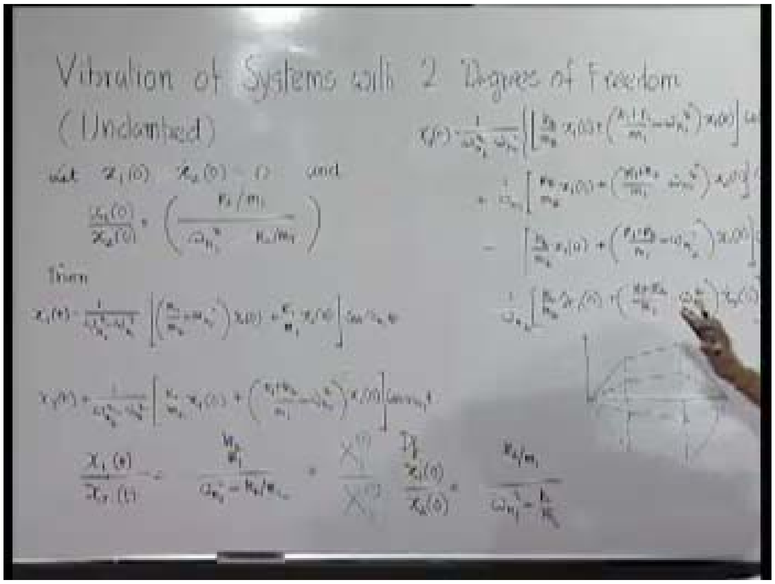
That is all quantities vary in a simple harmonic way. What is that condition let us see. Before example let both x_1 dot 0 and x_2 dot 0 be equal to 0 that is there was no initial velocity given immediately the sine terms go, next we give an x_1 0 by x_2 0. So all these quantities are known omega n_2 we have already found out, k_2 k_1 m_1 everything is known. So this quantity is obviously a known quantity. If we keep our initial displacements in this ratio, then what is going to happen, we will find that this term x_1 by x we will find that this term is going to be 0 and similarly if we substitute this here we can easily show I am avoiding the algebra that this will be all clear. So therefore the solutions will be then, so both x_1 and x_2 are simple harmonic functions of time with frequency omega n_1 that means it is in natural mode oscillation.

(Refer Slide Time: 37:11)



But this mode of oscillations we have to give the initial disturbance like this and we can show that all along, this x_1 by $x_{2.}$ This ratio will remain same as this and that is minus k_2 by m_1. So, whatever ratio I gave that ratio remains. So this is the first block, this is the second block and displacement is in this direction so we will find. They vibrate like this that means, always the ratio of the two will be quantities that means, the shape is given by the two ratio $x_{1.}$ So this is the first mode and so this is nothing but x_1 in the first mode by x_2 of the first and this is the first mode shape, the frequency is omega n. In a similar way, we could also excite the second mode. How? There if x_1 0 by x_2 0 we select as, then you will find that this term is 0. We can substitute and verify. Then again, we are getting same that means, the vibration of the whole system x_1 and x_2 as functions which will be simply proportional to cosine omega n_2 t and that will be second mode.

(Refer Slide Time: 42:08)



They will be very similar expressions where this will be cosine omega n_2 t cosine omega n_2 t and that mode will be again something like this. So we will see that one goes in this direction, at the same time other goes in this direction, so mode shape will be something like this, and the extreme positions will be something like this all the time it will vibrate in the opposite. So the ratio x_1 0 y x_2 0 will be a quantities. Now, why we are studying the normal mode oscillation we should understand first of all this is something which is mathematically practicable, because all the variations are simple harmonic functions of time with same frequency.

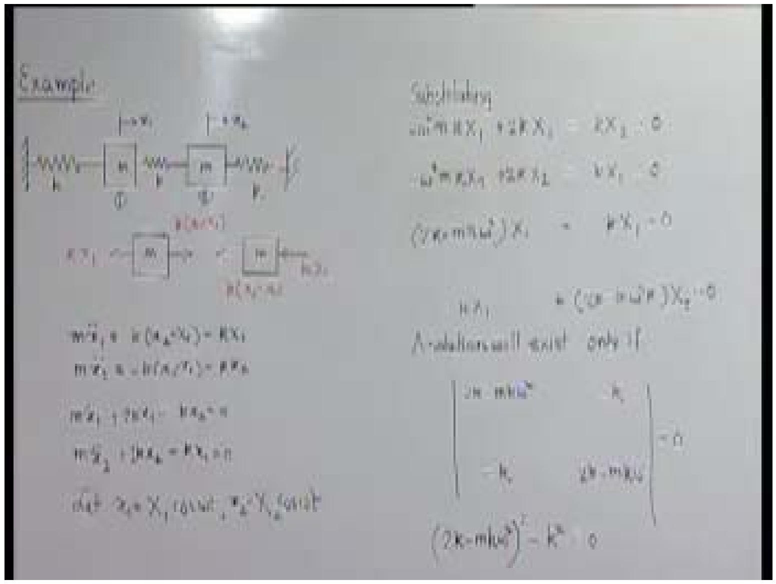
But we should not also forget the fact that, it is always possible to write any kind of displacement pattern of a system, in terms of its natural mode just like as an arbitrary function can be expressed in the form of a series of simple harmonic functions, like a series, here also it is quite easy to understand that if we give that any kind of oscillation can be always represented by a combination of its natural mode oscillation.

So, since a general oscillation can be expressed in terms of the simple harmonic natural mode oscillations, we will investigate only natural mode oscillations and again as we have done in case of single degree freedom system. Our primary objective will be to find out the natural frequencies and one extra change in such cases we have to also find out the mode shapes. Mode shapes means the ratio of the various amplitudes which will be same as ratio of the various instantaneous values of the coordinates and as many natural frequencies will have so many different ratio combinations will have.

Let us solve a particular example, of the case as we discussed freedom without any damping. What happens when damping is present then various modes which are there in the vibration of the system? The higher modes damped oscillates because, they execute larger number of cycles at the same time corresponding to the lower frequency mode.

Therefore, different modes dissipate or decayed at different planes that way the concept of mode itself become a very difficult thing to sustain in such situations and we will ignore damping. So, it is a system with two blocks and three springs, they are all identical. This is the first body, this is the second body and their displacements are represented with the help of x_1 and x_2. The equations of motion will be derived from the free body diagram, this being the free body diagram, both is identical masses. The equations of motion can be written as, for the second body, so we can rewrite this little bit k x_2 minus k x_1 minus k x_1 so it will be plus 2 k x_1 minus k x_2 and this one will be 2 k x_2 minus k x_1. Now the technique of solving this set of homogeneous equations will be that since we are studying only natural mode oscillations will assume that both x_1 and x_2 to be simple harmonic functions of time with the same frequency so let x_1 equal to X_1 cosine omega t and x_2 equal to X_2 cosine omega.

(Refer Slide Time: 47:38)

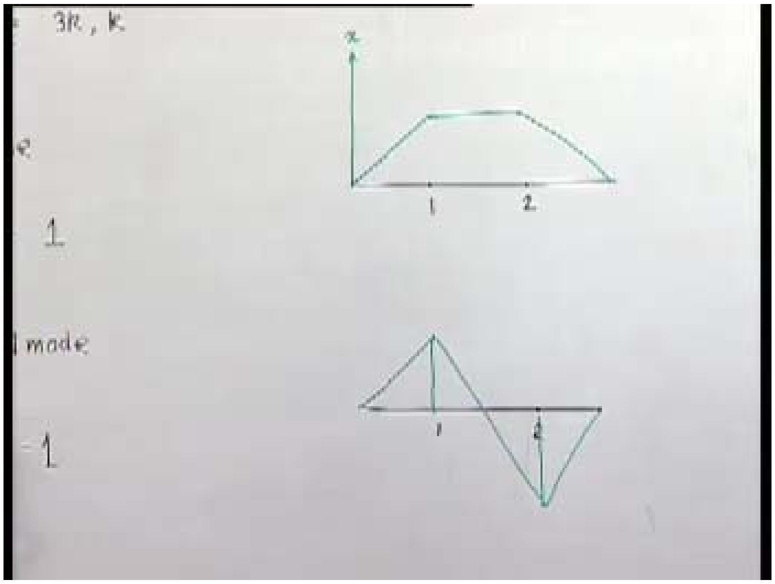


Now, as we know in undamped situations their motion can be either in face or out of face and both can be taken care of by algebraic sign of x_1 and x_2, if they are of same sign the motion of the whole body are in place, if they are opposite signs or 100 to the power 0. Now if we substitute this in these equations what we get simply minus omega square m k x_1 plus 2 k x_1 minus k x_2 equal to 0. Because omega t everywhere is there and I can take it out substituting x_2 and x_1 in the second equation or we can write it in this fashion, 2 k minus m k omega squared x_1 minus k x_2 equal to 0 minus k x_1. So we get in place of differential equations now what we have done. We have got a set of algebraic equations both are homogeneous. So therefore, a solution will exist only when the determinant of the coefficients is 0 and this will give us a characteristic equation. Now, as you can see here, we can solve the whole equation and we can find out the mode shape so directly I can give the solution also. So this equation we can definitely solve like this, or if we use this cross value we will get a m omega squared if we use the minus value then obviously we get the two natural frequencies omega n_1 is equal to k by m and second natural frequency will be 3 k by m. So natural frequencies we have got. Now after we get the natural frequencies for each natural frequency what will be the mode shape?

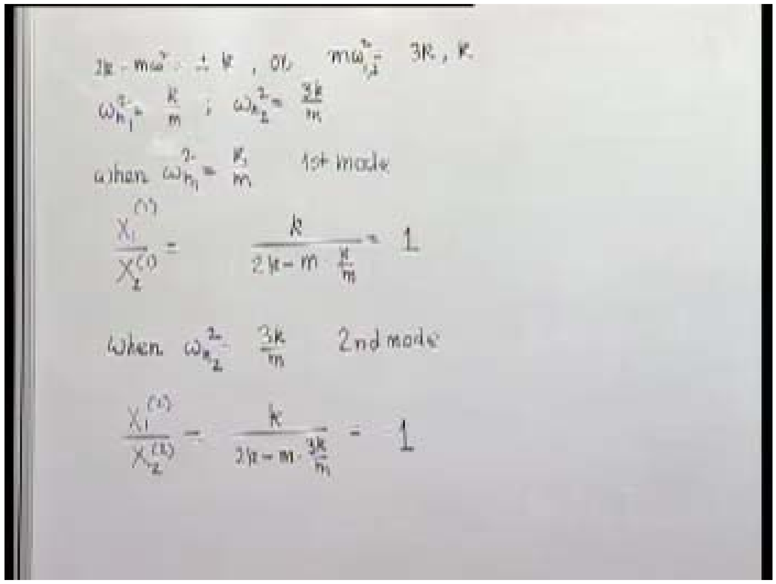
If we take this equation for example, if we substitute the value of omega we will get the ratio that means for the omega n equal to k by m let that be the first mode. So for this equation we can get x_1 in the first mode by x_2 will be equal to k divided by 2 k minus m omega squared. Now this omega squared will be k by m. When omega n_2 is equal to 3 k by m that is the second mode, then we substitute it here x_1 of the second mode by x_2 equal to k by 2 k minus m to 3 k because omega n_2. So this is simply minus 1 so second mode shape is the ratio is equal to minus 1 that means they are just equal and opposite here they are same.

Now if we plot the situation how it will look? Like this is the first body, this is the second body. Let us plot the longitudinal oscillation in this direction or representation. So x is not in this direction but I am plotting the displacement of the mass in this direction.

(Refer Slide Time: 50:18)



(Refer Slide Time: 52:35)

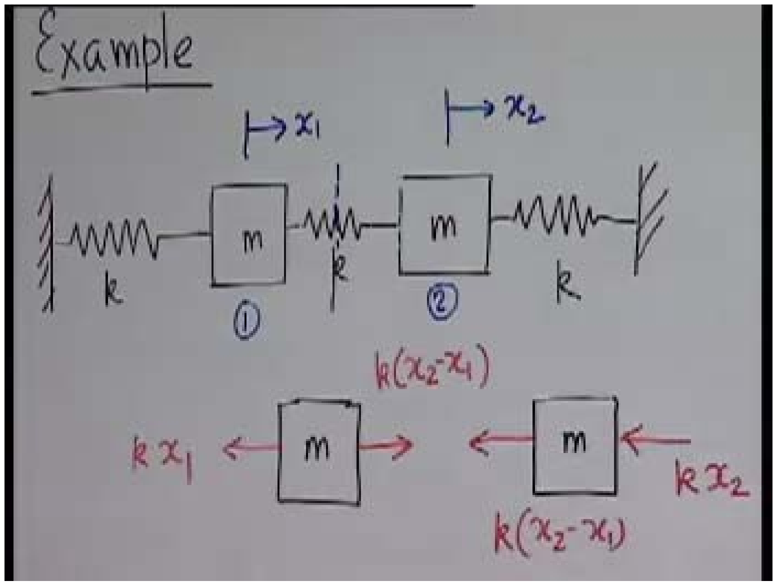


So for the first mode what will be the amplitude and also that means the x_1 by x_2 is always equal. In the second mode obviously we find if x_1 is here, x_2 will be equal and opposite, that will be the mode shape. Remember again that, it is only these two bodies are moving like this because movement in this direction. So let us try to, this being a very rather fairly simple case, whether it is possible to physical explain the results.

One possible way let us consider its oscillation, if they oscillate the same frequency same phase and same amplitude what will happen in that case? This spring will have no distortion, because the movement of the two masses are identical at every instant, so distortion being 0 there will be no force.

So then it is as good as this spring not being there and so the system will be equivalent to and the natural frequency obviously is square root of k by m this is nothing but 2 simple schema systems of identical length. The other mode is that, they are just equal and opposite, what happens if they are equal and opposite? That is, they are moving like this, if they are moving like this then midpoint of the spring will have no and so therefore we can consider the spring is hot and this there is a vault here this point is fixed. So it becomes again two systems, with this mass one spring of thickness k and this half of this spring's thickness is obviously the double because if we make the length of a spring as half then its thickness is because two halves of springs, if this is k prime and this is k prime together in series they should give us k and you know when things are in series.

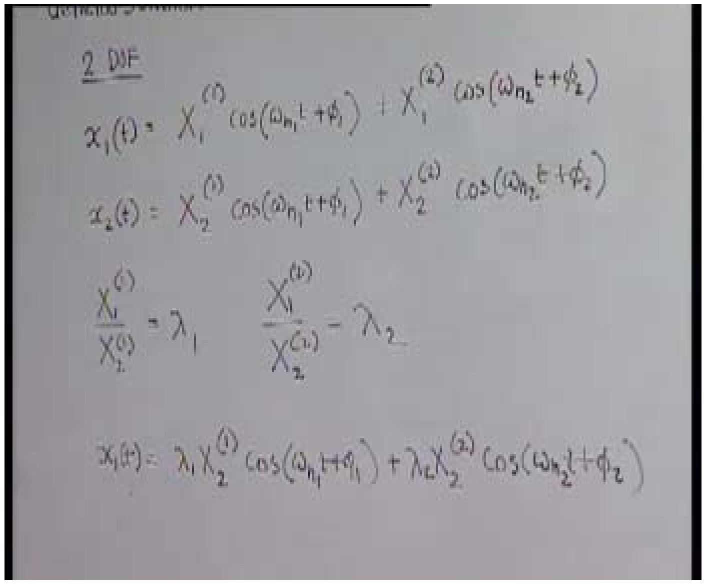
(Refer Slide Time: 55:11)



So, k will be equal to as you can see k prime, k prime by 2. So when k prime will be therefore, if this spring length is half, each half will have stiffness which is double the stiffness of the original, so this system is equivalent to now, that means if you total resultant stiffness with mass, it has to overcome to move is 3 k and therefore omega n_2 is going to be natural frequency of this system. There is another subsystem this side, that we also have seen we will see that physically also we can explain that there are two situations when the two masses will have the same frequency harmonic motion and how the physical expansion can be given in this particular case. However this kind of physical insight may not be possible in more complicated situations and we have to depend on the mathematical results.

Now if there is but in practice one message that there is no currently that will be motion which will be handling will be simple natural mode oscillation. But as I mentioned that a general motion can be represented in terms of natural modes, and we just look into the matter natural or normal mode whatever you may say.

(Refer Slide Time: 58:19)



So if a two degree freedom system there will be two natural modes and general solution. x_1 t can be written as we know of course that x_1 1 by x_2 1 is a quantity $lambda_1$ and mode shape x_1 in second mode will be quantity, then of course we can write this in terms of, and these equations are there. So how many unknown quantities will be handling will be because these things will be known after solving the natural mode oscillations is $lambda_1$ and $lambda_2$ will be known to us so there are not unknown quantities. Unknown quantities will be x_2 1 x_2 2 phi_1 and phi_2. So therefore x_2 and x_1 can be written in terms of this four unknown quantities and this four unknown quantities x_2 1 x_2 2 phi_1 and phi_2 can be from initial conditions. There will be four initial conditions.