Dynamics of Machines

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Module 9 Lecture 2

So far we have been discussing systems with no mechanism for energy dissipation. That means, the total mechanical energy in the system always remain constant. However unfortunately in reality such an idealized situation is never possible and all practical systems will be subjected to certain amount of dissipation of energy when it oscillates. Sometimes it may be very small almost we can consider it to be of negligible level. Sometimes it will be of reasonable magnitude, so that its effect is very pronounced. But you should always remember that all practical systems are subjected to this energy dissipation, which a common term is damping and therefore all practical systems are damped.

What we will do now, we will analyze or investigate systems oscillation assuming for a certain amount of damping to be present. Before you do that, before we make a mathematical model, which ultimately we will solve and we will investigate, we have to first see what are the various ways energy is dissipated from a system during its oscillation? The various mechanisms which are common are as follows.

One is a viscous drag which results from the friction of a solid object when it moves in a spin and when it can be treated to be laminar motion. That means, number is low or the system behaves in a linear fashion, we will see that very soon. So this kind of damping is called viscous damping.

The next mechanism is energy loss due to solid friction between sliding solid surfaces. This type of damping is called Coulomb damping. Say for example here, if a pendulum oscillates here, the pendulum its whole body, the rod extra, they are all having relative motion with the surrounding air or whatever fluid it may be immersed in. That will give rise to dissipation of energy and it will give rise to a viscous damping.

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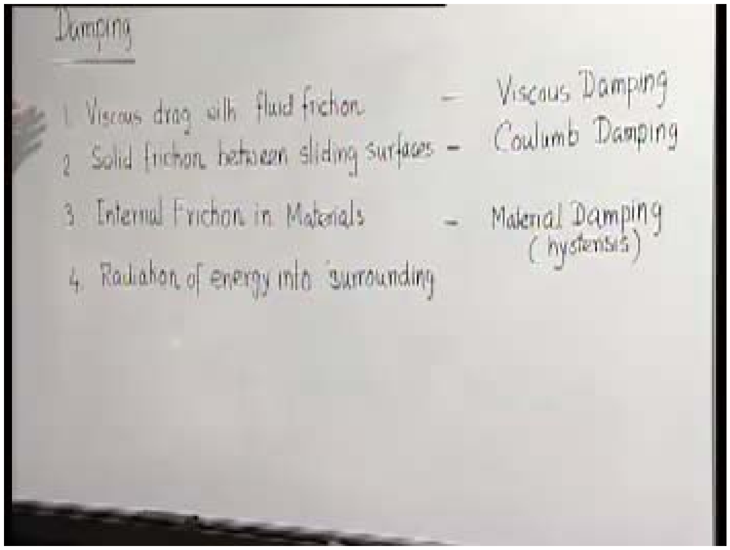

Sometimes we may find say for example, the pin of the pendulum in its hinge. If it is not properly lubricated, that may give rise to a solid friction between the housing or the journal of the bearing and the bearing itself of the spline. Similarly, there can be other situations where vibration or vibrating system may involve two solid objects rubbing against each other and that will give rise to Coulomb damping.

Now for example, if you take a simple beam and go to space, hit it and it start vibrating and you leave it there. Whether that vibration, neither there is not any fluid surrounding it, so that there exist any viscous damping, nor it is having any contact with any other solid objects to give rise to solid friction damping or Coulomb damping. But in such case, will vibration remain constant or whether the energy will remain constant? There will be no dissipation, no there also there will be dissipation because during any deformation of a solid object there will be internal friction that is a very complex mechanism we will not go into it. That how, when the layers of molecules rub against each other, the gain boundaries may have relative movement against each other. All these things give rise to certain amount of conversion of mechanical energy into lower forms of energy, like heat. Such damping or such energy dissipation is called internal friction.

This is called hysteresis damping or material damping. Other possibility of dissipating energy is in the form of radiation that can also give rise to energy dissipation. However, we should keep in mind that the first three are of much bigger importance where engineering systems are involved. There are also most commonly involved problems we will consider, we will involve viscous damping, the reason I will discuss. Then we will also discuss Coulomb damping or the dissipation due to solid friction. We will also discuss quickly about internal friction or hysteresis damping in material.

First let us take viscous damping. Let us consider this is the sectional view of a cylinder with a piston inside. Now piston is connected to one rod and cylinder is connected to another rod.

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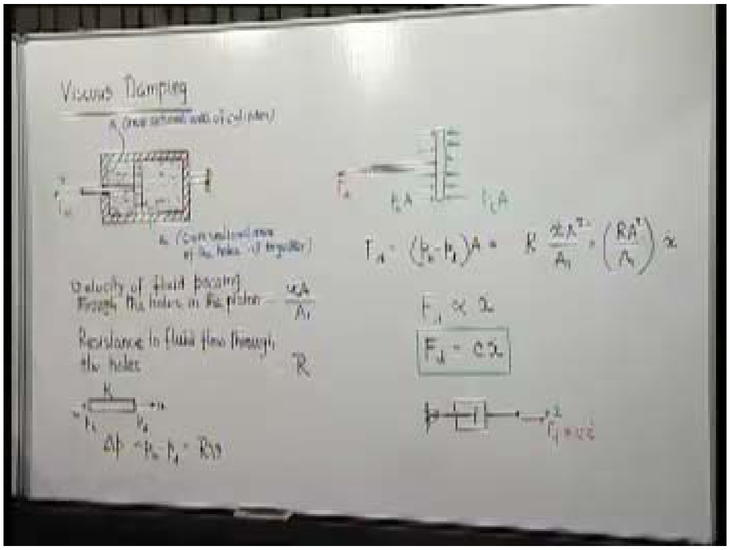

The inside of this is filled up with some kind of a viscous fluid, some oil or something like that. Now the cross sectional area is say A and A_1 is the cross sectional area of the holes all together, this piston has fine hole drilling to that through holes. The total cross sectional area of all these holes which have been drilled is said A_1. Now if I get this end fixed and try to pull this end with a speed say x, that end is fixed now. We know that, to draw it in this direction with a speed x dot, we have to apply a force F_d. How F_d depends on x dot that covered from, so what happens when we pull this piston in this direction? Fluid in this side goes through these holes to the other side because fluid is incompressible.

When we try to reduce this volume obviously the fluid has to go to the other side where a similar amount of vole is created. So the velocity of the fluid passing though the holes which have been drilled in this piston will be how much?

Now we know that, what is the amount of fluid which has to go for a unit time? It will be speed with which it is coming to this side into cross sectional area. You ignore the cross sectional area of this rod extra, which we can ignore. Therefore, the rate at which total volume of fluid must flow to this side, is how much? It is cross sectional area A and the rate at which this volume is reducing is x dot into A. The cross sectional area through which it flows is A_1, total amount so the velocity through these holes must be this now. We also know that the fluid resistance or resistance to fluid flow between, if it will be something which will depend on the cross sectional area of each hole, length of the hole, surface finish, but over all we designate it by quantity R. We know that any passage, if fluid flows at a rate V and the fluid resistance of this part be R, then the difference in pressure from this side to this side to P1 this side, say P high, P low.

That is the difference of pressure on the two sides will be given by the quantity or proportional to this, whatever we may say. Therefore, what we find now the piston, the two sides will have different pressure. What will be the pressure on this side, higher pressure will be on this side. The lower pressure will be this side because fluid is going from this side to that side. This side is higher pressure. How much is that? P high into cross sectional area A, again we have a merit approximate, ignoring the cross sectional volume on that.

On the other hand, pressure from this side, which will be a lower pressure, the total force will be and the force which we are applying here F_d. The whole thing has to be an equilibrium because there is no acceleration and so this force balance tells us F_d is equal to (Refer Slide Time: 16:28). This is equal to the resistance of the total resistance of the part, all the holes together, into the speed of the fluid passing through that, which is x dot A by A_1 into A square. Therefore, we find that this is nothing but R A square by A_1 into x dot.

Effectively what we find, that F_d is propositional to x dot. All these things of course are valid as we know only for laminar force. Therefore, we are considering that for a suitably viscous speed, the flow will be less laminar or we can write the constant of proportionality, which is represented by this, by a coefficient which we called c. The coefficient is a special property or coefficient for a particular damper, its dimension, kind of fluid, everything will depend on that. Therefore, we find that we get the damping force, linearly dependent on the velocity of the output. The viscous damping need normal form, provide a force which is a linear function of the velocity. That is why it is so easily amenable to solution, because the differential equation which we will get will be a linear differential equation that we can solve.

With this little introduction to viscous damping mechanism and the job of a damper, we will now take up our single degree freedom system and its damping parameter model. Generally viscous dampers are represented by this diagram. When this is moved with a velocities x dot, it needs a force F_d, which is equal to, which of course, this side fixed. Therefore, this damper here, its prominent property is providing a damping force. We will ignore its mass, we will ignore the elasticity of the whole system, that is any kind of stiffness which will be present. Therefore, this becomes again an ideal situation without any mass, without any stiffness. That means it is perfectly rigid. Its only job is to provide a damping force to the lumping of this damping parameter will be to this idealized situation.

Let us now take up the simple phase of a single degree freedom system with viscous damping and its free vibration. We take up now the case of free vibration of a viscous the damp, single degree freedom system. We just go one step ahead of our first and the simplest model where there was no energy dissipation.

This is the lump parameter model of the system. Of course, in its equilibrium position will be such when it has been left to itself. That means these spring has taken its equilibrium position.

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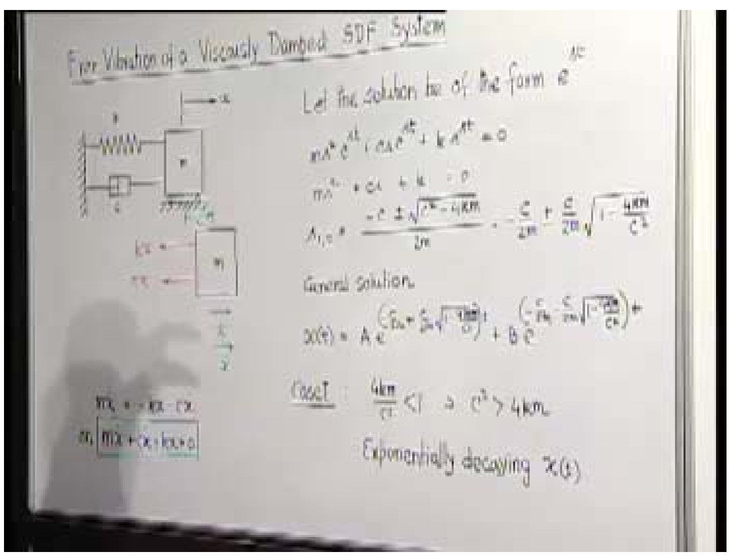

The natural length position, because if the spring is stretched or compressed, it will again slowly either go out or come back, so that this mass can come to that position when no force is acting on it. That means when it is stationary, so no viscous damping force and this spring is at natural length to known compression or tension force in that. When we displace it, mass is here, there will be two forces that will be acting. As we know, at any instant its position is x, so therefore there will be a spring force which will be k into x. If at this instant it is moving at a speed x dot, then here this damper will provide a damping force also like this, will be full. It will try to prevent its expansion. This will be the force and its acceleration will be algebraically positive acceleration in this direction.

Therefore, the Newton's second law for this will be m x 2 dot must be equal to the total force in this direction, which now is minus Kx minus Cx dot. Or this becomes the equation of motion of the mass when viscous damping is present. That means this term has come. Now to solve this differential equation. We proceed the standard way, that means let the solution be of the form e to the power st. When we substitute it here, we get ms square e to the power st plus cs e to the power st plus k e to the power st equal to 0. The characteristic equation what we get for s is a quadratic equation.

This quadratic equation will yield two values of this $s_{1,2}$ which you can write. You know minus c plus minus c square minus 4 km. Sometimes you can write it like this and the general form of solution or the motion that is xt, we can write as A e to the power e to the power minus C by 2 m plus c by 2 m 1 minus 4 km plus B. So, A to the power e to the power s_1 t plus B into e to the power s_2 t. This is the general form of solution. What will be the nature of solution? Let us first see before we proceed further.

We can see that a situation that when 4 km by c square, this quantity is less than 1. So case one, where 4 km by c square is less than 1, that is c square is more than 4 km. Under this situation, this whole quantity is a real quantity because this is real, these are all real and as you can see, since it is 1 minus something, this quantity will be less than c by 2 m. This is because this is less than 1 and since this is negative and this is positive, this whole quantity will be a negative real quantity. So e to the power a negative real quantity into t is nothing but an exponentially decaying torque with time. x, the first term, will decay as we progress.

Go to the second term, here again you find that this is a positive quantity less than 1. This is negative, this is negative and so this 1 again is a negative real quantity. Therefore, this is also nothing but an exponentially decaying term but here the decay will be fast. Both the terms are exponentially decaying with time, so we will have this kind of motion. That means, it is going to just asymptotically going to zero is called R square dt, which is not going to be a periodic motion, where the equilibrium position is crossed again and again repeatedly. There is no question of any period and this situation which results in this kind of a motion, this condition is called over damped.

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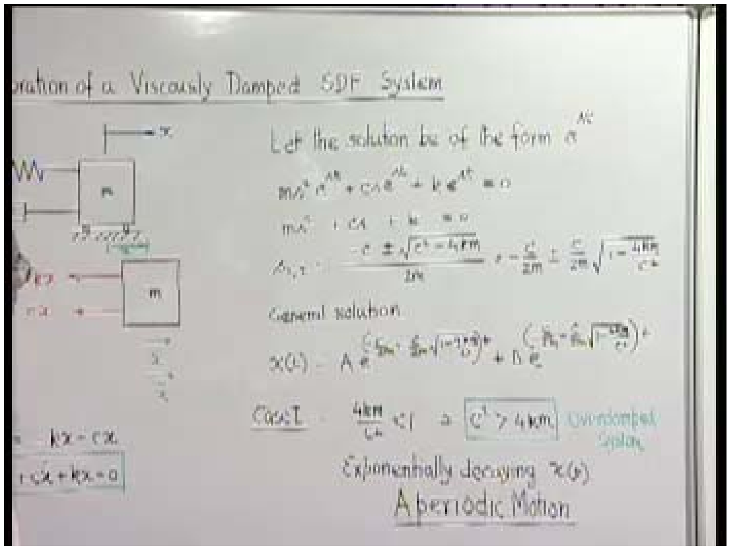

When the damping is very strong and it is so large that it is more than this quantity, then the system does not execute any oscillatory motion. It decays as the time increases resulting in a periodic motion.

The next possibility is case two when c square is equal to 4 km. This is a critical condition and with this condition, what we may say, which satisfy this condition c is called c_c and the system is critically damped system. In this case, when c square is equal to 4 km, that is when c is a particular value which you could indicate by the letter c, we call it critical damping coefficient. With this situation the system is called critically damped system.

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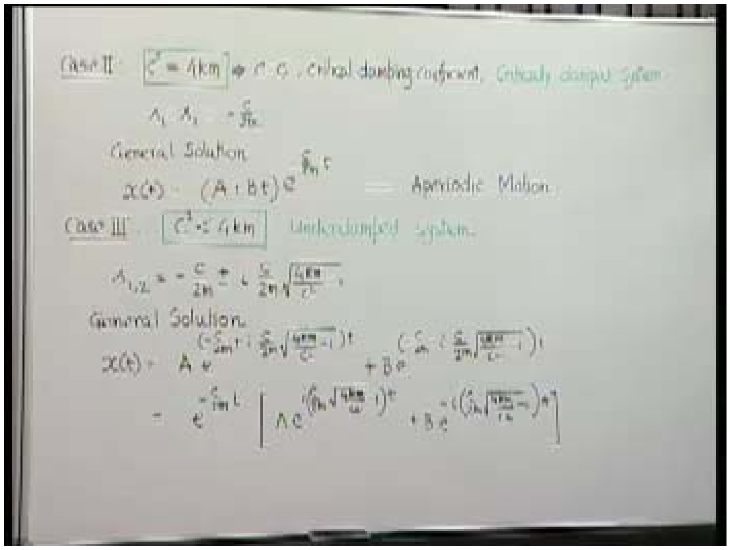

What will be nature of solution in this case? We know that s_1 will be equal to s_2 will be equal to minus c by 2 m. When the two roots are equal, the general solution is xt is equal to A plus Bt into e to the power st, which is minus c by 2 m into t. Here again we see that as t increases, this increases perhaps linearly. Of course we have to find out what will be nature of A and B, we will discuss that later. But this is an exponentially decaying term and so as a result here also we get a periodic motion. Here for example in an over damped phase, we displaced the mass and leave it, so it will go and approach the equilibrium position asymptotically. Here also, if you just pull it and leave it, it will again go a periodically and asymptotically approach the equilibrium position. The only difference with the over damped phase, this will be here and it will approach equilibrium position at the shortest possible time.

In situations of where we want a system not to execute oscillatory motion, there are many such situations. We will take up some examples like say door closing, when you open a door and if we leave it, which has to execute an oscillatory motion, it will go through the door frame with the maximum velocity which you do not want because then it will then bang somebody, which is not a very desirable situation.

On the other hand what we would like, we would like it to go and close at its equilibrium position with zero velocity or negligible velocity. Therefore, we will try to put damping which will provide a periodic motion. At the same time we do not want this closing to take place enormous amount of time. We want this to close as quickly as possible, say the room is air conditioned we want the door to close in shortest possible time. If somebody comes in and he leaves the door, it is going to close at the shortest possible time. Therefore, in such situations, we provide the system with critical damping.

Another example is cannon recoil system. When cannons are fired, they get a recoil and then if it is only a spring, then it will keep on oscillating. We do not want that, you want the cannon to recoil and then while it goes around this side, the damper is attached. It should go asymptotically at the quickest possible time and again due to high critical damping, same as the case which shock absorbers, in many ways we saw in suspension system, where we want the system to approach the equilibrium position without oscillation, at the quickest possible time.

That is where the importance of critical damping lies. In fact, this, we keep it slightly above critical damping because due to temperature rise or something, if the viscosity of the oil which has been put inside the dashpot, they reduce. Then it may go below the critical damping and its system may have oscillation.

Therefore, generally it is get very near critical damping slightly apart. It is now important to see that what happens in the other case that is the only case left. When damping is small and it is less than c square root of km, then the system is called under damped system. What will happen to this, we will have roots will be, if c square is less than 4 km and obviously this quantity is positive or more than 1. Whatever we put inside this square root is a positive quantity and that is ok. But we have multiplied the whole thing by minus 1 and so this has come. Therefore, now we find that the two roots are complex. There is a real part and there is a complex part of the solution.

These will be the general solutions. We can rewrite it slightly in different form. Before we proceed further, it will be desirable to have certain symbols or certain concept developed here. It is not generally very convenient to say c square is more than or less than or equal to 4 km because this particular critical damping coefficient, what we are telling, which is only for the particular configuration here. A different configuration can get a different relation as we will see while solving problems. There should be a better way of telling whether a system is damped or under damped or over damped or critically damped.

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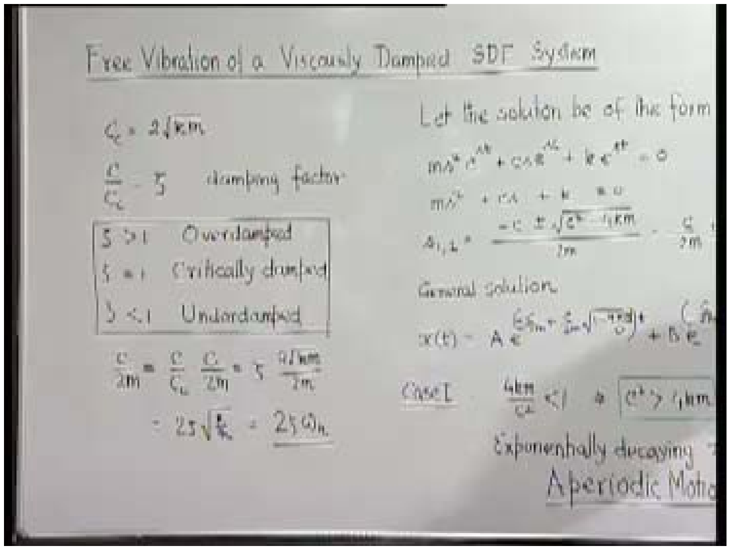

What you do, at least for this particular case, say if we find that c_c is equal to 2 square root of km and this quantity c by c_c, the ratio of the actual damping coefficient and the critical value, which just distinguishes between the aperiodic and periodic motion, as we will see is defined as damping factor. When zeta is more than 1, it is over damped. When zeta is equal to 1 critically damped and when for a system zeta is less than 1 it is under damped. Before we proceed further, just to comment on this nature of solution what we have got. As you can see, this is an exponentially decaying term but inside this we have a quantity into e to the power I theta plus B into e to the minus I theta. We all know that this represents a harmonic function of time. We will write it down later, after we use this concept and do the necessary changes here. Therefore, here we will find quantity c by 2 m can be written as c by c_c into c_c by 2 m and here this is zeta of st and c_c is 2 square root of km for this particular system, we are doing divided by 2 m and this is equal to 2 zeta. This becomes square root of k by m and is equal to, we know it is the undamped natural frequency of the system.

That means the system we are investigating if we take out the damper whatever will be the natural frequency will be the square root of k by m, wherever we have c by 2 m, we can replace it by this term. Therefore, for the three cases, let us write the solution using this concept. For case one, the general solution will be A e to the power minus; c by 2 m is 2 zeta omega$_n$. We have already seen c by 2 m is 2 zeta omega$_n$ plus, again c by 2 m is 2 zeta omega$_n$ square root of 1 minus. This is critical damping coefficient square c_c square which will be 1 by zeta square plus Be to the power minus 2 zeta omega_n minus 2 zeta omega_n square root of 1 minus 1 by zeta square.

This is the general solution, we can write it in this form. Then e to the power minus 2 zeta omega_n t is common in both the cases. So it goes inside A outside e to the power. Now if I write it z raised to the power e 1 minus zeta square or zeta square minus 1 here, we get 2 omega_n zeta square minus 1 t plus B e to the power 2 omega_n zeta square minus 1.

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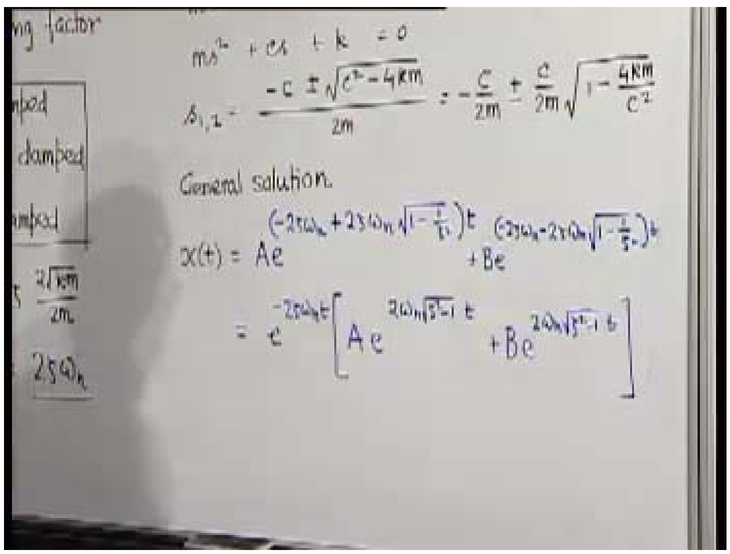

Depending on the value of zeta, whether it is more than 1, equal to 1 or less than 1, our solution will depend. Of course when it is equal to 1, then we have to follow this form of general solution that we have to keep in mind, not this form.

When we have the case three, that is zeta is less than 1. xt can be written as e to the power minus 2 zeta omega_n t and A e to the power. Now zeta is less than 1 so this quantity is negative. We can write this in this form 2 i omega_n 1 minus zeta square t plus B e minus 2 i. This is nothing but of this form, if we express e to the power i theta and e to the power minus i theta, you will ultimately get it to be something like A_1 sine. But in this 2 cancelled, sine omega_n 1 minus zeta square into t plus B 1 cosine omega_n. Now observing this, what we have got now in the under damped case, we find that the system is oscillating with time. But if it had been all of this, it would have oscillated like the case when we solved first without any damping.

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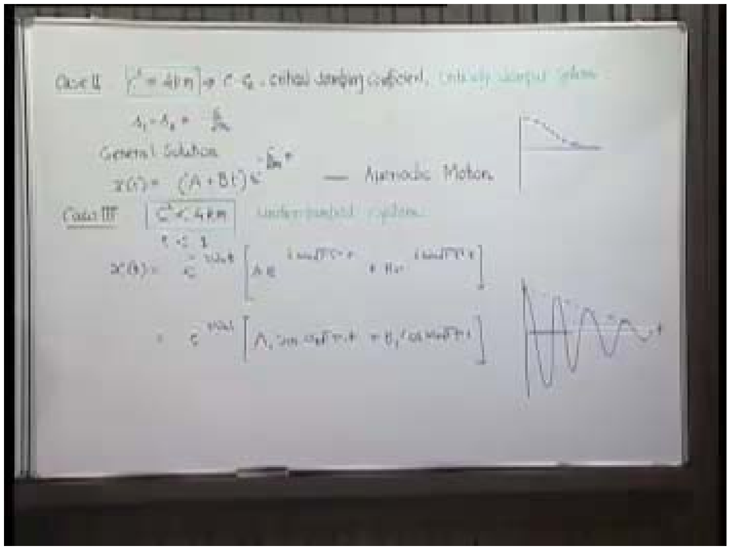

But the whole thing is multiplied by an exponentially decaying function with time. The resulting motion we find is slowly decaying, whereas in the previous case is we know that they are aperiodic motion, same thing in the over damped case. Now we have oscillation but the magnitude of oscillation is gradually decreasing. Therefore, what we have done till now is that for viscously damped simple spring mass dashpot system, depending on the magnitude of the damping coefficient of the dashpot, there can be three possible situation. One is called over damped. When the damping factor is more than 1, then the resulting motion of any disturbance is a periodic motion; whereby, the system approaches its equilibrium position asymptotically with the critical damping coefficient. When zeta is 1, it is the similar kind of wave here. Only thing, it happens with the quickest possible time. On the other hand when the damping factor is less than 1, system is under damped and it executes an oscillation when disturbed, but with a diminishing magnitude as time progresses.

We also noticed another important thing besides the fact that the vibration diminishes in its magnitude with the progress of time. The frequency of oscillation, which we call damped natural frequency is also slightly lower than the natural frequency of the system, if we take out the damper. Without damping it will oscillate little faster with higher frequency, whereas if damping is present, then it will be little less than the natural frequency without damping for the same system. This is called damped natural frequency which is nothing but omega_n into 1 minus zeta square root.

If we want to investigate further cases we will look into the general solution. Before we attempt the general solution a further discussion of the impact on the resulting motion of the nature of the initial conditions on which A and B will depend. Before that let us quickly look into the matter of under-damped vibration as we have seen here. Now if a system vibrates, which is given by x or say we can write it another form. That equation can be written in this form. This 2 constant A_1 and B_1 can be replaced by 2 constant as x and fi to m series when discussed the harmonic oscillation in the earlier lecture. The amplitude, for example, suppose this is the energy amplitude and say the next 1 is n plus 1, 2 successive peaks we are taking. What is the time taken for a system to go from one peak to another? Obviously it is nothing but the time period.

How much is this time period? Time period we know the 2 pi by the damped circular frequency or undamped natural frequency into 1 minus. So your x_n by x_n plus 1. How much? x_n is this for a particular time so therefore, it will be e to the power minus zeta omega$_n$ t into x. What will be the time for the next peak? It will be t plus tow and this is nothing but e to the power zeta omega$_n$. How much is zeta omega$_n$ tow? You can see it from here, that omega$_n$ tow is equal to 2 pi 1 minus zeta square here. Therefore, if I take natural log of this quantity, we get 2 pi zeta.

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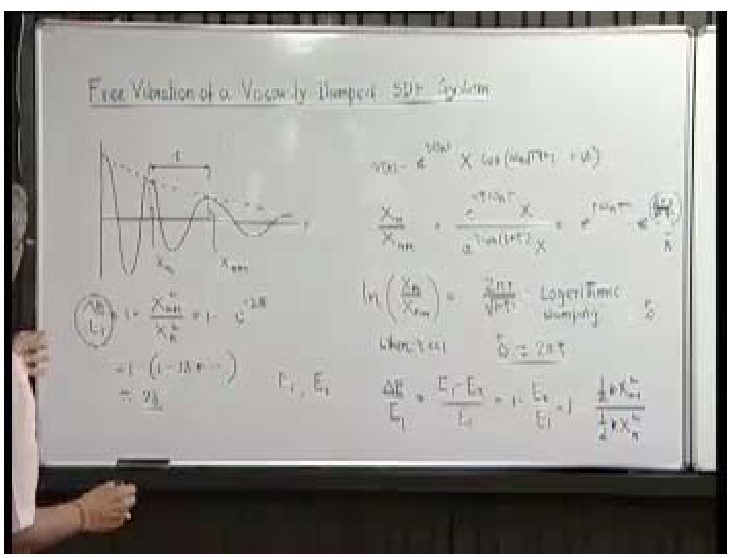

Now this is a measure of the intensity of the damping that means how fast the amplitude is reducing. Higher the value of this ratio of the two successive amplitudes, larger is the damping and when you take the natural logarithmic of the ratio of the two successive, you can see it does not depend on the end. It depends of any two successive amplitude will provide this quantity. This is called logarithmic damping and this is a measure of the intensity of the damping. As you have seen, which is of course it has to depend on zeta, as zeta represents the damping. When zeta is small, which happens in many of the situations then, this quantity logarithmic damping delta that is the standard symbol is equal to 2 pi zeta.

Another method of representing the intensity of damping is by how much energy is being dissipated power cycle and its ratio with the maximum energy during that cycle. When the system is vibrating, any cycle you take say either this or this. Any cycling, the energy is obviously maximum at the beginning of the cycle because energy is continuously being dissipated. Of we take the loss of energy divided by the maximum energy, suppose at the beginning of the cycle, energy is E_1 and at the end of the cycle the energy E_2.

This is E_1 minus E_2 by E_1, this is this quantity which is again nothing but 1 minus E_2 by E_1. We know that the peak position all energy is in potential form and obviously it is nothing but this strain energy in the spring. That is half k and x say n plus 1 square, if that is the energy at the end of the cycle.

That means with the next peak divided by energy at the beginning of the cycle, that is the previous peak. What we get here is nothing but delta E by E_1 equal to 1minus x_n plus 1 square by x_n square and this is equal to 1 minus, how much is x_n plus 1 by x_n? e to the power minus zeta omega$_n$ t or since omega$_n$ tow is equal to 2 pi zeta. This will be 1 minus e to the power and this quantity is delta. This will be minus 2 delta, if we expand it 1 minus, what will be this infinite series so approximately 2 delta.

Again we find that this quantity, the amount of dissipation of the energy for a torque, during the cycle divided by the maximum energy during the cycle is nothing but double the logarithmic decrement. This is another way we can represent energy dissipation to the amount of energy dissipated per cycle and you can see the relationship is again there, with the logarithmic decrement. The measure of the intensity of damping, which we do with the help of damping factor but during experiment when we do, we try to figure out how quickly the amplitude is decreasing. We take two successive amplitudes and find out the logarithmic decrement. Sometimes it may happen that two successive amplitudes are very close to each other and therefore there may be a large amount of error involved.

What we can do then, we can take a p number of cycles in between. We can take the ratio of logarithm and take the logarithmic of x_n by x_{n+p}, just the next 1 but after p side. Obviously we will find this will be p tow. That means $omega_n$ tow is 2 pi by 1 minus zeta square. So it will be 2 pi p and obviously this is going to be or simply p delta.

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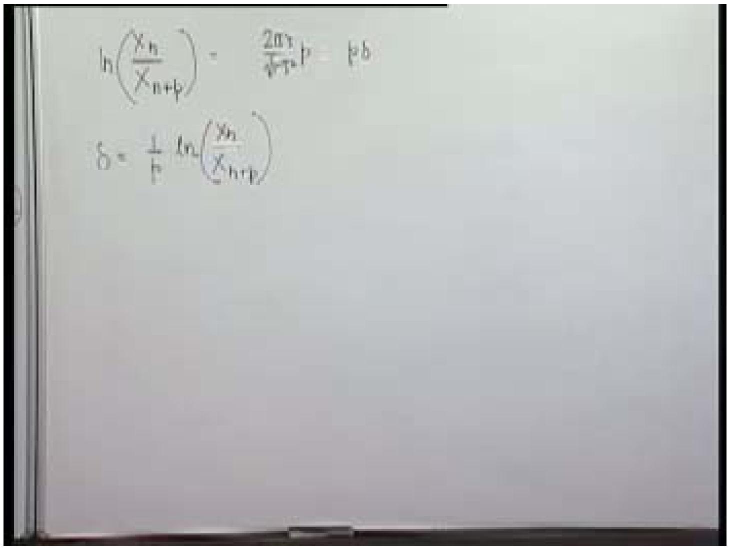

Delta can be found out in such cases and finding this ratio between two amplitudes widely separated can be much better in the accuracy. That is another technique by which logarithmic decrement is found out. Now if you want to find out the damping factor, you can do so from delta by using this equation. The delta is given by this or this to get zeta. If zeta is really very small, then we can even use this directly. You divide delta by 2 pi, you get zeta and so on. Then you can also find out the amount of energy dissipated per cycle and all types of investigation are possible.

What we will do in the next lecture is that we will investigate in great detail that what kind of initial conditions can lead to what kind of motions based on the amount of damping present in the system.