Dynamics of Machines

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Module No. # 09

Lecture No. # 01

So far, we have been considering with single degree freedom, it is the systems which consist of only one of each element. That means, one block representing the inertia and one spring which represents the restoration. However, there can be single degree freedom systems, which consist of multiple numbers of bodies representing inertia; multiple numbers of springs of different types representing the whole effort that brings the system to equilibrium restoration.

In these cases, the technique to solve the problem will be, to convert the whole system into a simple spring mass system, but it should be dynamically equivalent to the original system. Therefore, in this lecture we will take up the discussion on technique for finding out the equivalent inertia and equivalent stiffness.

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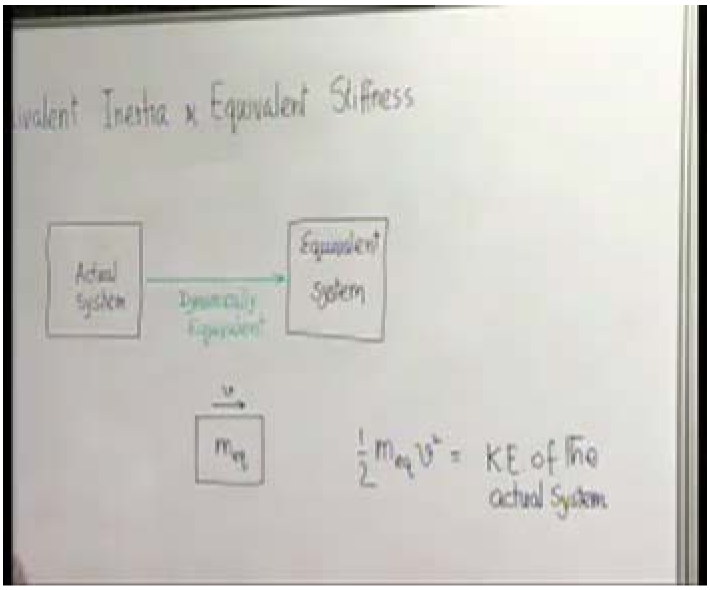

We have a system; the actual system and you will convert it into an equivalent system, but why do you do that because it must be dynamically equivalent. Dynamically equivalent means, if we apply certain force for certain moment, then the acceleration produced in the system will be same as that of what would have been produced if the same force of moment had been applied to the actual system. How do you find out that or how do you ensure that?

Since, the system has to be dynamically equivalent, we must always keep in mind that they also should be kinematically equivalent that means, the velocity displacement and acceleration of the various points or members in the original systems must be same all the time, as that of the equivalent system.

You will find out how we can do that and let us consider this; this is the equivalent system. Select this equivalent system with the inertial mass $m_{equivalent}$, $m_{equivalent}$ is chosen in such a way that if its speed or velocity, whatever it may be at this instant is v, then this kinetic energy is equal to the kinetic energy of the actual system.

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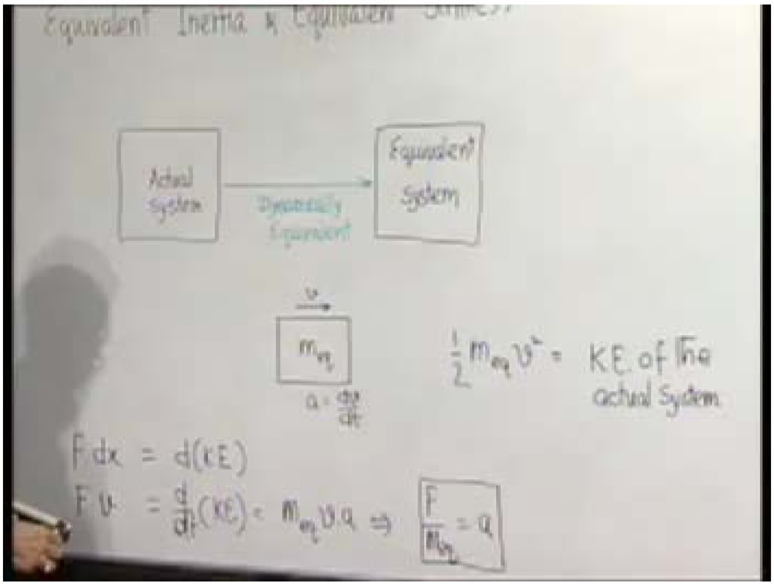

If we apply a force, we know that force will work with a small infinity distance dx, here dx is the amount of work done; where will that work go? That energy will go? It will increase the kinetic energy. If we consider, the potential energy remains same, then this will be nothing but gain in the kinetic energy, you could assume that potential energy remains same.

So, if we write this by dt then we get, F into v will be equal to d by dt of kinetic energy (Refer Slide Time: 05:32). Now kinetic energy is same as this; so this will be nothing but $m_{equivalent}$ into v into a, which is equal to da divided by dt; a is nothing but da by dt. Since, v is not equal to 0, from this we get, F by $m_{equivalent}$ is equal to the acceleration. You have produced same acceleration by applying same force, mass is chosen in such a way that the total energy of these each energy for a same velocity is identical with that of the original system.

Therefore, the technique of finding out the equivalencies is by keeping the total kinetic energy of the actual system and representing that by a single mass $m_{equivalent}$ in this form. Similarly, without going into details, we can also tell how to find out equivalent restoration or equivalent stiffness by giving the same deflection to a particular point, when a force is applied. The total amount of spin energy in the system will be same as that of the equivalent system.

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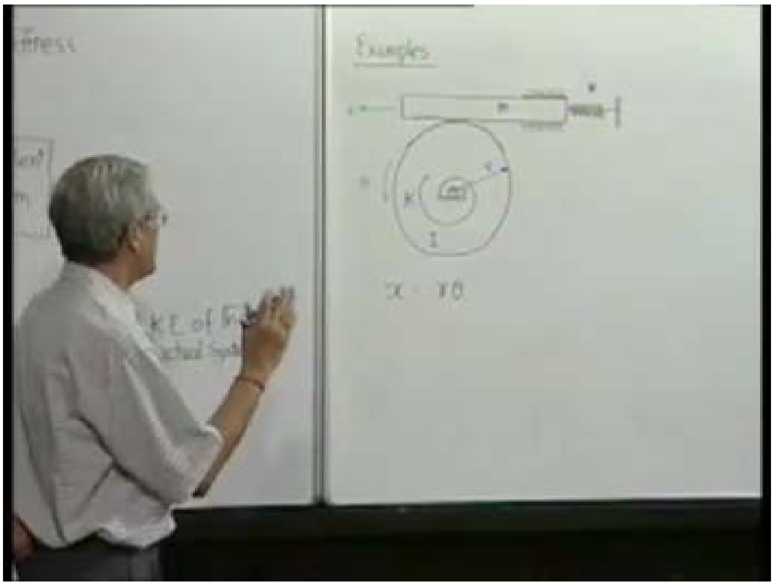

If we take up some simple cases, let us take up first case, where we have one mass m, which is connected by a linear spring k to a fixed wall. At one portion of the lower edge or the lower surface of mass, which is like rack and that is engaged with a gear or a wheel, a part of this is having a key.

The radius of this disc maybe r, its moment of inertia is I about an axis of input O (Refer Slide Time: 09:23). Further to this, this disc is connected to a fixed wall with the help of a torsional spring with its thickness as capital K, plus the system consist of two bodies for this inertia, one is moving in a linear direction that means, it is having displacement x and another object which is having a rotation correspondingly; rotation theta. This has a mass m, it is connected by a linear spring to a solid wall. This disc whose moment of inertia is I, because it can rotate about this, it has a radius r, and it is connected to the foundation by a torsional spring K. Its displacement is indicated by rotation theta, now this theta and x are obviously not independent.

Because it is a single degree phenomenon system, if any one quantity is specified, then it should completely specify the configuration of the system. So, either we mention theta or we mention x and they are related by this x, which is nothing but r theta.

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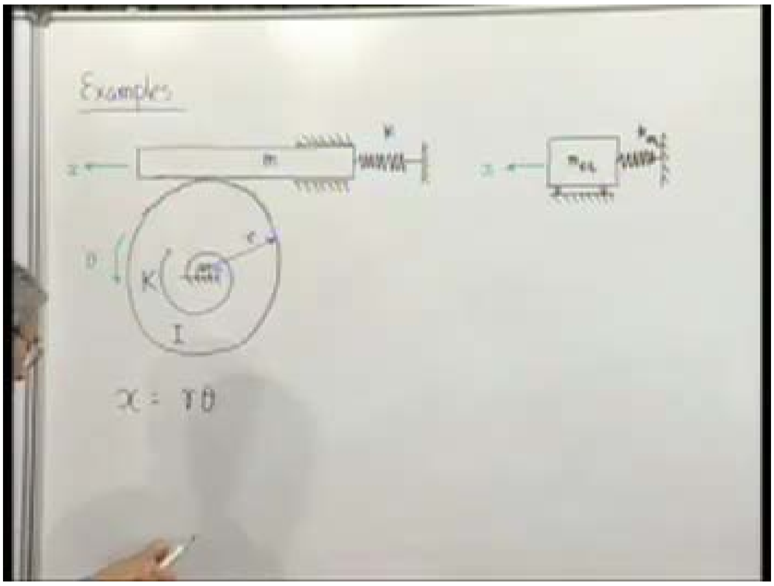

From their equilibrium position, if we give a displacement x to mass m, the disc automatically rotates by an angle theta due to the severing action here, which is related by this relation. This is a single degree freedom system though it has two bodies.

If you want to find out an equivalent, we have to find out two types of equivalent system. We may consider x displacement to be the primary thing (Refer Slide Time: 00:11:42). It is a simple spring mass system, with a mass $m_{equivalent}$ and the connection to the wall by another equivalent spring, whose thickness is $K_{equivalent}$.

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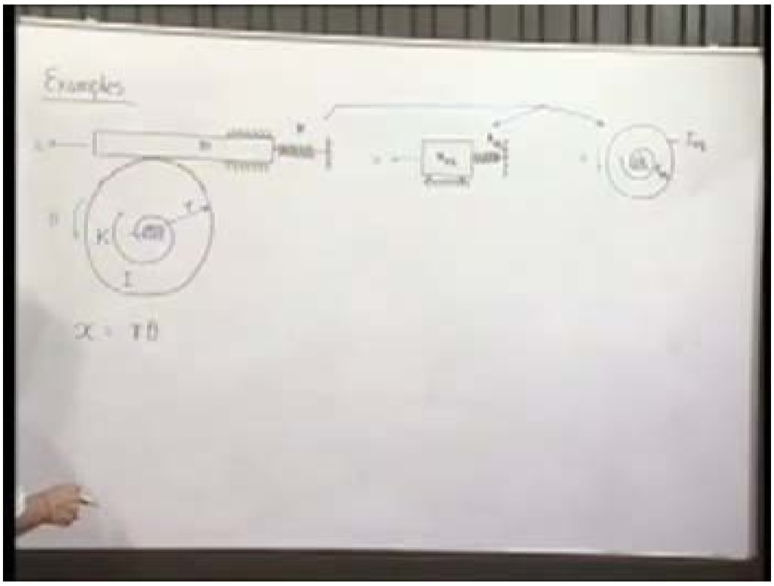

But the displacement parameters will remain same as that of the original body, because its kinematics must be same. Now one may say, why do you take up this, which has the primary body? Instead of that we can take a rotation and we can represent the equivalent system to be a rotating system, which is connected by a torsional spring capital $K_{equivalent}$, the moment of inertia of this, which is $I_{equivalent}$ and its angular displacement will become equilibrium position; both are possible. This means, equivalents can be either of this; it is purely our choice (Refer Slide Time: 12:47).

Now, what will be the values of $m_{equivalent}$ and $K_{equivalent}$? As I mentioned, the equivalents is determined in case of mass; the total kinetic energy of the system and the kinetic energy of the equivalent system should be same.

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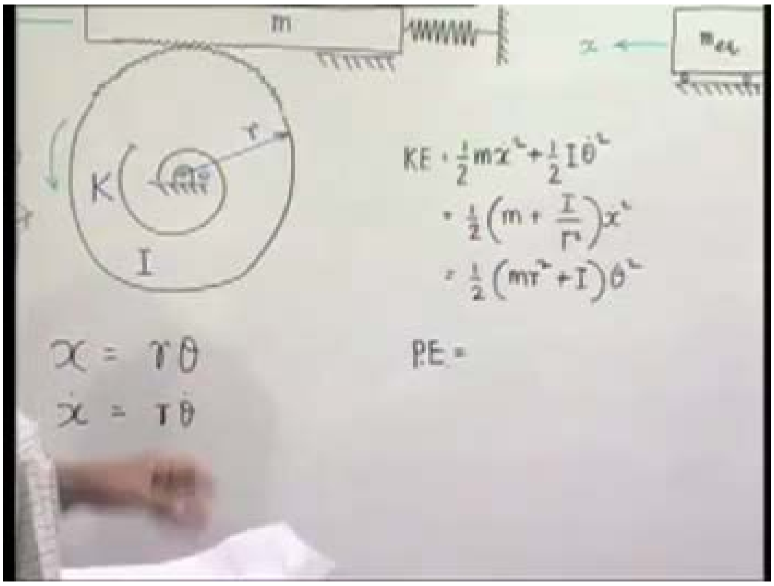

What is the kinetic energy of the original system? First, linear kinetic energy of this mass m is equal to half m x dot square. The angular velocity of this spring k is equal to theta dot therefore, x dot is equal to r theta dot (Refer Slide Time: 13:48). So, kinetic energy is equal to half m x dot square plus half of I theta dot square, which is same as half into x square of m plus I by r square , where theta dot can be written as x dot square by r square. You can rewrite this as half into theta dot square of m r square plus I, because x dot is r theta dot, both equations are possible.

Similarly, if we consider potential energy for any displacement, then there is no gravitational potential energy involved, because this disc is remaining in the same location though it is rotating, so its potential energy due to gravity is not changing.

This mass is the center of mass; it is not changing to any level because we have placed it in a horizontal guide (Refer Slide Time: 14:47). Therefore, the potential energy due to gravity of that mass remains constant, but we can trip the potential energy, we can keep their position to zero and increment to zero all the time.

There is steady energy in the two springs that is; the linear spring and the torsional spring, which provides the potential energy. The distortion of stress of this spring from its equilibrium is x and the equilibrium position was known under this spring condition.

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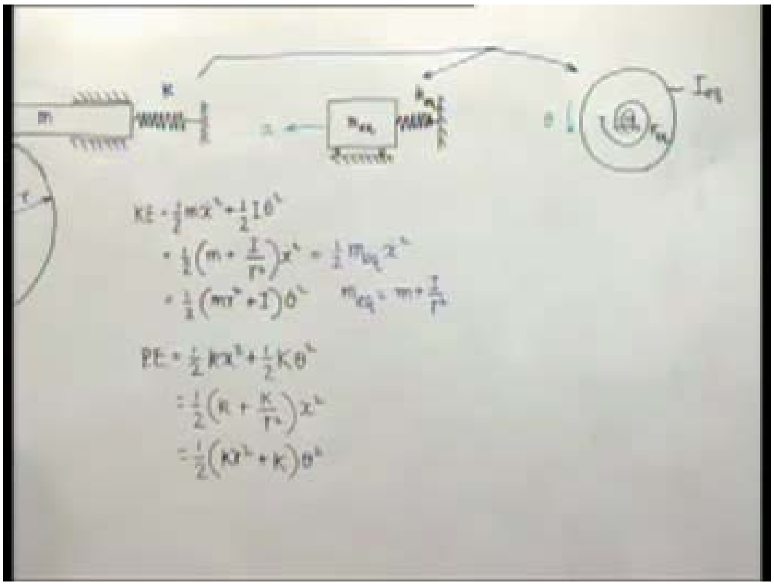

Therefore, in the linear spring, spin energy is equal to half of K x square. Similarly, if we consider this, during or at the equilibrium position, this particular torsional spring was also called torque spring. The potential energy stored or the spin energy stored in the torsional spring will be this (Refer Slide Time: 15:44).

By using these relations, we can write it like this, half K plus; theta is equal to x by r, which should be capital K by r square. We can also write this as half k x square and x is equal to r theta. So, the kinetic energy will be used for the determination of the $m_{equivalent}$, where as Potential energy is used for finding out the $K_{equivalent}$. That means, if we say that data's are used for two different colors; one for this; one for this (Refer Slide Time: 16:33).

So, I will say that since we have taken $m_{equivalent}$ system to be a linear guban, we will take up this. Therefore, these should be equal to half $m_{equivalent}$ into dot square. That is the kinetic energy of equivalent systems with the same velocity of ((force)) that you have to always keep in mind.

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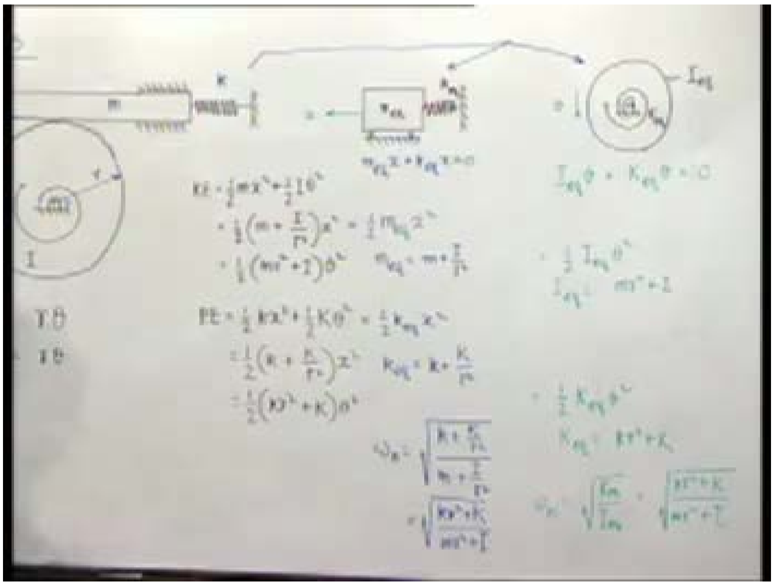

We get $m_{equivalent}$ as (Refer Slide Time: 17:16). Now, $K_{equivalent}$; this is equal to half of $K_{equivalent}$ x square. Therefore, we get $K_{equivalent}$ is equal to. What will be the equation of motion for this? $m_{equivalent}$ x two dot plus $K_{equivalent}$ x equal to zero; omega n will be equal to square root of $K_{equivalent}$ by $m_{equivalent}$ or simply this. If we try to solve the problem by considering an equivalent system, which is irrigating angular motion, then let us use this. Then we will take this, therefore its equivalent total kinetic energy is equal to half I_{eq} theta dot square. Equating these two we get.

Similarly, x total potential energy is half; $K_{equivalent}$, is nothing but; the equation of motion on this is nothing but $I_{equivalent}$ theta two dot plus $K_{equivalent}$ theta (Refer Slide Time: 18:50). This is coming directly from Newton second law, the total torque acting on these quantities on that angular displacement theta is $K_{equivalent}$ into theta and that must be equal to $I_{equivalent}$ into theta two dot by Newton second law; this is nothing but the inertia.

Obviously, omega$_n$ will be equal to $K_{equivalent}$ $I_{equivalent}$, you get the same result, and it cannot be different, as far as the frequency is concern (Refer Slide Time: 19:50). Thus, we have seen that the system when it is complicated it is always desirable to modulate in the form of either a simple spring mass system or simple disc torque null spring system. Then, we can determine the equivalent mass and equivalent stiffness, by equating the kinetic energy of the original system with the kinetic energy of equivalent system and so on.

Next thing, what we have to see is, sometimes the restoration elements like spring will get combined which means, we may not have a single spring like this. Here, we find a single torsional or a single linear spring, sometimes a composite structures maybe there, which consists of a number of elastic body. In such case, the resultant stiffness of the spring of the restoring element has to be found.

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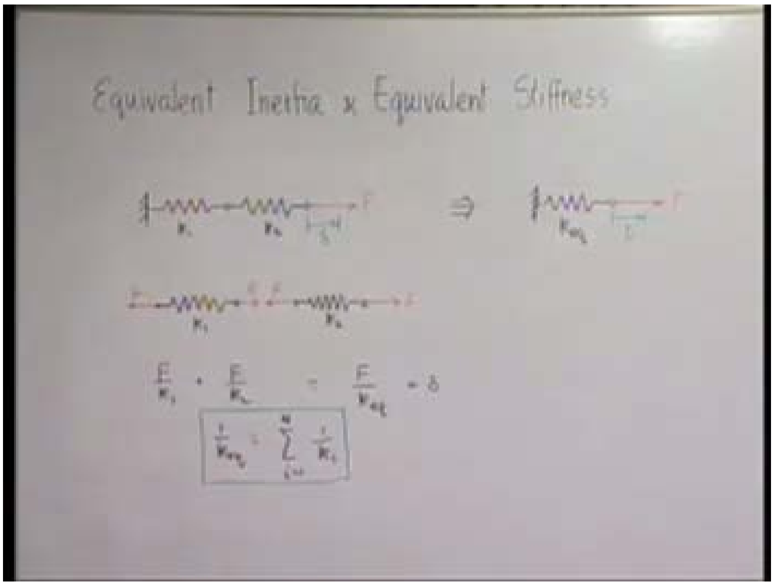

Next, let us consider the case of a composite spring system and how we can find out the equivalent system, which can be represented by either a simple linear spring or torsional spring. First let us consider the springs in series, the simplest composite system will be having two spring of stiffness K_1 and K_2 respectively and they are connected in series. Our job is to find out, what will be their equivalent system, we have to find out the equivalents; what should be the condition of the equivalents. As we mentioned that the energy stored in the system here, energy stored in the system here, could be same (Refer Slide Time: 23:55). In other words, we can also tell that the deflection cause here under this force should be same, as the deflection cause here.

So, we will not understand the difference whether there are two springs or three springs, we will find that we are applying a force and this is causing a displacement delta. Same thing here, we are applying a force causing a deflection delta therefore, we thrift that the stiffness are identical.

Here the stiffness are known, let us find out that under the action of force, what is this deflection. It is very clear to see that this displacement form this original collision is nothing but the stress of this spring plus, the stress of the second spring (Refer Slide Time: 24:18). What is the force transmitted or sustain by these two springs? It is the same force there. If you draw the free body diagram, it is quiet trivial but the force acting on the second spring is F is the same that of force acting on the first spring. Therefore, the stiffness of this is K_1; stiffness of this is K_2; the amount of stress in these will be nothing but F by K_1; the amount of stress in the second string will be F by $K_{2.}$ Since, they are arranged in series, the total stress will be some of total of this; this should be same as that of equivalent spring, whose amount of stress will be under the same equal force, which is defined.

We will find that 1 by $K_{equivalent}$ is equal to some of total of 1 by K_i, where i is equal to one to N. If there are N number of springs connected in series, with stiffness $K_{1,}$ K_2 and K3 up to K_N, then the relation between the equivalent stiffness of a single spring, which will produce the same degree of elasticity or stiffness, which is given by this simple derivation.

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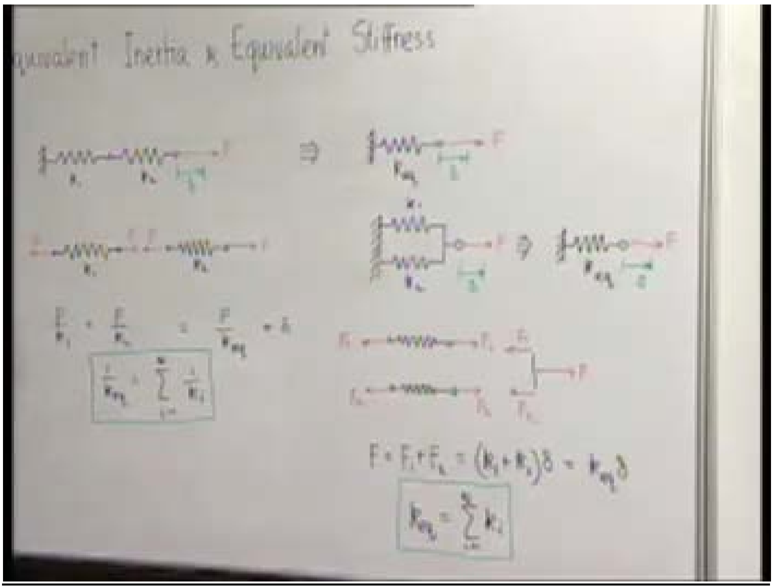

The other possible case is that when springs are in parallel, again we will consider these two springs, which are now in parallel and subjected to a force F. We have to find out an equivalent stiffness for a simple spring, which can represent this stiffness of the original system (Refer Slide Time: 27:00). That means, again we have to find out the deflection of this point, under the action of this force F, which will be delta, which should be same as the deflection of this point, under the action of the same force, this will be equal in general.

If you draw the free body diagram, you find that this spring is subjected to a force F_1; this one is subjected to force $F_{2;}$ this is subjected to F; this side is F_1; this is of course mass placed and has no resistance (Refer Slide Time: 28:14). Therefore, one thing is very clear that F is nothing but F_1 plus F_2. How much is F_1? Now, both this spring's have undergone same amount of stress, which is delta. Therefore, the force in these will be K_1 into delta; force in these will be K_2 into delta. Therefore, this is nothing but K_1 into delta plus K_2 into delta, we can take delta outside.

We also know that this force is same as K equivalent into delta. Therefore, when these springs are in parallel, $K_{equivalent}$ is nothing but sum of total of stiffness, but this is the basic rule (Refer Slide Time: 29:22). Now, in a real system there maybe still more complicated system, some are in parallel, some are in series, all these things, but I think one can understand the following of these two rules; any kind of complex system can be handled and can be represented by these simple springs. We have shown it in the case of linear springs; in the case of torsional spring situation. These systems can be found out; and equivalent thickness can be found out.

Now, in this energy technique, which we have been discussing so far, it can be used to solve certain other types of situation, which cannot be handled by simple approach. One such example is, perhaps it is appropriate to discuss at this juncture. If a system has any spring and it is desired that the mass of these spring is also considered at least approximately, then how can we find out the natural frequency of such system.

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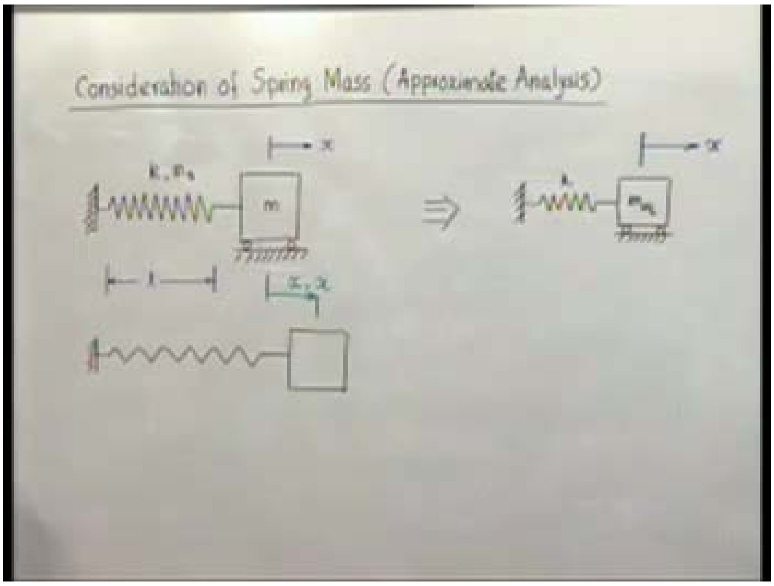

Though, we have mentioned earlier that while solving problem we always lump the parameters and what we are actually handling is a lump parameter system. While lumping, we thought, originally in such simple cases the springs are very light and they may be consider mass less, the block is the only thing whose thickness is extremely high and it can be consider to be rigid and that represents the inertia only. But, there can be situations where it is desirable, who considered the string mass. Now, we will solve that problem.

This is our system that means the simple spring mass system, the block has a mass m which off lets freely in the horizontal direction. (Refer Slide Time: 32:30) The spring has the stiffness k and a length of the mass m_s and the length of the spring l. We have given the mass of the spring as m_s and you want to find out the natural frequency of the system. So, what we will do as we have been thinking all the time that we should have lump parameter system.

We will solve that system, where the spring has been replaced by another spring, whose stiffness is in same state, but which is mass less. The effect of the spring mass has been taken care by an equivalent inertia. We will obviously use the same technique as before; we have already seen that the equivalent inertia is found out by considering the total kinetic energy of the replest system or the equivalent system, which should be same as, the total kinetic energy of the original or the actual system, under the same kinematics condition that is, at the same velocity the equilibrium position is the same.

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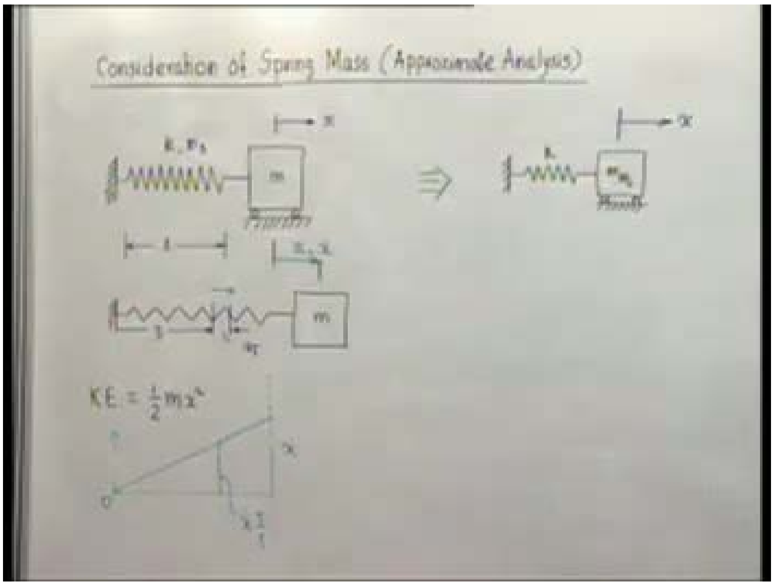

Now, let us consider this one, when it deformed the original equilibrium position is this and the displacement at this instant is under consideration x. Its velocity of force is x dot in this direction, which is highly potential. Let us find out, under this condition what is the total kinetic energy of these? Total kinetic energy of this is obviously; first, the kinetic energy due to the block, which is half m into velocity of the block this is equal to x dot square.

How much is the kinetic energy stored in this spring? You have to find out by considering each element of the spring. (Refer Slide Time: 36:04) Let us consider this, element at a distance zeta form this distance, whose length is dz. If that particular element has some mass, then the point is what is called the velocity of this element?

Here what we can do, we will assume that the formation of this spring is linear; this is an assumption. Though it is very different form the actual case, but exactly it is not. We assume that in the fixed end of the spring the displacement is zero and when it comes here, obviously the displacement is x.

Similarly, velocity also will be proportional; velocity is the same at this end, which is equal to x dot; velocity at this end is zero so velocity at the zeta will be how much? It will be x dot into zeta by l, if we have taken a linear distribution of spring. What we can do, we add the kinetic energy in the spring. First, the mass of the element is dz by l into m_s; that is the mass of the element. Mass for a unit length of the element is m_s by l. (Refer Slide Time: 37:49). The length of the element is dz or d zeta. So, d zeta m_s by l is the mass of the string.

What is the velocity of this element? We have found out x dot into zeta by l, this is square of that. This is the kinetic energy of the element; the kinetic energy of the whole spring will be sum of total of the kinetic energy of all the elements. That means you can just integrate.

How much is that; this is half m x dot square plus half; this m_s by l cube goes outside, and integration of x dot also goes here, so it become x dot square (Refer Slide Time: 39:00). Now, integration of zeta square d zeta is zeta cube by 3 and in the limits you get this, which is same as half m plus m_s by 3 x dot square.

So, this is the total kinetic energy of the original system. Kinetic energy of this is half and that must be equal to the kinetic energy of the original system. Thus, we get $m_{equivalent}$ is equal to m plus m_3 by 3.

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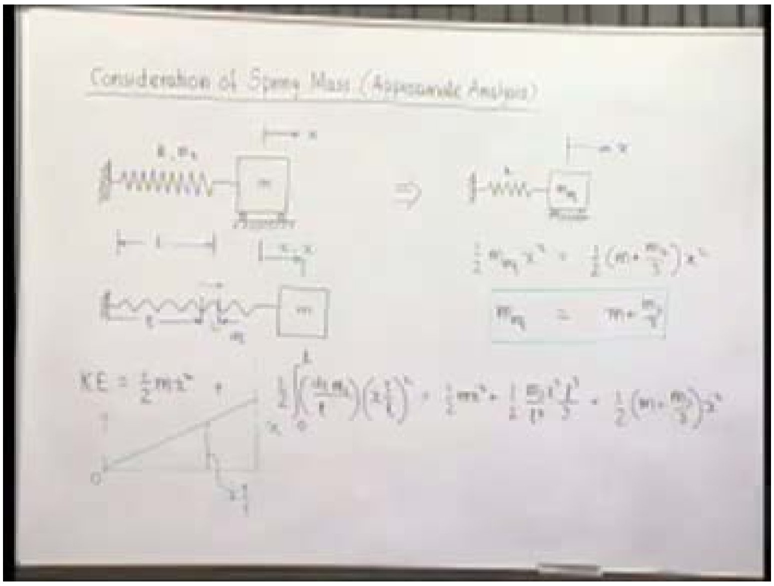

We must remember that this relationship which we have got actually might be approximate, because when this approximation is considered, then the displacement of each element is linearly proportional to its distance from the fixed end, which is not exactly correct. But the area introduced is very small, so everywhere there is certain degree of approximation involved. Therefore, you find that taking care of the spring mass is very simple, only thing what we have to do, we have to add one-third of the spring mass to the mass of the block and therefore, the natural frequency can be easily found out from this (Refer Slide Time: 40:47).

So, you have seen that the energy technique or energy approach can really help us in solving many problems, which may not be that convenient to be taken care of using direct approach.

Next thing, which we will take up in today's lecture is representation of this harmonic oscillation in a particular way, which is called phase plane representation.

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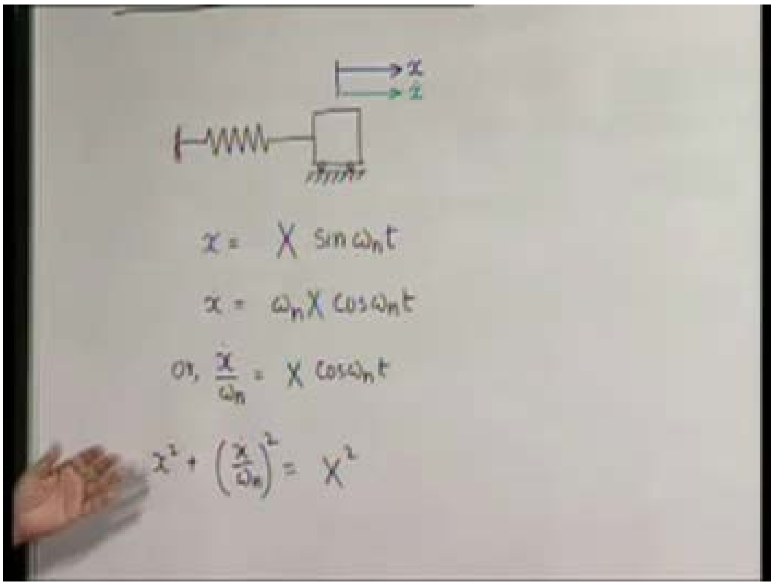

The phase plane representation sometimes can be very convenient in solving certain problems and I think it will be appropriate to introduce this particular topic at this juncture. Whenever, this application comes in future we will just remember the discussion here and apply the technique and the concepts which we are discussing.

This is the spring mass systems; say for example, in a simple spring mass system the mass is executing how many oscillations, it can be angular oscillation and also the equations are same or similar.

For all small oscillations, we know that we can represent x by a harmonic function, this we have used by considering the energy so, the harmonic oscillation of this object is being represented by this function of time.

We know x dot will be, (Refer Slide Time: 43:54). The instantaneous speed of that will be this, if you define the quantity then it will be instantaneous velocity divided by instantaneous frequency, it will be x cos omega$_n$ t, where x is the displacement, where the instantaneous velocity divided by natural frequency-circular natural frequency and square them, then obviously indicate function.

So, if you plot instantaneous displacement and instantaneous velocity by natural frequency, which is constant for a system, then obviously we will get a circle, because this is nothing but the equation of a circle.

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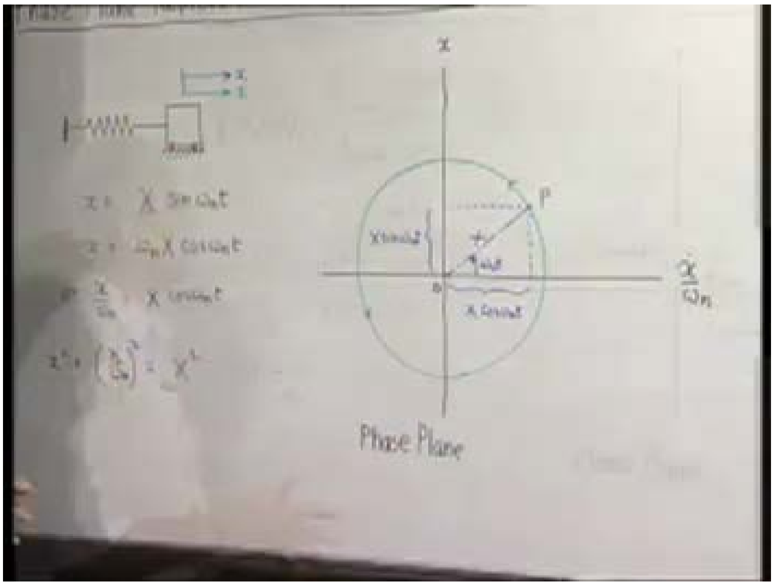

Therefore, this plain which we are not constructing is not the physical plane, because one axis represents the velocity and the other axis represents displacement. So, this plane is not a physical plane, this plane is called the phase plane. Why it is called phase plane? Because every point here identifies what is the position and what is the speed that means, the base or the state of motion of the system and that is what called as phase plane.

Here, we have a harmonic oscillator; the points representing its state of motion will describe this circle, with this as the amplitude (Refer Slide Time: 46:14). So, displacement at any instant is nothing but x sine omega$_n$t. If we consider this to be at the 0 position, this is nothing but x, which happens to be velocity divided by omega$_n$. The phase plane diagram for a simple harmonic oscillator is a circle, whose radius is the amplitude. One thing should not be forgotten is that there is an arrow; this arrow has to be in this way because when velocity is in the positive quadrant, this one here (Refer Slide Time: 47:17). When velocity is in positive, x dot is positive identically, x must increase.

During this half cycle, x is increasing because velocity is positive. On this side where velocity is negative, x is decreasing from this value to this value that is why the arrow heads are unique. It must be represented like this (Refer Slide Time: 47:36). This concept is the presentation of the harmonic oscillation, which can be made with the help of circle in the phase plane. This is the very useful concept mainly because circles can be drawn very quickly and conveniently. It has other use also for example; let us take a simple example.

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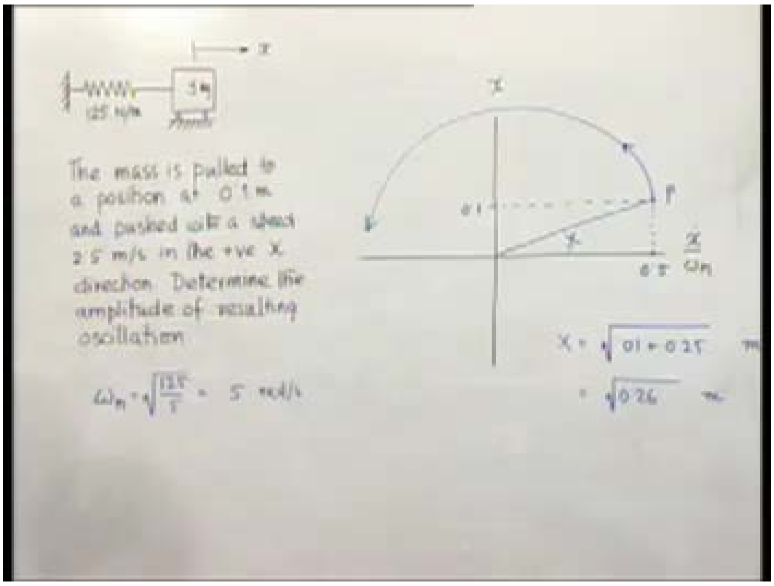

Let the system with 5 kilo gram mass, this is say 125 newton per meter, just the hyper frequency phase to its thickness. So it is taken. It is a simple problem, there is a mass of 5kg connected to a spring, which is again connected to thick wall and spring is also connected to wall, the stiffness of spring is 125 newton per meter, then the masses is pulled to a position in this direction at 0.1 meter, with a speed 2.5 meter per second in the positive x direction (Refer Slide Time: 50:40). Determine the amplitude of resulting oscillation. Now, you can solve the problem by other approach also, but the objective here is to the use of phase plane diagram.

The state point; if we can find the single point which represents the state of motion of the object, our job is done. Wherever the point is, the actually assuming motion will have to be a circle, with o as its center. Say for example, what is x? x is 0.1 and what is this, the speed x dot by omega$_n$ (Refer Slide Time: 51:30). Now, we can find out omega$_n$, this is equal to square root of 125 by 5, which is equal to 5 radians per second.

So, x dot in meter per second divided by 5 radian per second will give you simply meter, which is same. Therefore, we have to use the meter per second. Here, 2.5 by 5 is how much, it is equal to 0.5. In the amplitude of motion the fencing motion is going to be, a circle, starting from this; this is the starting point where the displacement is 0.1 (Refer Slide Time: 52:57); x dot by omega$_n$, where x dot is 2.5 meter per second and omega$_n$ is 5, so it is equal to 0.5. Therefore, this is the starting point and it will describe as starting point, x will be equal to square root of 0.01 plus 0.25 meters. This problem can be solved quickly by using the technique of phase plane, without even solving any equation. More use of phase plane technique will be coming in the future and we will take up the problems when it comes.