Dynamics of Machines

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Module - 11 Lecture - 1

We have been discussing the cases of systems capable of oscillation. When the system is disturbed from its equilibrium position and subsequently allowed to oscillate freely of its own, that is when we call them, free vibration. The primary objective of that kind of an analysis had been to find out the frequency with which it oscillates of its own, that is what we call natural frequency. The importance of that comes from the fact that as we will see very soon. Any kind of situation where an external excitation matches in frequency with the natural frequency of the system can result in disastrous consequences. That is why, it is better to know beforehand what the natural frequency of a system is. It should be such that it never comes anywhere near the frequency of any kind of excitation the system is expected to be subjected.

In this lecture, we will start with cases that are far more encountered in our engineering design. When a system is excited by an external agency, then what will be the response of the system when it is subjected to this external excitation? This we call forced vibration. Again to keep our discussions simple at the beginning, we take up the single degree freedom system subjected to unexcitation force. We should also beforehand know what does it mean by forcing?

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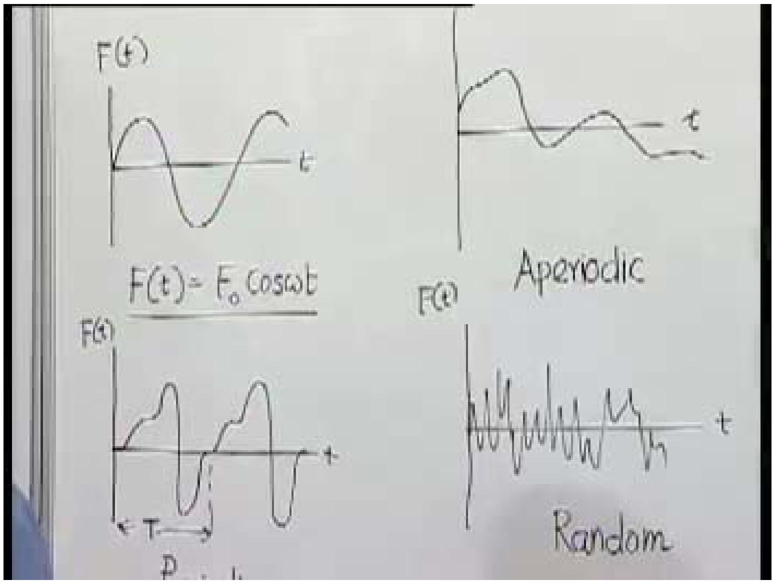

Excitation of a system can be of various types. First of all, we will classify the situations according to the nature of the forcing function or the disturbance. If the forcing is purely a harmonic function of time, say if we call this is the force, so this is called harmonic excitation (Refer Slide Time: 03:39) because the excitation, the function of time is a harmonic function of time.

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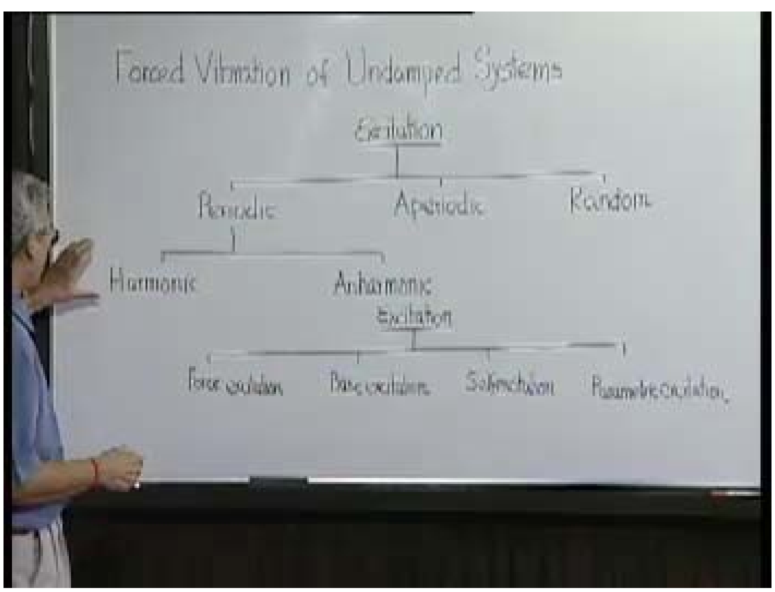

But before that, we should classify, in general excitation as periodic that means, where the forcing function is a function which repeats itself after a fixed period. This kind of excitation which repeats after a fixed time, which we call time period, is called periodic excitation. Another kind of excitation can be aperiodic excitation. Aperiodic excitation can be edited like this which does not repeat itself after a definite time. There is a very special case which we will not able to discuss but nevertheless very important called random. There are many cases where the excitation is of random nature, which has certain special characteristic. What is random? But we will not go into those things, because we will keep our discussions to the simplest possible situation.

However, this periodic excitation can be further classified, as I was mentioning initially into two cases called harmonic, other is anharmonic. The second one, it is periodic but it is not a simple harmonic function of time (Refer Slide Time: 06:25). The classification of excitation can be again done from a different point of view. Here, we have been bringing it from the point of view of the nature of the excitation as a function of time. The other way of looking at the classification problems is what the manner in which it is excited. For example, I can take this and try to vibrate by applying a force. Sometimes, a system can be oscillated or disturbed by applying some kind of a disturbing motion to its support or the wall to which it is connected. Sometimes, a system vibrates by itself, only thing that we have to supply is force of energy. Sometimes, it is excited by a periodic variation of one of the parameters either stiffness or inertia or something.

There is another way of classifying excitation based on the fact that from where or how it is being excited. There we can have force excitation, the other one where the support on which something is resting and that is oscillated is called base excitation. Other one is self-excitation, there are cases of course, which we call parametric excitation. Therefore, our problem of force vibration can be any kind of combination. It can be a periodic, aharmonic or base excitation problem.

What we should do? As always it is advised to take up the simple possible equation. We have taken the simplest system, an undamped single degree freedom system, but vibration now excitation also we will take the simplest that we will take harmonic excitation and excitation is given in the form of either a force or a moment. Another reason why harmonic excitation is a very important problem is that any kind of periodic function of time can be represented in terms of a Fourier series where each term or component is a harmonic function of time.

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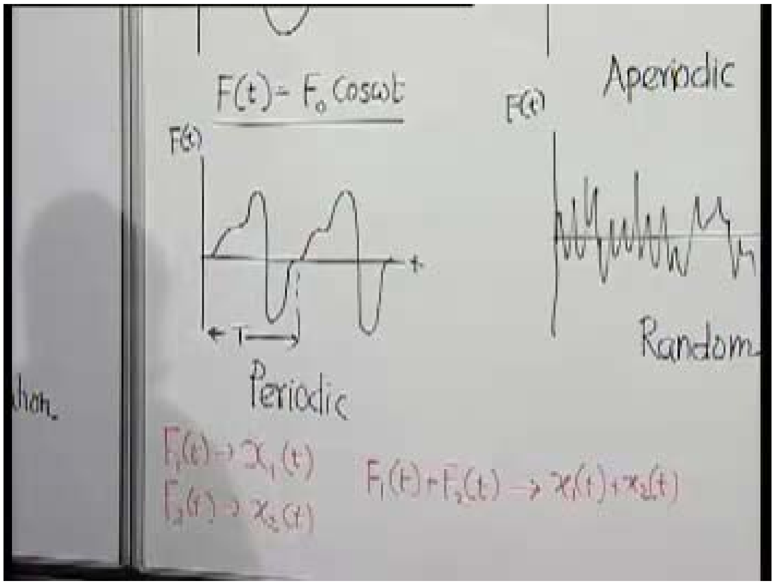

As we are dealing with only linear systems, if a particular forcing function gives a response x_1 and another forcing function F_2 gives solution x_2, then when both are combined they will give the response x_1 t plus x_2 t. This is a very important result, but valid only for linear systems.

Therefore, if this period be function of time can be split into a series of harmonic functions of time, then for each harmonic function of time, we can get the response and so the resultant response will be the total of all the responses. If we know how to solve the problem for the case of harmonic excitation, we can solve for all periodic functions of time. Even aperiodic function of time can be represented in terms of an infinite Fourier transformation and they are of course not discrete frequencies, but the frequency spectrum will be continuous and we will see such cases. The conclusion is that within this problem can be solved using the result of the system response to a harmonic function because this can be also be represented in terms of harmonics functions of time using

Fourier transformation. Therefore, if we are handling linear systems, knowing its response to harmonic functions of time can give the result of any type of coefficient.

With this little introduction, let us solve the problem one by one. We will first take up the case of an undamped system with a single degree of freedom which is excited by a force that is a simple harmonic function of time. It is a very simple problem. Let us find out the value of or the expression for x, that is, the response of the system.

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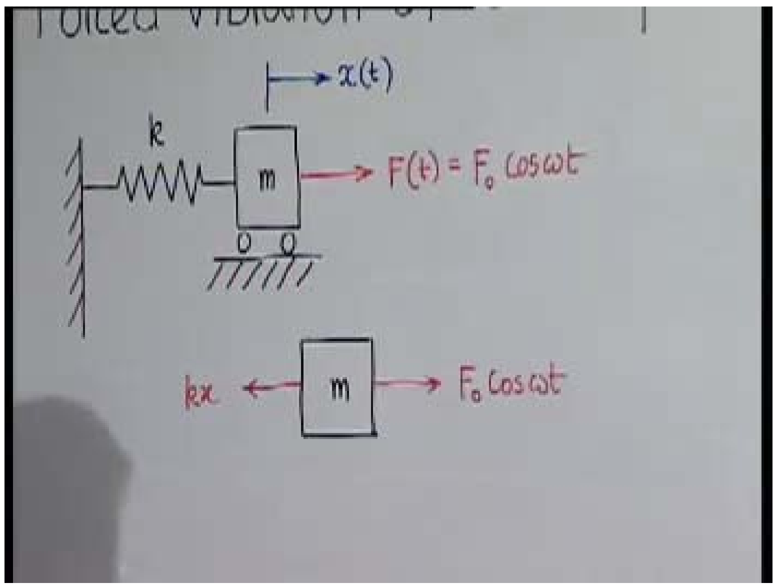

At any instant of time, the forces acting on this will be kx. So, total acceleration in this direction x two dot multiplied by m has to be total force in that direction.

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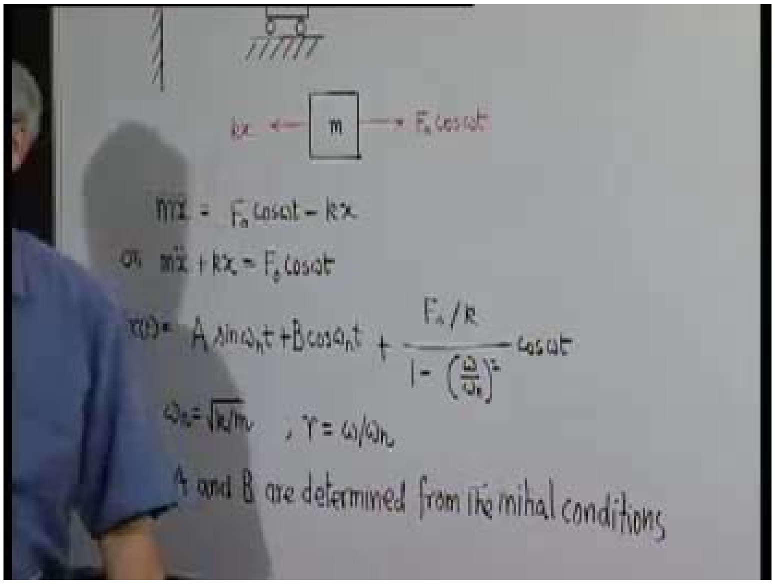

This is the differential equation which we have to solve now and we all know that the general solution will have two parts. One is the complementary function, which is the solution of the homogeneous part of this equation, that is mx two dot plus kx equal to 0 plus the particular integral which for this case, we have to find out. The complementary function, that is the solution of x to the homogeneous part of the equation is we know that already from our previous lectures. This is the complementary function and this is the particular integral. Here of course, $omega_n$ is a gain with natural frequency of the system.

We also define another quantity r, that is the frequency ratio given by the ratio of the forcing frequency with the natural frequency. Now, these constants A and B will depend on the initial condition. Initial conditions mean the starting velocity and solution. Let us now try to solve this problem.

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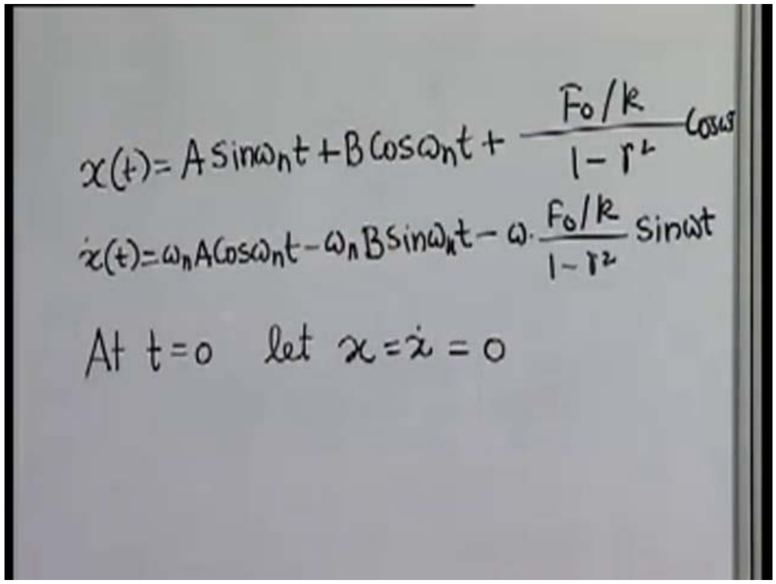

We know x (t) and x dot t from the above figure. Let us consider, relax condition to begin with, so at t equal to 0, let both x and x dot equal to 0.

This is something which is possible in forced vibration. In free vibration, if we give initial condition as x and x dot, both are 0, system will be continue to remain at rest. Since we are disturbing the system, it will oscillate. So putting this, what we get is 0 equal to B plus (Refer Slide Time 17:38).

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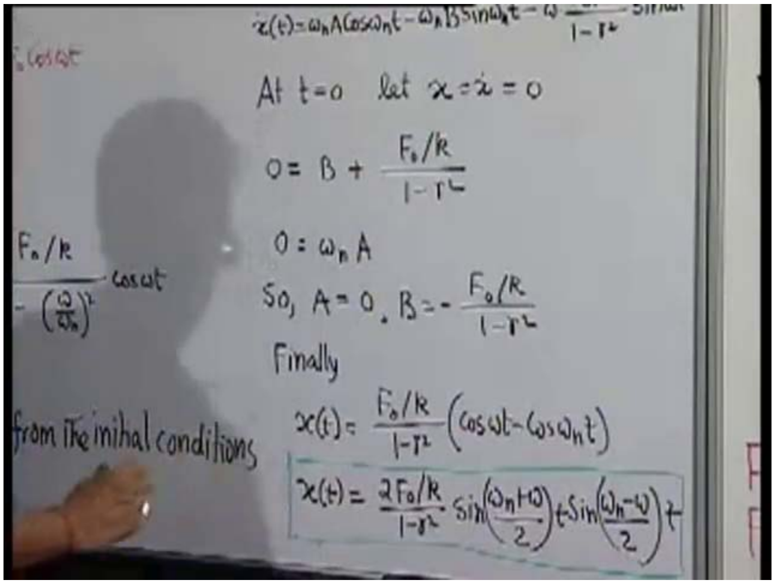

If we put t is equal to 0, x dot equal to 0, then 0 equal to omega$_n$ A. So, we get the two conditions, A equal to 0 and B equal to minus F_0 by k divided by 1 minus r square. The solution becomes as mentioned in the above slide. It can be written in the slightly different form, this pure trigonometric manipulation. This is our final solution that how perfectly undamped single degree freedom system will respond to a purely harmonic floating function. Let us plot it.

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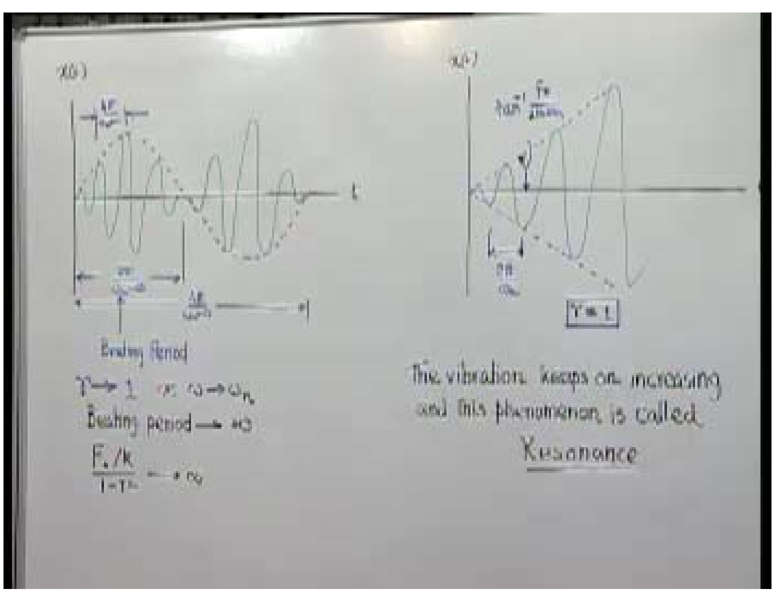

Now, one thing we can find is this is a constant for a given value of r or r is nothing but omega by $omega_n$. This is again a harmonic function of time where the frequency is almost approximately equal to omega because this, when $omega_n$ is not very different, it will be something like that. On the other hand, this is another harmonic function of time which is changing with a very lower frequency because it is the difference of the frequency. In a crude manner, we could say that this is a harmonic function of time where the magnitude or amplitude fluctuates slowly with this form and that is how we are getting (Refer Slide Time: 21:49).

The fluctuation of the amplitude is a slow process with a time period 2 pi by $omega_n$ minus omega. This whole thing will be 4 pi by $omega_n$ minus omega when it completes 1 and this fluctuation phenomena is known to you, perhaps which we call beat and the beating period is this one (Refer Slide Time 22:25). Suppose, what happens when we start from a low value of omega, omega is nearing 0 that means r is nearing 0. This will be approximately $2F_1$ by k and this will be approximately sine $omega_n$ by 2 and this will be more or less same. But what we find here is that as omega increases and approaches towards end, when r approaches 1, what happens to the beat frequency? Beat frequency is this (Refer Slide Time 23:13). So, when omega approaches $omega_n$, beat frequency or beating period approaches, since beat period is 2 pi by this, the beating period will approach infinity. Another thing we find that the amplitude with the maximum value to which it fluctuates depends on this quantity and here when r is approaching 1, then this is gradually excluding amplitude which depends on this (Refer Slide Time 24:18). That also is approaching infinity. Therefore, if we want to plot this under this situation, this is a case where r is equal to 1. (Refer Slide Time: 24:45). This angle is tan inverse and of course this is still. Now, we find when r is equal to 1, that particular case, bit period is infinite. That means it never comes back, it keeps on increasing. The maximum value to which we reach also keeps on increasing. It can be shown mathematically, we will not go into those things. This amplitude will keep on increasing linearly and the rate of increase can be decided by this angle of the tangent to the… [Refer Slide Time: 26:05].

This particular phenomenon, what is going to happen, if such system is there, then it will keep on increasing in its oscillation amplitude indefinitely. That is what we call resonance. We should remember the earlier caution that a forcing function when it is acting, you should be careful that the forcing frequency never really goes anywhere near the natural frequency of the system. That is why, when something is being designed its natural frequency should be estimated and it should be ensured that the system is never subjected to a kind of forcing function or excitation where there is a chance of resonance and these are the kinds of things, perhaps you have heard that when a military marches through a bridge, this left right thing and normally on a solid ground, they keep on marching, all in unison, that means everybody is in place. But, once they cross a bridge they are always told not to be in place or unison and they are not supposed to march, they walk back because in case all these hundreds of soldiers keep on marching with the same frequency in same pace, then the forcing function will be enough and it can match the natural frequency, at least go somewhere near the natural frequency of the bridge and it can cause the disaster and therefore this is the point which we wanted to. For the comment view that which is held sometime that if the resonance takes place, there is amplitude or vibration is infinity, which is not so.

Actually, what happens in resonance is that the intensity of vibration keeps on increasing, indefinitely. If you allow infinite time, if the system is capable of taking infinite deformation, it will go to infinity. But mainly what happens that the system gets damaged because of large scale oscillations taking place. However, this is very idealistic situation because in nature it is impossible to make a system which is absolutely free from any energy dissipation. Therefore, to make our analysis somewhat more realistic, what we can do is that, even if we consider the system to be notionally undamped, that means, we do not show a specific damping agent there or damper. But still we have to keep in mind that there is certain amount of dissipation, it may be very small and so in reality what happens, even the very minute amount of damping present in a system will damp out the complementary function or the free vibration part and how the response will look will be this. (Refer Slide Time 30:15)

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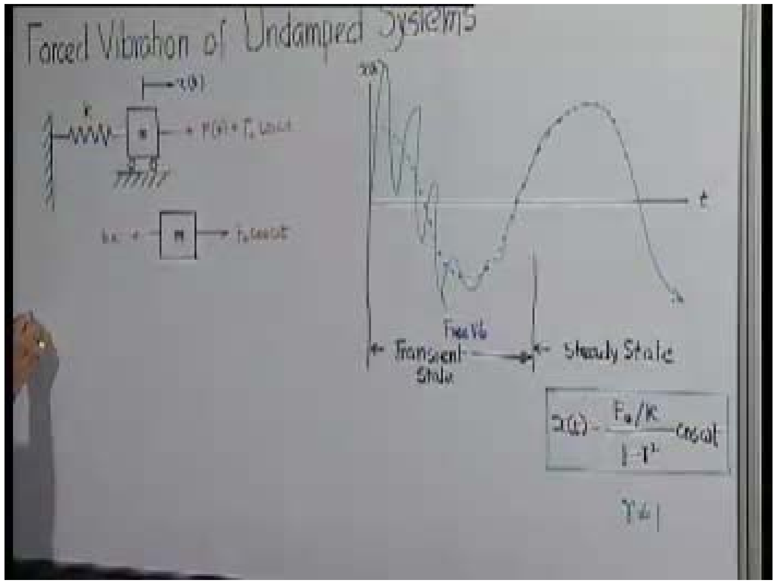

Of course I have shown this x in short time, but in reality, it may happen that it will continue for a large time. Eventually the initial free vibration part will gradually damp out and what will remain is only the complementary or the particular integral. The response in the steady state situation will keep on maintaining its nature and character that is called the steady state.

In the steady state, the solution or x t is in a particular integral part because the natural vibration part which was represented by the complementary function has gradually died down, that is, the response and this is what we call transient. It is a passing phase, so this does not repeat itself. After it reaches the steady state, there is no further change and there the response is this. (Refer Slide Time 32:36). In general, our primary concern is the response of a system in the steady state, which in this particular case happens to be this, of course, r is not equal to 1.

Now, we see this F_0 by k here, we find that the forcing function is cosine omega t F_0 and response is also similarly, some quantity into cosine omega$_t$. Therefore, if we plot in the steady state function will look like this.

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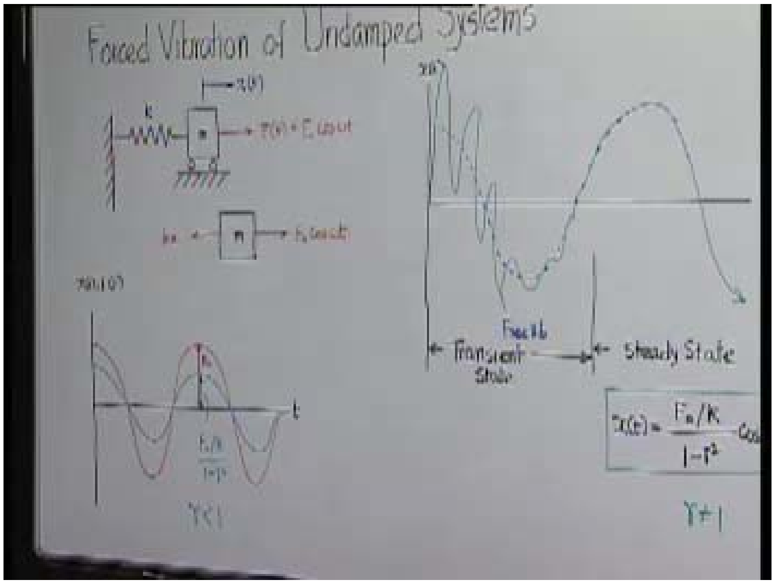

If we plot both x and the forcing function in the same graph, it will look something like this and the response will look, where the magnitude of this is F_0 and magnitude of this is F_0 by k. This is the situation when r is less than 1. When r is more than 1, then what will happen?

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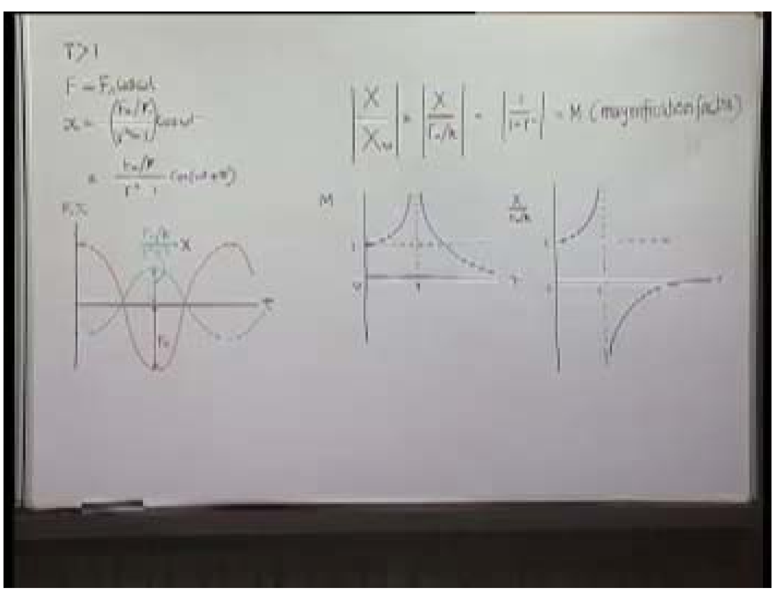

This case x will be as shown in the above slide. I want to keep this always a positive quantity, so we can write this as r square minus 1 and take minus sign here. We can write this as F_o by k divided by r square minus 1 into cos omega t plus pi. Therefore, what will happen now is that the forcing function and the displacement or response will be just out of phase.

When the forcing frequency about the natural frequency of the system, that is r is more than 1, then the response will be out of phase by 180 degree of pi as indicated in this. Now, what is this quantity F_0 by k? This is nothing but the static deflection of this. (Refer Slide Time 36:47); if applied this force, that is F_0 static energy, x F_0 by k is the displacement of the mass from equilibrium, if the same load is applied in a static way not dynamic.

On the other hand, if the same force is applied dynamically, the deflection or the maximum deflection of this is nothing but the amplitude of oscillation, which is F_0 by k divided by its quantity. This ratio, the amplitude of oscillation, which is nothing but this divided by the displacement, which is static when the force is statically applied is and we take its magnitude, that is X divided by F_0 by k magnitude and this is nothing but as can be seen here. This is X divided by F_0 k is nothing but 1 by 1 minus r square. This is called M or magnification factor. It means that when a load is applied dynamically, the displacement of the system gets magnified compared to the situation when the load was applied in a static manner.

How this magnification factor looks like? This magnification factor when we plot as a function of r, it starts at r is equal to 0. Magnification factor is 1, r is equal to 1 and it shoots up to infinity. Again, when r is more than 1, it will become negative, but we are taking only the mod. So, it becomes positive and it asymptotically approaches 0, as r tends to very large values. Of course, if we just want to plot the ratio of the amplitude with the static deflection, then when r is less than 1, X is positive and this quantity is positive, it goes. But, when r is more than 1, then obviously X is negative because this quantity is negative.

If you take the mod of this, then we get back this term. This is called the steady state response of a single degree freedom undamped system, which response characteristics is indicated by this plot where the magnification factor or the amplitude at a particular frequency is given. For the whole range of possible frequencies or the possible frequency ratio r the magnification factor has this kind of characteristics. This is the simplest situation which we had discussed.

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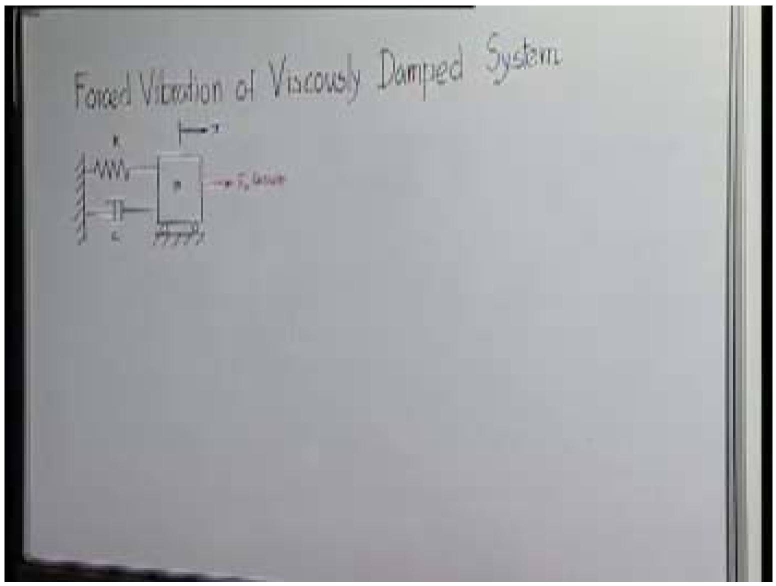

Next, I think if we have to take up more realistic cases. What we will do is that we will now consider a viscously damped system and subjected to a harmonic force. We will now take up a more realistic case where there is some certain amount of viscous damping present and again we will subject it to a simple harmonic function of time as before. This is the situation or a lump parameter model, its viscous damping, its thickness, mass subjected to simple harmonic force, a harmonic function of time F_0 cosine $omega_t$.

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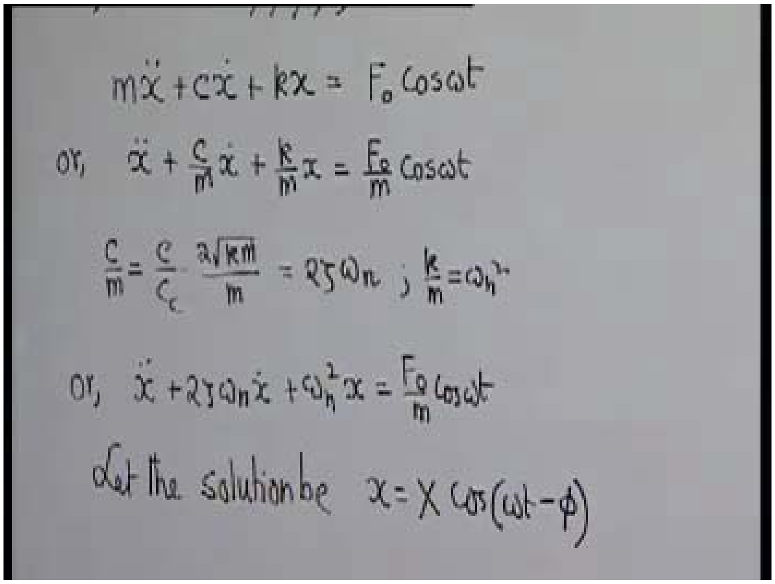

We can write the equation of motion without going into again the free body diagram. We can write, but it is not going to be 0 as in case of free vibration or the external forcing function.

We have seen and we can keep it in mind that any kind of response of a simple system to harmonic function or harmonic force like this will be also again a harmonic function of time with the same frequency. This is a principle which we should keep in mind that the response to a harmonic force will be a harmonic function of time with the same frequency. That is the only thing which we have to keep in mind. But before we go further, we will write it in a slightly different form. We can write this as or dividing the whole thing by m.

Now c by m can be written as c by c_c, that is, a critical damping coefficient into c_c, which is nothing but in this particular case 2 into root of km by m. That is equal to zeta, damping factor into 2 and root over km by m, is nothing but root over k by m, that is, omega$_n$. Similarly, k by m is omega$_n$ squared. This equation can be again written in this form x two dot plus 2 zeta omega$_n$ x dot. We can always write the solution. Let the solution be a harmonic function of time with the same frequency. We can write X equal to that is the amplitude cosine, now a harmonic function of time with same frequency.

But, there is now guarantee that it will be with the same phase. So, we cannot write cosine ω_t with a different phase here, rather what we will do, we can see that this is again harmonic function of time.

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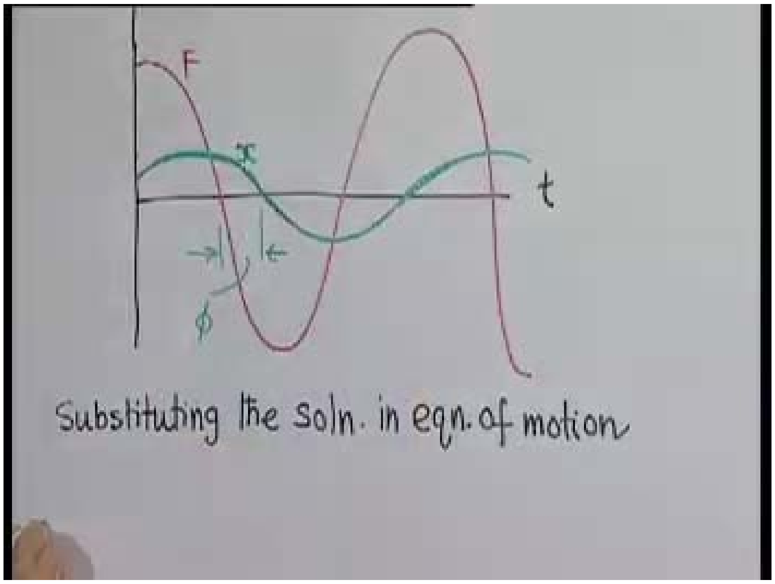

They respond, but not at the same phase as what we had in case of undamped system. This is the most general situation and I think there should not be any problem with this; this is the response (Refer Slide Time 48:16). Our job will be to find out the magnitude of the amplitude X and the phase that is what we wanted to do.

What we will do now is we will substitute this solution which we have assumed into the equation. Substituting the solution in equation what we will get is x two dot. If we differentiate this twice, what we will get is minus omega square X cosine omega_t minus phi into, there is nothing else. Therefore, this will be the first term plus minus twice zeta omega_n omega.

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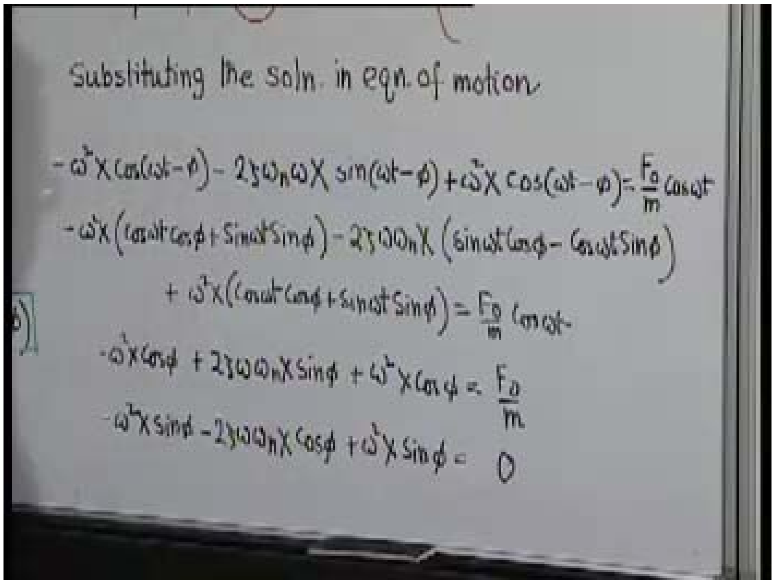

This is x dot multiplied by 2 zeta omega$_n$ x dot, is minus omega x sine omega t minus phi and they were multiplied and this is equal to F_0 by m into cos omega t. Therefore, we have converted the differential equation into an algebraic equation. What we have to solve is for two quantities: 1 is x, other is phi. The technique is generally that we will see on both sides of equation, the coefficient of cosine omega t on both sides must be same. Coefficient of the sine omega t on both sides of the equation should be same. If we split this, what we will get is (Refer Slide Time 50:56). If we want to equate the coefficients of cosine omega t on both sides, what will be the coefficient of cosine omega t on this side? It will be minus omega square X cosine phi plus 2zeta omega$_n$ X sine phi plus omega square X cosine phi.

This is the coefficient of cosine omega t on the left-hand side and that must be equal to the same in the right-hand side, which is F_0 by m. Again, we have to equate the coefficient of sine omega t (Refer Slide Time 52:38) and the coefficient of sine omega t on the right-hand side is 0 from this. If we take sine phi and cosine phi, we get so we get tan phi is equal to (Refer Slide Time 53:47) and if we divide the numerator and denominator by omega$_n$ square what we get, 2 zeta r by 1 minus r square, that is, the second equation.

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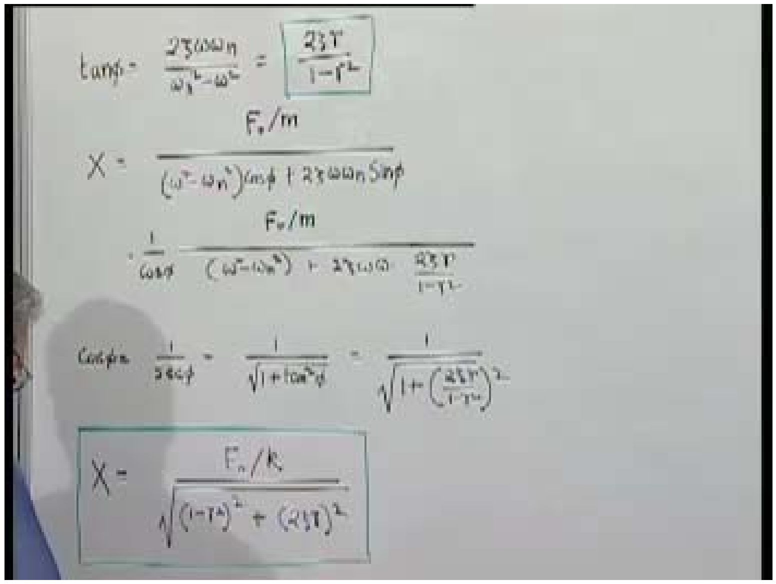

The first equation will tell us X is equal to (Refer Slide Time 54: 10). So, we take X common and divide F_0 by m by the coefficient of X. So, if we take cosine phi common outside, we get F_0 by m by cosine phi, if we take 1 by cosine phi, then it is omega square minus $omega_n$ squared plus 2 zeta $omega_n$ into tan phi. Tan phi is already determined, is 2zeta r by 1 minus r squared.

We can easily find out cosine phi. It is equal to 1 by sec phi, that is 1 by square root of 1 plus tan squared phi. Now, tan phi has been already determined. If we substitute this here, finally, X we will get as (Refer Slide Time: 56:08). After simplifying we get this. This is the response, it request two things to be given. O is the amplitude which is this, the frequency is same as the forcing frequency and the phase difference is this. (Refer Slide Time: 56:51)

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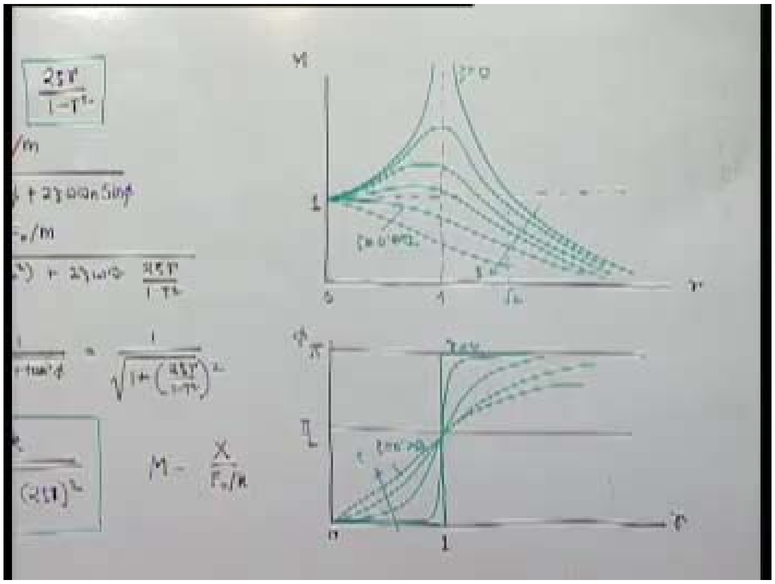

The characteristic looks like this, if we plot it, magnification factor is defined the same way here, X by F_0 by k. As we can see, when r is tending to 0 when it is simply, this is 1, this is 0, so it will be 1. This way now of course, we have a parameter like zeta, for different values of zeta, we will get different result. So, zeta is increasing in result and this is zeta equal to 0.

Let this, so far at the magnification factor, phase difference characteristic, if zeta is very small it will be like this. Now, as zeta gradually increases and so on, so zeta increases and this is the case when zeta is 0.

There are special cases where you know this is where it is tangential, this is a special value of zeta, which is 0.707, same thing, this is zeta 0.707 here. We will discuss about this things later. The frequency response characteristic of a viscously damped system is like this, it depends on the magnitude of zeta. The peak value is now restricted, it is no longer infinity, [Refer Slide Time: 1:00:03] even at omega is equal to omega, which is no longer infinity. The peak value is also not coinciding with r is equal to 1, gradually it is shifting.

For zeta is equal to 0.707, the peak value of the response is at r is equal to 0, that we will prove later. Therefore, this is the general characteristics here. Again, we find the phase difference is always 90 degrees at r is equal to 1 and it is asymptotically approaching pi. So, this is the situation. What we will do next is to look at this problem from a different angle using a different approach, which we will be taking up in the next discussion.