Dynamics of Machines

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Module-10 Lecture-2

So far we have discussed the free vibration problem of single degree freedom system with viscous damping. One reason why we did it first is that with viscous damping the fundamental equation or the governing equation of motion remains linear in current, so it is amenable to solution; second point that it also represents a very large number of real life situations,where we have a viscous damping.Nevertheless it is very important to consider at least the few other forms of damping mechanisms, particularly, the energy dissipation due to the rubbing of solid surfaces which is present almost in all cases. Of course, with this type of damping which we call Coulomb damping, as I have mentioned earlier; the other type of damping like internal friction in the materials and so on we can consider.

First we will take up the free vibration problem of single degree freedom systems with Coulomb damping. Of course the resulting equations will be linear but fortunately for this system, which is possible to get some solutions without much difficulty. Other types of damping we will not takeup and when the situation arises, we will make some comments on that, so far as the results are concerned. But free vibration problem with Coulomb damping we will take up in today's presentation.

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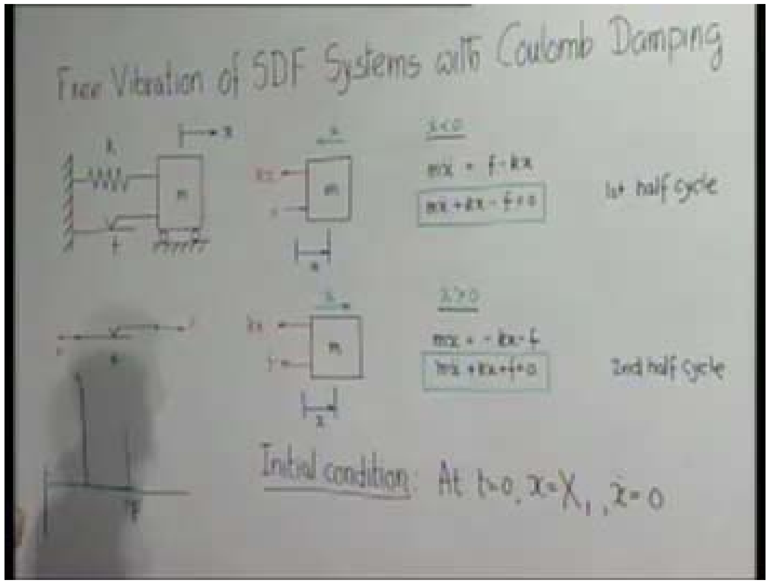

So the lump parameter model of a single degree freedom system with Coulomb damping can be stated like this. The spring that is the restoring element as the stiffness k block has mass m and the solid friction is represented by that device here, where this force F is the force required to cause a sliding. Whenever the force is applied to this device, if you think a movement you want to cause, what we will find is, if gradually we increase F, nothing will happen until and unless we reach a force F. Similarly, in the other direction also, when we try to press them, force must exceed the magnitude F, so that a relative sliding takes place. Therefore when this force is below F, then friction automatically adjusts itself and no movement takes place. Now what will happen if you want to solve this problem? What we will have to consider first is the equilibrium position. The situation is not like that, that is if you leave it at any position, it will come to a position where this spring will have its natural length.

We define because there will be always a small region in width, the spring force whether it is tensile or compressive is less than the friction force, then wherever we keep it, it will stick there. Therefore, we have to be very particular about the point that the equilibrium position we are considering when the spring takes its natural length. For the two cases, when the mass is moving from this side to this side, that means x dot is negative, frictional force, is acting in this direction on this.

When it is moving in this direction, that is in case x dot is less than 0, then the spring force, irrespective of its position, suppose its position form equilibrium position is x, it will be in this direction, which is proportional to the stretching or the formation of the spring. The force on this, when it is moving in this direction, will be in this direction. Its magnitude will be constant because we all know that once sliding starts the friction force between these sliding surfaces remains constant.

In this situation, when x dot is less than 0, what will be the equation of motion? Mx two dot is equal to f minus kx or mx two dot plus kx minus f is equal to 0 for the first half cycle. How we are doing it? We are stretching this to some position and leave it and allow it to oscillate. So during the first half cycle, that means from this extreme position to the other extreme position, x dot is negative and this is the equation of motion.

In the next half cycle, when the mass is moving from left to right x dot is positive. Then again at the similar location where it was at a position x from the equilibrium position, the spring force will be again kx. Since it is now moving in this direction, the friction force on it will be the opposite direction. Friction always opposes the relative velocity.

The equation of motion for this situation, in the next half cycle, when x dot is greater than 0 will be this or this, we have two equations, the whole cycle does not operate at the same equation of motion but in each half cycle the equations of motions are different. To solve the problem, we have to solve it probably for each half cycle.

What we will do, we will take up the problem when the mass is taken to one initial position capital X_1 and then left and what happens? Initial condition to start the motion is at t is equal to 0; x is equal to $X_1$and x dot equal to 0. It is one extreme position here. If we now leave it, what will be the solution for the first half cycle?

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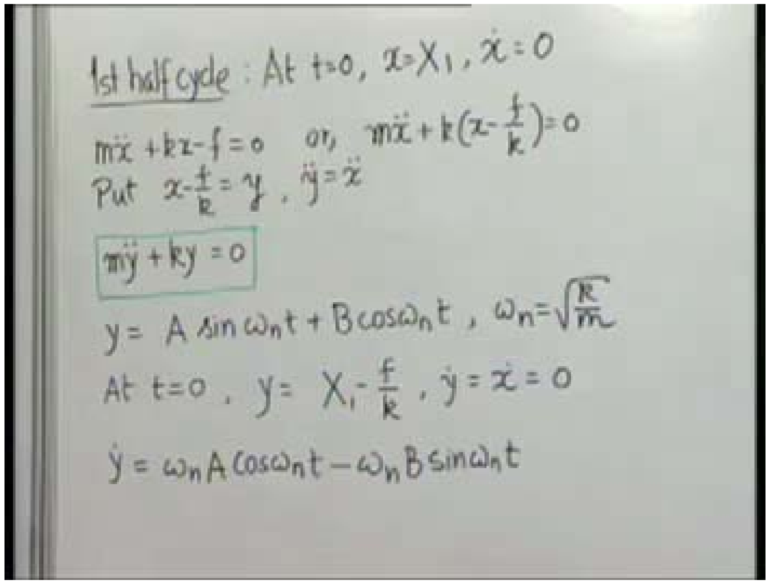

What is the equation of motion? This is the equation of motion because once you bring it to this right most position and leave it; the mass will go because of the spring force towards left. That means x dot is negative and this is the equation of motion which will be valid. So, mx two dot plus kx minus f equal to 0, can be written as or this. Then of course y two dot will be equal to x two dot, because f by k is constant. Thus the equation of motion in terms of the new variable y becomes this. We find that in terms of this new variable y, it is a final equation which we normally get. Therefore what will be solution of y? We know that general solution for y will be, A sine omega$_n$ t plus B cosine omega$_n$t, where omega$_n$ is again the natural frequency of the system without any damping. Now to find out A and B, we have to put the initial condition in first half cycle.

First half cycle initial condition is at t is equal to 0, x is equal to X_1, x dot equal to 0. So putting this at t is equal to 0 in terms of the new variable y. What will be y? y will be capital X_1 minus f by k, this is the relationship what will be y dot equal to x dot equal to 0? If we differentiate this to get y dot equal to omega$_n$.

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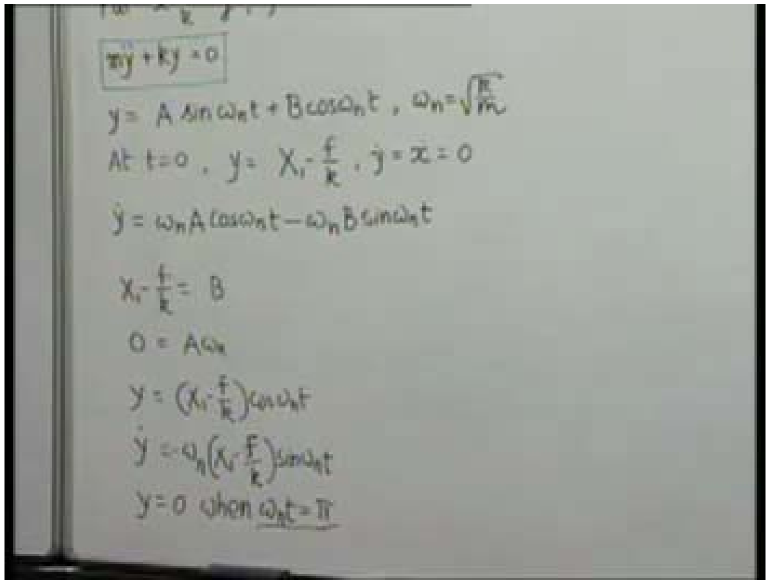

Put the initial condition at t is equal to 0, y is equal to this much and at t is equal to 0, y dot is 0, putting it here we get y dot is equal to 0. The general solution we get A is equal to 0, this goes, so B is equal to now, when it will stop when y dot becomes 0 again; when y dot becomes 0 when t is equal to 0 which is the initial point or when sine omega$_n$t is 0. Next time when sine omega$_n$t becomes 0 again, when omega$_n$t is equal to pi, that is the end of the first half cycle. If omega$_n$ t is equal to pi, what will be corresponding value of y?

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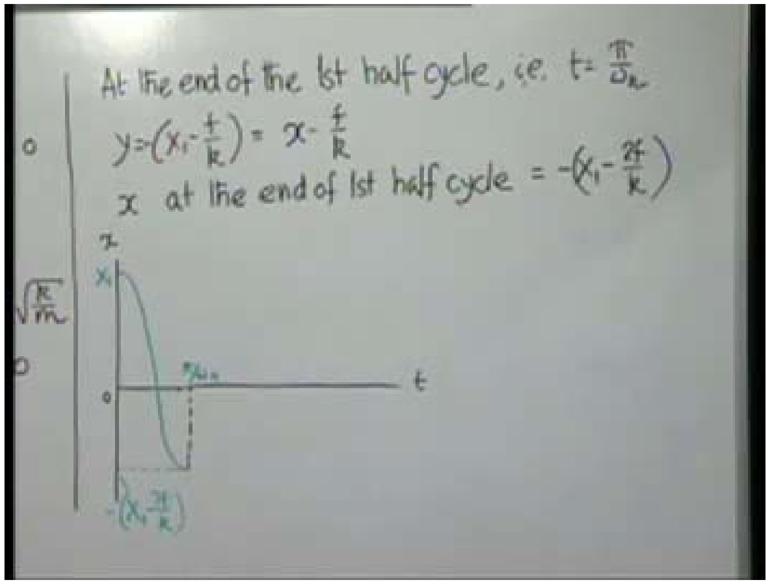

If we put omega$_n$ t is equal to pi, cosine pi is equal to minus 1 therefore, y becomes this one. So y is equal to x minus f by k, from this we get x at the end of first half cycle is equal to how much? It is minus X_1minus 2f by k.

That is if we plot we start from here, which is X_1 will come here. Again it stops here at pi by omega$_n$. First half cycle is over, all along x was decreasing. That is why the x dot is negative. But the peak amplitude, what we had at the beginning, which was X_1, now this is less than that. How much is this, which is X_1 minus 2f by k but the whole thing is negative.

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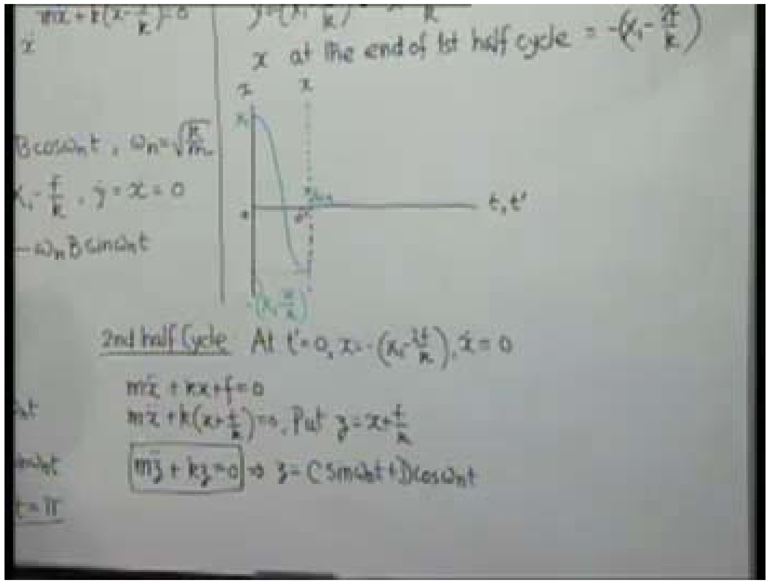

Next let us take up the second half cycle that means what happens when the motion starts from here. Again velocity is 0, x is equal to this much. What will be the next half cycle, second half cycle? In second half cycle what will be the initial conditions? Let us have another time at t prime equal to 0, x equal to minus X_1 minus 2f by k and x dot equal to 0.

We are measuring now with a different time coordinate t prime and this is x and this is the new origin of the time. We are freshly starting our clock at this. What is the equation of motion? It is mx two dot, now when x dot is greater than 1, because now x is increasing and velocity is positive so this will be the equation of motion.

We can rewrite it like this and put a new variable z equal to x plus f by k. So x two dot will be equal to z two dot plus kz equal to 0. We get same equation with a different variable z now. Solution of this will be again z equal to C sine omega_n t plus D cosine omega_nt, C and D will be found out from the initial conditions.

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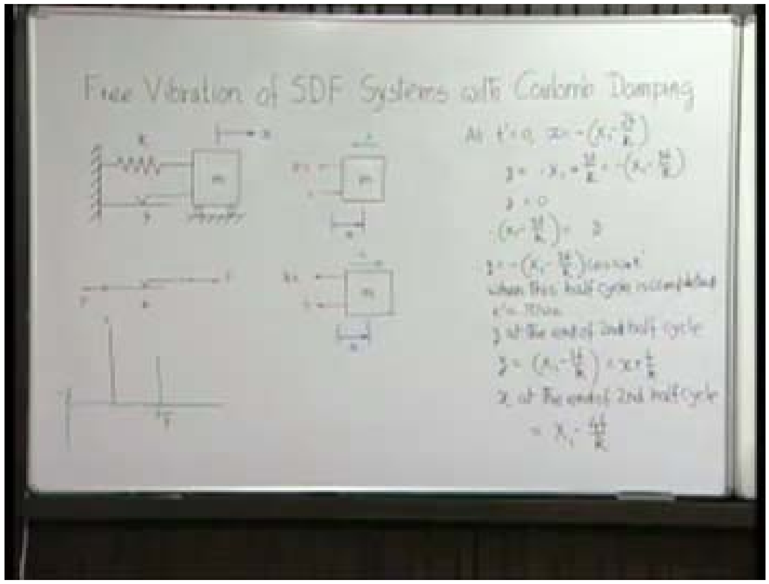

What will be initial conditions? At t prime equal to 0, x is equal to minus. So, z equal to this x plus f by k. If you do it, it will be minus X_1 plus 3f by k that is, minus X_1 minus x, 3f by k and z dot is equal to 0.

We will again get the way we did it for this.We will find that when we put z dot equal to 0 at t prime equal to 0, here we should put as t prime because which is our new time coordinate. From that we can easily find that our C term will be 0, as we had A, 0 here. Therefore and your minus X_1 minus 3f by k is the value of z at t prime equal to 0. So this will be equal to, so z is equal to minus, now after one half cycle that is, when the second half cycle is completed, that means when z dot become 0.When sine omega$_n$ t prime will be 0 that means omega$_n$ t prime will be equal to pi, z at the end of second half cycle will be that, if you substitute this here, we get minus 1. So minus 1 minus plus and obviously, z is equal to x plus f by k. x at the end of second half cycle is equal to, how much X_1 minus 4f by k.

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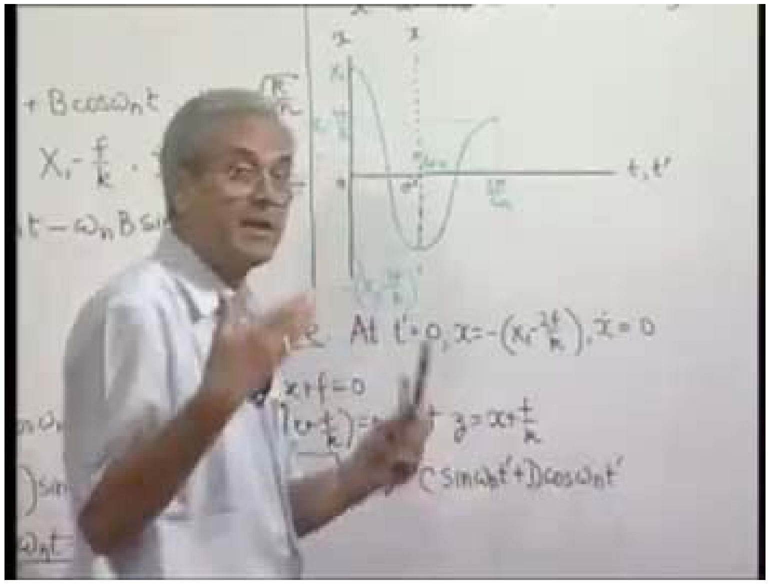

So when you plot it, again the peak value of this displacement reduces by 2f by k. If from minus or that is, if the value was X_1 minus 2f by becomes X_1 minus 4f by k. After every cycle, we find that the amplitude decreases by 4f by k. Similarly, in the next half cycle also it will decrease by another 4f by k. That means in each cycle the amplitude is decreasing by a fixed amount which is 4f by k.

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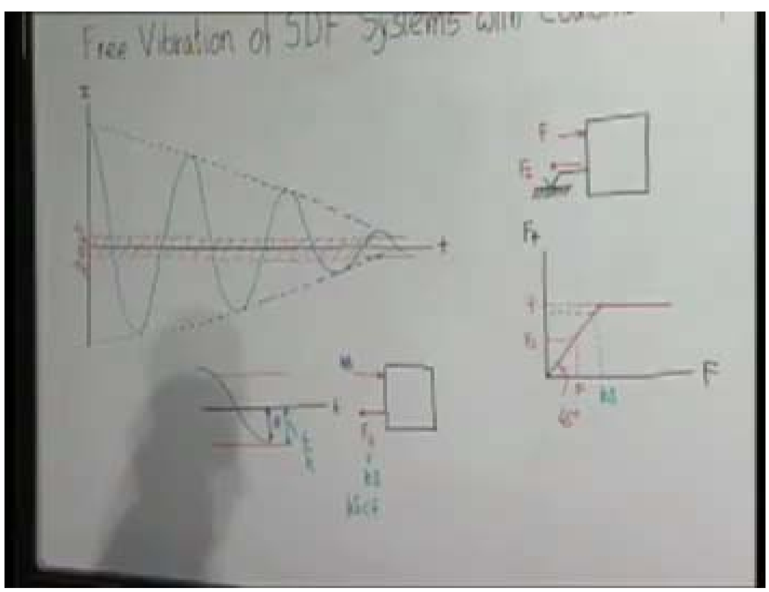

The displacement diagram of this Coulomb damped system is like this. The peak amplitude you know. If we join, we get a straight line. The question is now that how far it will continue doing this. Let us consider that it oscillates like this and let us draw a band of width 2f by k, upper side it is f by k. Lower side it is minus f by k on the t axis. We know that when from here to here it reduces from capital $X_1$to A_1 X_1 minus double of this that is 4f by k. It continues like this and so what happens, I will show it in a magnified way, as soon as the mass or the block stops somewhere in this region, say came like this and say it stopped here. Now what happens, the velocity of the mass is 0 that means it stopped.

For this to move there must be some resultant force acting on it. But what is happening mass has come here on the other side of the equilibrium position and spring is supplying. How much force, it will be, if this deflection is say delta, it will be k delta and it is compressive because delta is in the negative direction. Whereas, the friction force, if it now tries to move in this direction, friction force will act in this direction, which we call F_f, How much will be the friction force? We know that the characteristic of solid friction is like this. That is it always adjusts the friction force to be applied force up to a limit. The friction force, if it is being pushed in this direction by a force F, then it is subjected to a friction force which is F_f.

We all know from our basic mechanics that if we plot F_f versus this, whatever there will be a 45 degrees line that means at every instant f is equal to F_f, up to a limit which we call the limiting sliding force, beyond which friction cannot apply any object. Therefore, if the value of k delta is less than F_f which is apparent from here k delta because this much is how much f by k, so F_f that means friction force will be equal to k delta because k delta is less than f. Thus, maybe somewhere we are operating here.

Whatever force is being applied, which is k delta, friction force will immediately balance, which means that the system will not move. Therefore, whenever the system stops, it cannot move beyond that point, if that stopping point happens within this band. This is what I mentioned once if you remember that real systems always come to a total stop, velocity becomes 0. Primarily, because in all real life systems, there is a certain amount of Coulomb damping prism, if it is viscous damping theoretically, it should go at freedom to oscillating with smaller and smaller amplitude.

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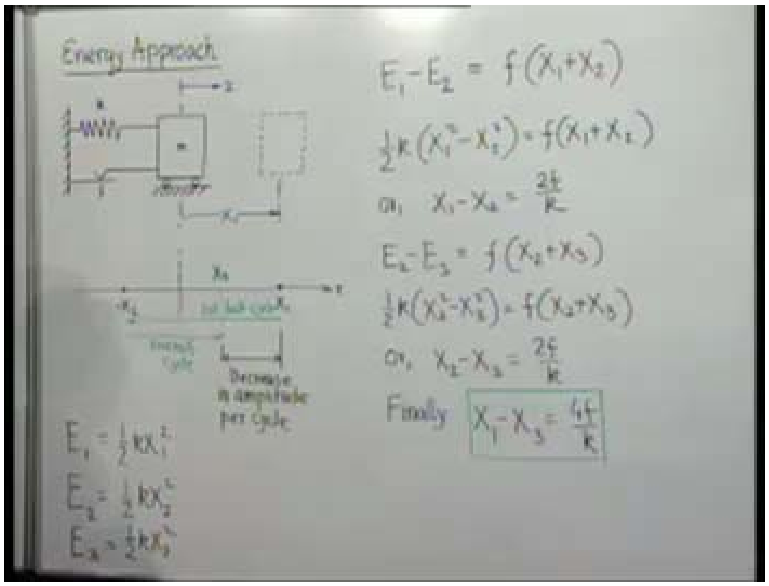

This decreasing amplitude per cycle, which we have, could be found out in another method, which is of course much be simpler as we will see now. We solve the problem by the direct approach, solving the union mechanics of the problem. Let us solve the same problem from the energy approach.

Again we take up the same problem, that we pull this far, to the position here. The displacement from the equilibrium position is capital X_1 and release it, then what happens is that this is the starting position, if we put the location of the mass first, it is here, capital X_1, then in the first half cycle, it comes here and here it is equilibrium position. This is the first half cycle. It comes from here to here, in the second half cycle, it goes from here to here, and let this value be X_3. Like that, in one cycle it comes from here to here and then to this. Let us find out the drop in magnitude per cycle. What is the total energy when we pulled it and we left it?

Since the velocity here was 0, the whole mechanical energy was in the potential form that is the spline energy of the spring, because gravitational potential energy is not playing any role, as the whole thing is in the horizontal direction. Energy at position one was simply half k X_1 square, because X_1 is the amount of space in this field. When it came here, the spring becomes of natural length then it started being compressed. When the block reached the other extreme position, then the spring was under a compression of X_2, so E_2. Then again when it went, and it stopped here, the energy at location three was finally half.

We can see the energy is decreasing. Why? The reason is simply because when the mass is coming from here to here, which is also dissipating some energy because of this friction. What is the amount of energy loss due to friction when it travels from here to here? Simply since the friction force is constant, because for any two sliding bodies the friction force is constant. So the work done due to the friction or the loss in friction will be simply the amount of distance travelled multiplied by the friction force which is constant.

Energy loss due to friction in the first half cycle is nothing but, the difference between E_1 and E_2. How much is this energy loss is nothing but the work done by the friction force, which is constant f multiplied by the total distance travelled by the block. This is again nothing but X_1 plus X_2. We know therefore half k into X_1 square; if we substitute E_1 and E_2 here we get this. Straight forward we get the decrease in amplitude or the extreme values of the displacement in the first half cycle is stated for what we have by k.

Similarly if we do this, if the next half cycle, the energy decreases from E_2 to E_3, E_2 minus E_3 is the amount of work loss by friction, which is the friction force into this distance travelled, substituting or finally X_2 minus X_3, adding these two, thus we get the same relationship for same, that the amplitude decreases by an amount 4f by k in every cycle and it decreases by 2f by k in every half cycle. We can get the whole thing, but one thing we must remember, which has only given information about $X_{1,}$ X_2 and X_3, what happens in between, we have no idea. So therefore, to get a complete idea about what is happening, complete displacement diagram, we have to solve the differential equation.Energy approach can only give some information about the extreme having.

We should not have the misconception that we can avoid Newton's or Newton mechanism and use only energy approach to get the complete solution; it will not give the complete solution or full information. We will only get the decrease in amplitude; whereas, the solution part we had by solving the differential equation, it was the complete displacement diagram you should draw.

Now I think this complete displacement diagram or complete information about motion we can obtain also by another method. With a technique which we will use, that has been explained earlier. That is the phase plane approach.

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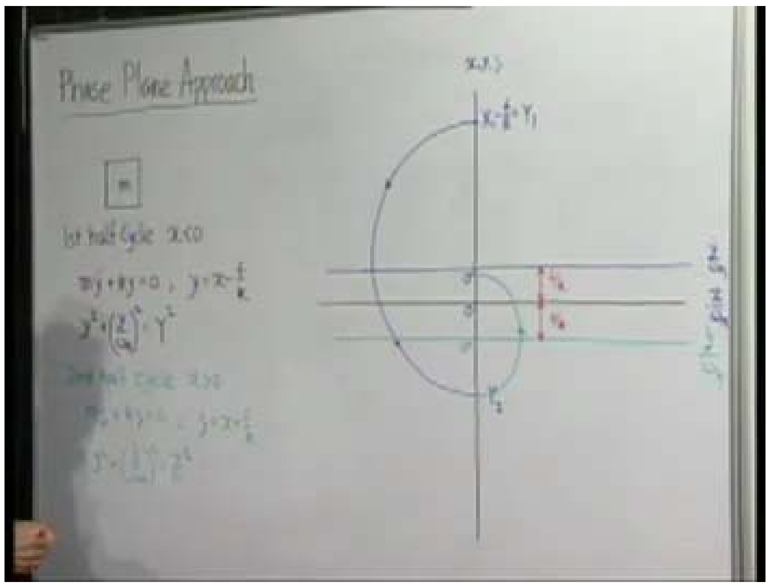

We will solve the same problem. We will get a complete solution, not just the extreme values of the amplitude. We will solve the whole problem using this phase plane technique discussed earlier. First, let us find out the system in masses movement. We have seen that when in the first half cycle x dot is less than 0, and the equation of motion was my two dot plus ky equal to 0, where y was equal to this If we plot x by the blue and this is the origin for x, and this side is this, now x dot will be same as y dot in axis. Then we know that here the solution is something of the order of, we know that y squared plus y dot by omega$_n$ squared will be a circle. We have seen that this equation solution always leads to a circle in the phase plane, where the axis are, one is the displacement other is the velocity divided by omega$_n$ that you should remember.

Only thing what we can do now is, since y is x minus f by k, we have to shift the origin here, so what we will do originally, I did it here but it could be found here, whereas, this is the origin now, new origin.

Now we have a new axis, where this is the y-axis. If this is the origin, this is minus y. Therefore, wherever we start, say if we start at the extreme position here, x is equal to X_1 so y is equal to X_1 minus f by k plus $Y_{1,}$ is the starting value of y. In the first half cycle it will describe a semi-circle coming here with O prime as the center and this has the starting point, so here this is Y_2. Since it is semicircle, you should know the magnitude of this and magnitude of this will be same. If you go to the second half cycle, x dot is more than 0, equation is mz two dot plus kz equal to 0, where z is equal to x plus f by k. Therefore now the new axis, with new origin will be here. This will be the z axis and this will be z dot by $omega_n$. Now we know that this is z square plus z dot by $omega_n$ squared will be another circle or we should say square not essential.

In this next half cycle, this will be the center and this will be a semi-circle. If these are the center and this point, so we can draw it, so it will go there and we find that the mass has stopped with velocity 0. All the velocities are 0 here, and it is within this band of f by k. Therefore, motion will stop and to plot what we will be the x versus t, we can easily extend this. We can see that means at any position, if t is 0 here, then what will be the value of t here? That we can find out by taking any point, say first point is this, second point is this, this angle we say $theta_1$.t_2 will be $theta_2$ by $omega_n$.

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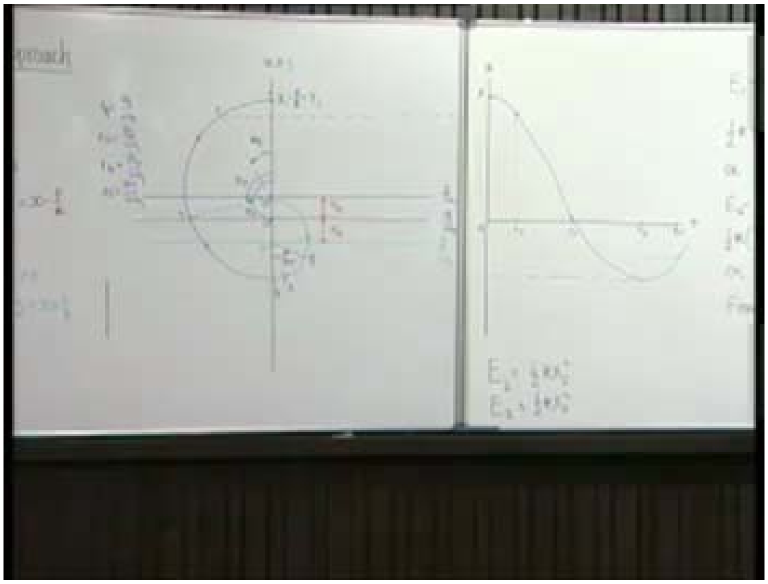

When we plot an xt diagram or rather I should use this side, we get a displacement diagram where one axis is x, other axis is t. Originally we started from here, at t is equal to 0, then at t is equal to t_2, this will be the value, t_2 is equal to, is this, so the point will be here. Similarly, we can go to this t_3 and t_3 will be theta_3 by omega_n that is the amount of time for the phase point to come from here to here, so t_3 will be something like, say here and the value will be somewhere here. We find that it will come like this then next let us go directly to this. That is theta_4 and this is the forth point and t_4 is equal to theta_4 by omega_n. That is something here and here we have $g_1$to the other extreme end.

First half cycle we have plotted will be like this. Second half cycle, when we plot then time we have to calculate from here, this is $t_{5,}$ or say fifth point, so this is theta_5 now.t_5 will be theta_5 by omega_n and that is this measurement of time is from now here. This is t_5 actually, not this one. For that of course we can give different color in here. It will continue, so we can get a complete displacement diagram graphically from the phase plane diagram, this determination of time, how to do it corresponding to various points on the phase plane diagram can be also done this thing. Thus in many situations, the phase plane approach can give us a solution, and without solving the differential equation we can get the displacement diagram.

I think we come to the end of our major discussion on free vibration of single degree freedom system. As we have noticed, two changes which we have done. In case of undamped system, our primary objective was to find out the natural frequency of the system; in case of damped systems, which is not only the finding out the frequency of oscillation, but it is also finding out the way the amplitude decreases or the displacement diagram we want to solve.

We have seen in viscously damped system, the time period is slightly increased because the damped natural frequency is slightly less than the natural frequency of the same system without the damper. In case of Coulomb damping, we have seen that the natural frequency or the time period it remains same as the undamped system. We have further seen that in viscously damped system, the amount of damping determines the nature of response to its initial disturbance. We have seen if there exists for each particular type of system a value in the damping coefficient, which is critical, if the damping coefficient of the system is above that, and then the response is aperiodic. But if it is less than the critical value, then the system is periodic though decreasing amplitude time.

In the next lecture we will start the determination of the response of a system to outside oscillating disturbance of force.