Dynamics of Machines

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Module No. # 08

Lecture No. # 03

In the previous lecture, we have solved problems on free vibration of simple single degree freedom system without damping by using Newtonian approach. It means that we consider the body, which is executing oscillatory motion subjected to the various process both disturbing and the restoration, and then they form the equation of motion. The solution of equation of motion will be subsequent result, which is a simple harmonic function of time. Sometimes, it may be inconvenient to draw the free body diagram or to conceive the problem as one suitable form for writing down the equation of motion in a state for one manner; we will take some example very soon. There is another approach for solving free vibration problems of single degree freedom system without damping.

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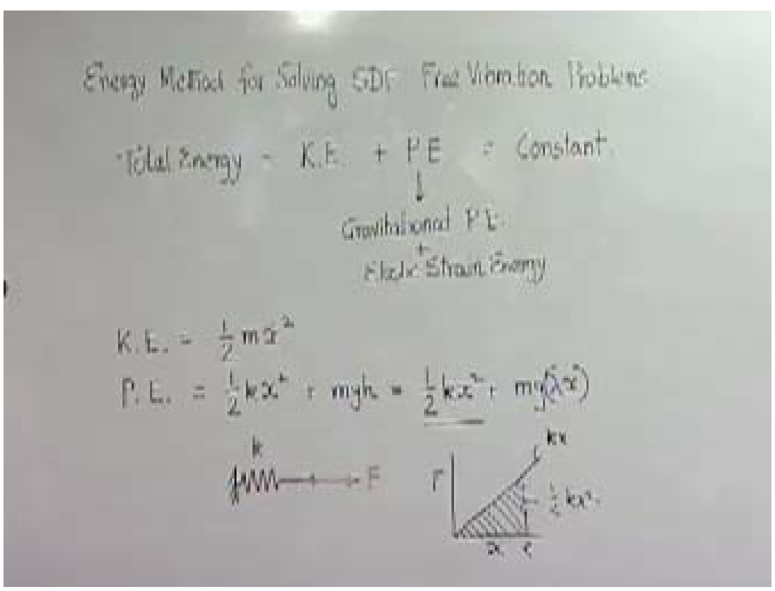

In today's lecture, we will discuss that approach; it is the energy approach, in solving simple free vibration problem. At the root of the whole philosophy or approach or method, you can say the fact that without any dissipation of energy total mechanical energy in the system remains constant. The total energy is the sum of kinetic energy and potential energy, which remains constant. Potential energy of course, will consist of gravitational potential energy plus elastic strain energy. We know that for a simple single degree freedom system where the motion of the body is represented by one coordinate that is x, the kinetic energy of such a body will be written as half m x dot square, where x dot is the E. Similarly, potential energy of a body will be half k x square plus mgh, where h is the height above the equilibrium position, g is the acceleration due to gravity and the potential energy is this (Refer Slide Time: 04:36).

Actually in fact, in many cases this can be written as some function of x that is, height will be dependent on x; it cannot be exactly x, sometimes it will be exactly x, sometimes it will depended on a linear pair. I think this is well known, because we know that if we take string of stiffness k and then it is subjected to a force at any instant, then we know the clock F versus X will be a straight line. It is nothing but kx; if you displace the mass up to x from equilibrium position then work done is nothing but area under this half, which is half kx square, this is known. The string element will contain this much of strain energy that will be potential energy and if it changes its level, the mass, then this will be change in level and this will be gravitational potential energy (Refer Slide Time: 06:13). Therefore, the total energy of the system will be kinetic energy plus potential energy.

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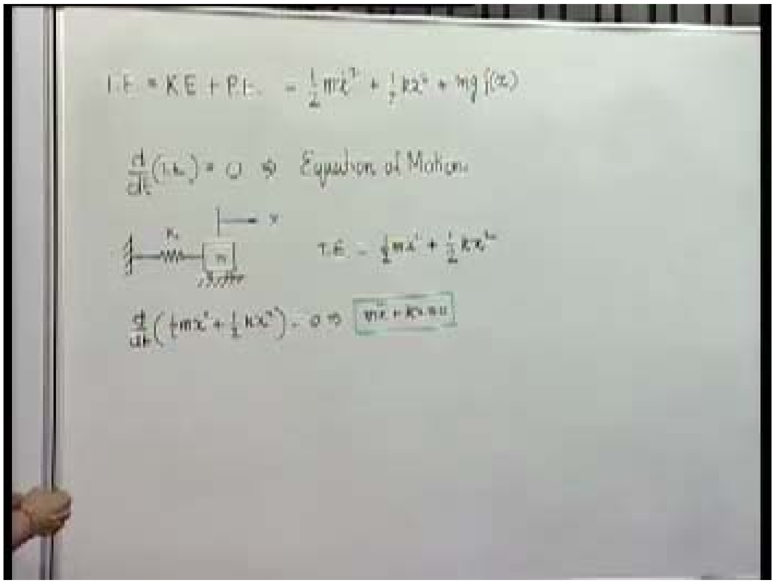

Total energy is equal to kinetic energy plus potential energy, which is again nothing but half mx square plus half kx square plus mg into lamda x. Now, we will see that there are two possibilities. We can do one thing; that d dt of total energy is equal to 0; that is one approach that means, differentiate this and remember that in most general situations, it will be a function of x. By differentiating this we will get an equation in terms of x and its derivation, which is nothing but the equation of motion.

To derive the same equation of motion without taking help of the free body diagram and the Newtonian approach. The simpler possible case - we will take the simple string mass system. This is the equilibrium position and the displacement co-ordinate from the equilibrium position is x. At any position when it is deformed, the total energy will be how much? Half m x dot square, which is the kinetic energy, since it is in the same level there is no change in gravitational potential energy. We can consider the gravitational potential energy to be a constant or 0 only at the strain energy in this string (Refer Slide Time: 09:00).Applying this conservation principle, this is mx two dot plus kx equal to 0**,** the same equation of motion is derived by using Newton's second law and then it has to be solved.

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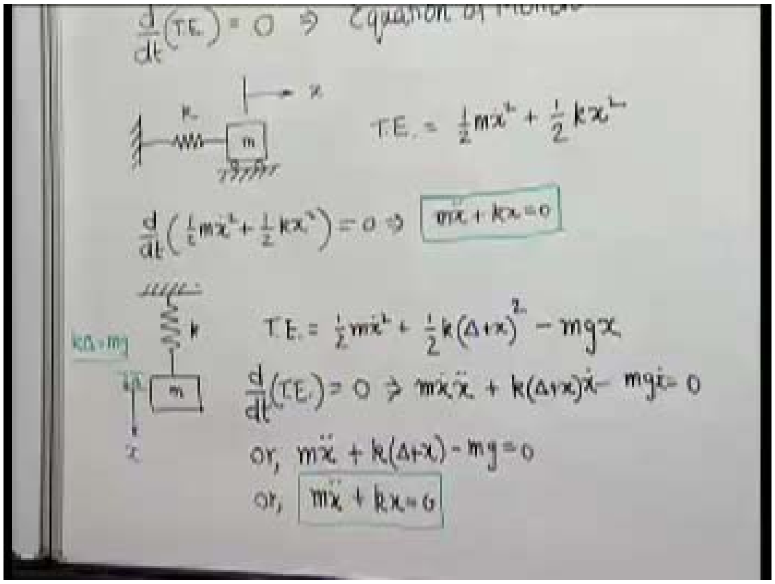

Similarly, we take up another example; where, if this happens to be the natural length of this string, then obviously it will be under the action of space in-between the state of gravity. Let this be equilibrium position, so that it has under gone as state of delta as we have seen before. We measure the displacement co-ordinate from its equilibrium position where the string is no longer in natural length. In this case, what will be the total energy? If it is k, this is m; total energy is kinetic energy, which is the same as half m x dot square and the amount of space in the string will be half k into delta plus x. Delta is the space from its natural length to the position when it was in equilibrium under the action of gravity.

The total stretch of the string, since we are measuring x from it already spaced condition which is equilibrium position, the total state in the string is this (Refer Slide Time: 11:14) and total amount of energy will be this minus; now, it is falling in height. So, its potential energy to be 0 at this location that is equilibrium position, then potential energy will decreased by or it will be minus mg x. When it has gone to a position x from its equilibrium position, so this is gravitational potential energy; this is the strain energy in this string; this is the kinetic energy. Now, again taking the conservation principle and setting total energy to the constant it should be 0 and this will lead to what situation?

Differentiate it with respect to time, m x dot x two dot plus k delta plus k delta plus x minus mg is equal to 0.

Here, you have to write x dot and then x dot you are differentiating with respect to time. This term after differentiation become this; this term after differentiating with respect to time become this (Refer Slide Time: 12:53). Obviously, all these x dot term go out and we get m x two dot plus k delta plus x minus mg equal to 0. Now, we remember the condition that in the equilibrium position when the amount of stretch was delta, then this was the condition when k delta is equal to mg the string force was balancing the gravity. If you use this we get finally m x two dot plus; k delta is equal to mg; so it gets cancelled. So, we get again the same differential equation. You may think that these are several cases; what is the need of doing it in this round about the way using the energy approach. We will not take up this case, where it may not be so easy to solve the equation of motion to find out the natural frequency; because, ultimately our objective is to find out the natural frequency of oscillation when the system is disturbed from its equilibrium position.

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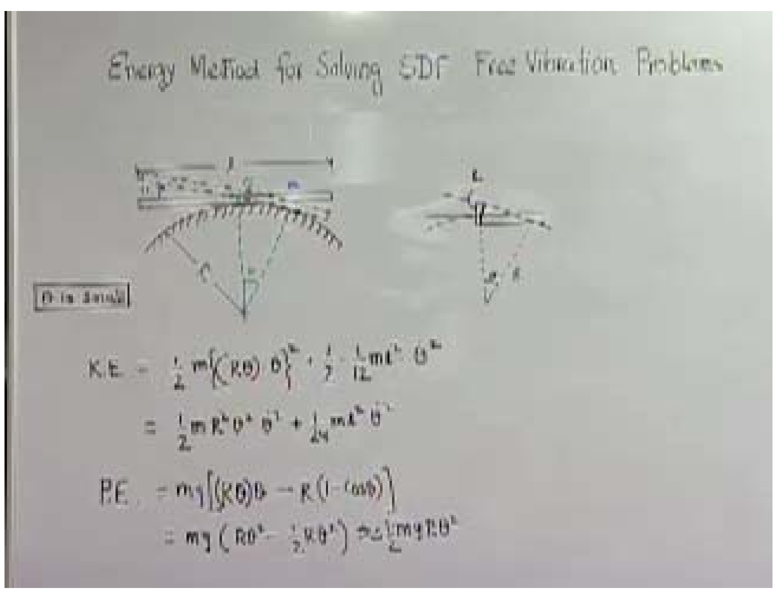

Let us consider this example: a thin rod of length l and mass m is resting on a cylindrical surface. This is the equilibrium position and we also mentioned that the friction between this rod and the cylindrical surface is enough to prevent the sliding. So that it can only roll. Now, if we dip it little bit to bring it to this position (Refer Slide Time: 16:18); this is dip it and left it; so we know that it will oscillate. Now, we have to find out the natural frequency of oscillation of this. Here, it may not be so easy to find out the natural frequency. Remember, here also it will be a single degree freedom system, because without any rolling or without any sweeping in fewer rolling condition, the only thing what needs to be described or to explain its configuration is angle theta from the horizontal position.

Let us see, if we can find the equation of motion using the energy approach. Now, the cg of the rod was here and now it has gone here. So, in the displaced condition the potential energy has been changed. If this is the point of contact, we know this angle is again theta. This is 90 degree, this is 90 degree; so if this is theta and then this must also be theta (Refer Slide Time: 17:53).

Therefore, let us first find out the kinetic energy. Now, for kinetic energy of rigid body we know is the kinetic energy of body because there is a particle and situated at the center of mass; that means, half m into velocity of the center of mass square. Take care of the rotation; it will be half into moment of inertia of the rod and the body passing through it about an axis passing through the center of mass into theta dot. What is theta of this (Refer Slide Time: 18:44)? Theta of this point is 0 because it is in contact with this body without any sweeping and this point velocity is 0. Velocity of this point will be simply this length into theta dot. So, half m when you first take the transitional kinetic energy half m into; how much is this length? This length is nothing but R into theta that is length of R, not approximately, exactly.

Then theta dot is the angular velocity and this point velocity, so this term will be velocity of this point (Refer Slide Time: 19:52), which is 0, plus this square. This is the linear velocity of the center of mass; that is the length of oc into angular velocity that will be linear velocity plus half into moment of inertia of the rod about its center of mass is half ml 1 square half ml square. So, half into moment of inertia about the center of mass into theta square; we can write this as half m R square theta square, theta dot square plus 1 by 24 m l square theta square, because it is the kinetic energy.

Next, you have to find out the potential energy; potential energy has no string element, therefore, there will be no strain energy stored in the system only there will be gravitational potential energy because center of mass of the system is changing height. How much is the changing height? This is approximately this much; in reality it will be R theta and this part. Now, we have to keep in mind that theta is more; without this, problem becomes very complicated non-linear. As a matter of fact all non small oscillation, large oscillations are non-linear and this we have to keep in mind. This arc link and cosine of theta is equal to approximately one, is approximately the change in height.

How much is this? This much is equal to this length into theta (Refer Slide Time: 22:09), minus we have to approximate this much that is the l. This approximately is equal to R theta is the length into theta is this total arc length minus this much is what R minus R cosine theta minus R 1 minus cosine theta; now which we already know, this is the total height change multiplied by mg. We know already that this is equal to m into g R theta square minus 1 minus cosine theta. If you explain, it is 1 minus half theta square. It will be 1 minus it will be minus half r, where this is the approximately mg half r. This is an approximation considering theta to be small.

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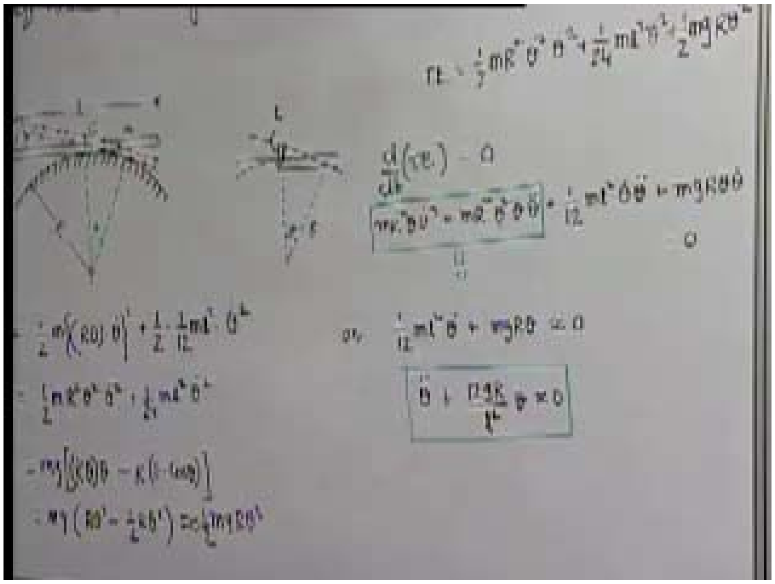

Total energy is half m R square into theta square into theta dot square plus 1 by 24. This is the total energy and we apply now the conservation principle it says the total energy is constant; that is to differentiate it with respect to time and it is equal to 0. This leads to what; now differentiate this, it will be m R square theta into theta dot mu plus m R square into theta square into theta dot into theta two dot. That is the first term plus 1 by l2 ml square into theta dot theta two dot, plus mg half theta equal to 0.

Now, we have to keep in mind the theta is small theta, dot also will be small. So compare these two terms as you can see; this theta square, theta square term; theta dot cube term they can be consider being very small and then neglect it. What we have left is 1 by 12 ml square theta two dot plus mg half into theta is approximately equal to 0. Removing this, we get the final form (Refer Slide Time: 26:27). This becomes our equation of motion or small oscillation. We can see that energy approach can be very convenient in solving some problems, which may not easily available to the Newtonian approach which can be done, but it will be more complete.

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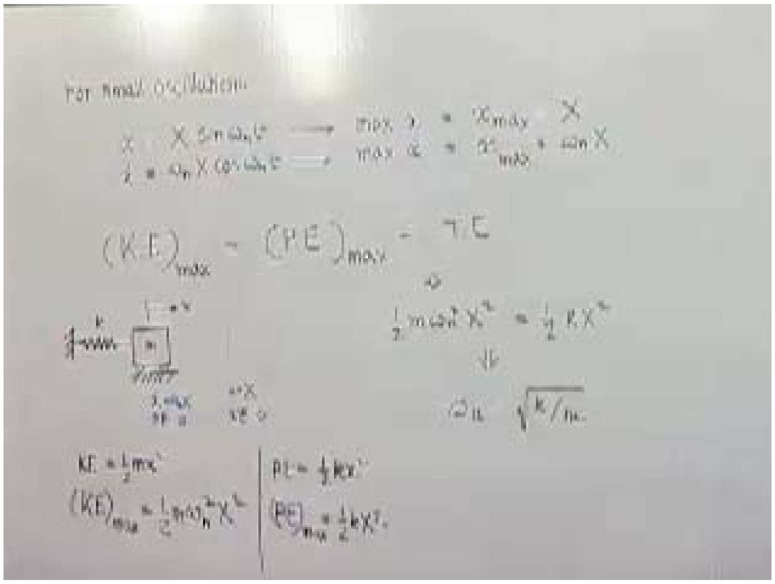

There is another approach; the energy method we will see is that what we have used here is the principle of conservation of energy. Now, principle of conservation of energy when differentiate, when you use this way, we get the differential equation. But, there is another way where you can look a problem like this. When a system is oscillating it is undergoing changes continuously. Now, when it is passing through equilibrium position, then if you consider the equilibrium position of the system to have the minimum potential energy is 0. When it is goes to end, then obviously it stops; that means its kinetic energy is 0. For a small oscillation always remember that - any oscillation can be written as - any oscillation that is angular, linear; if x is the parameter displacement co-ordinate, then x has a function of time that can be always written like this, that we have seen. Velocity of speed will tell us maximum x that is x_{max} is equal to capital X of the amplitude of oscillation.

Similarly, maximum speed or $xdot_{max}$ is $omega_{nX.}$ Now, what we have mentioned just little while ago is that, this oscillating system where the total energy is constant it is passing through equilibrium position. When its potential energy is zero and it is reaching the extreme end, when its kinetic energy is zero. In equilibrium position the whole energy of system is in kinetic form because the potential energy is zero. Similarly, when it goes to the end, then its total energy is in potential form because the kinetic energy is zero and since the total energy remains constant.

We can write that kinetic energy in the system is maximum value has to be equal to potential energy of the system $_{max}$ and both are equal to total energy. To know what we can do, we can take this approach that we know for all small oscillation, oscillation is going to be simple harmonic; only thing what is unknown we have to find it out is natural frequency omega.

What will we do now? We will find out the expression of kinetic energy in the system and find out its maximum value and find out the expression for the potential energy of system and find out its maximum and we will equate them. Since, the kinetic energy term contains omega$_n$ we get an equation omega and that way you can find out the value of omega. We take our simpler problem for example, will again take this one. At this location x dot is equal to omega$_m$ x and when its goes to extreme end here x is equal to X, when KE is 0 and PE is 0. What is the expression for kinetic energy? half m x dot square. So KE max is going to be half m x dots maximum value is omega$_n$X square and potential energy is here gravitational potential energy remains same; we can put it equal to 0. Strain energy is the only potential energy is half k x square.

Now, if we equate the kinetic energy max is equal to potential energy max, this will be given as in this case half m omega$_n$ square capital X square is equal to potential energy maximum to half A x square, x square cancel and this gives us directly the expression. This is another angle of the whole things if you can use this concept that we find out the maximum kinetic energy and maximum potential energy keeping potential energy at equilibrium position is zero and equate. That gets an identical equation not a differential equation. Whether, on the other hand to get the equation of motion the differential equation, you have to solve. This of course, you may think very crucial problem let us take another problem. If you take a slide t you have problem which may not be the very convenient for Newtonian formulation.

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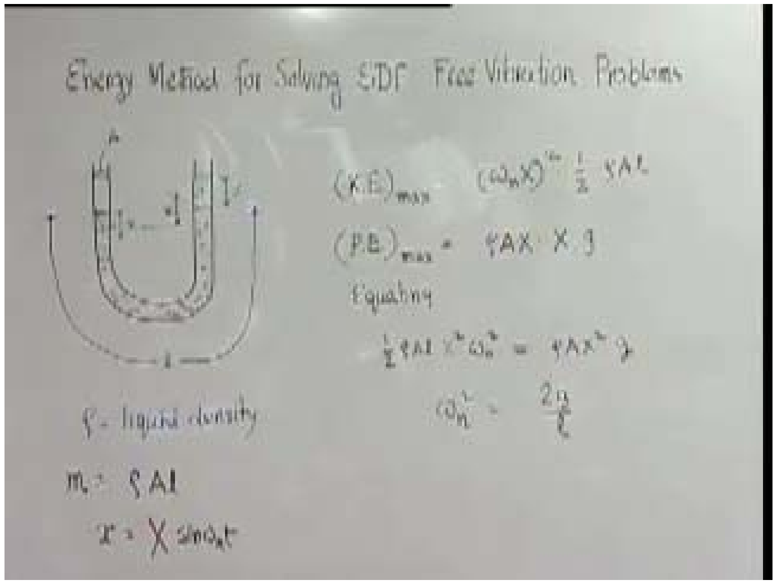

Let us consider the problem of u tube, which is filled with the liquid of density rho. The constructional area of this is tube is constant and it is equal to A. The length of this liquid column that is total length is L; thus, we get the mass of the liquid column which is nothing but rho into the volume of the liquid column Al; this is the total mass. Now when we displace, this will be equilibrium position. Let us consider that displaced and this end we have displacement by x obviously, other end we will go up by the same amount. This is one extreme position where you release and then you leave it, so oscillate. What happens is this water column gets oscillated as you move from our experience.

We will apply now, again the principle which has a concept and find out the expression for the maximum kinetic energy and find out the extreme for the maximum potential energy. All along we have to keep in mind that x is nothing but as shown (Refer Slide Time: 37:09) something either sine or cosine. What is the maximum kinetic energy of this column? It is when it is passing to the equilibrium position velocity is maximum and we know velocity maximum of this stage. This is the square of the speed at equilibrium position when it is maximum multiplied by half and mass A into L this is the maximum kinetic energy.

This has to be equated to the maximum potential energy that means when it has reached one of the extreme end. Suppose, when this x is equal to capital X that is the extreme end when it stops. How much is the increase in potential energy? Now, if you can fix now this amount of liquid has been taken here, so we get this consider and how much is the change in height? This is half x; this is again half x; so the total change in height is half x. What is the maximum change in height? That means this block has placed here, when small x is capital X, maximum value then that will be maximum potential.

How much is increased in potential energy? First of all the mass that is rho into A into X; when x is equal to x that is the volume and rho will be done this is mass of this block that we are shifting by how much? We have been shifting by amount X into g. This is the maximum potential equation. Equating the two x half rho A L X square omega$_n$ square is equal to rho A X square y. Here, we get an algebraic equation in omega$_n$ which omega$_n$ square is equal to x square goes A goes rho goes(Refer Slide Time: 39:53)so what remains 2g by l. This is again so nicely done using the principle of energy, where at we equate the maximum kinetic energy to maximum potential energy.

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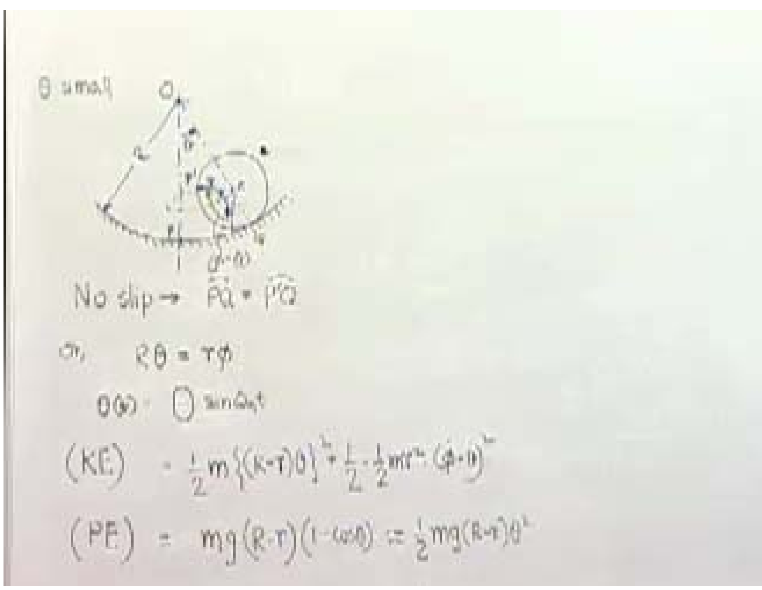

Another example: we will take up this problem is like this; there is a cylindrical gravity and inside that roller of radius small r and mass m is rolling and you know that it will oscillate like this. We have import the condition that there is no sweep between the inner surface of the cylindrical gravity radius r and the outer surface of the cylinder. It is the case of rolling, in this point when it was starting here and it has rolled then it has supposed gone there; that means, when the cylinder was at this point the point of contact p on the cylinder surface has gone now to this.

The condition of pro-rolling censor, that this p and the point is q, the prime q, this r must be equal to pq their total length must be because there is no slip between this. No slip condition tells us that arc PQ will be equal to arc P prime Q or how much is this? This is capital R into theta, theta is the center of the line drawing from the center of the six cylinders O to the center of motion cylinder t, that theta tells us this position or we get PQ is equal to R theta is equal to the prime key. The prime key is equal to it is small r it to be consider this angle to be small phi it will be this is the condition that we get from this pneumatic constrain.

Now, we consider theta to be a function of time and that can be written as capital theta into sin omega. When it is free also our objective is to find out $omega_{n.}$ We will apply the same principle of equating potential energy and kinetic energy that is maximum. What is the maximum kinetic energy? Now, maximum kinetic energy will be when it is passing through the equilibrium position and it will have a transitional energy.

It is given by the mass of the rolling cylinder into half of l square, it will be a half m - and what is the speed of the center it will be - R minus r into theta dot if this is the speed square, this is the transitional part of the kinetic energy of cylinder. Rotational part will be half and its moment of inertia is half mr square into theta dot angle of velocity theta dot. We have to find out the angular velocity of the rolling, therefore, you have to find out, now I explain this little bit. What is angular speed of this? When it goes here, what is the amount of rotation? We will find the amount of rotation is given that originally this radius for example, t time it was vertical here, t was here and this was the position of this radius.

Now, it is here therefore, the rotation is given by this angle, which is phi; that is the total angle minus this is nothing but this. So, amount of rotation of smaller cylinder is not phi but i minus say its angular velocity of the rolling cylinder if phi minus theta of time derivate that is phi dot minus r. That is the kinetic energy maximum for the value is not a maximum value it is kinetic energy. Potential energy similarly, you can write potential energy will be only gravitational potential as you can see and that is m into t. How much mt; it was here, it was here, now to here, it is R minus small r, minus R minus small r minus cosine and it will be and when theta is small, because small oscillation then this can be expanded can be written as half mg R minus small r theta square, 1 minus cosine theta is half square neglecting the higher order. Now you have one quantity phi you have to remove; from this equation, we find phi is nothing but R by r into theta. So, phi dot minus theta dot is nothing but R by r minus one into theta dot.

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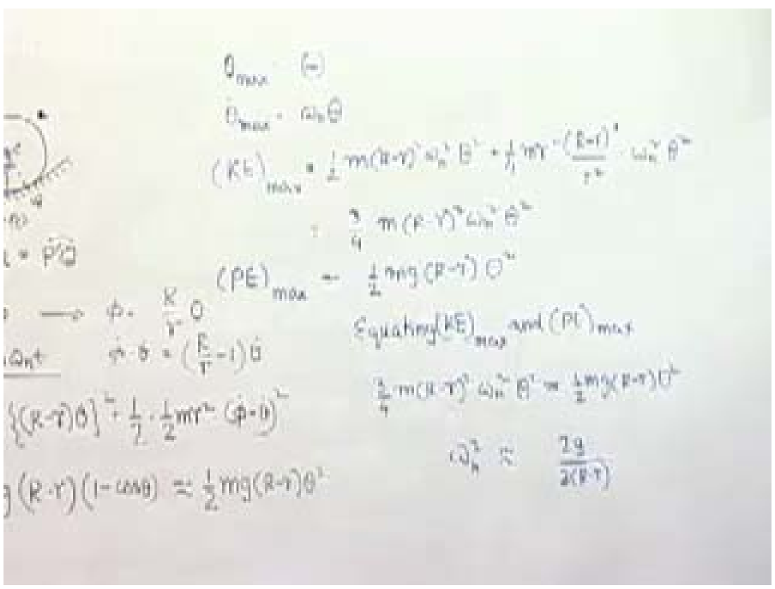

Thus, we also know, hence theta is like this it will give as theta maximum. So KE maximum will be equal to half m R minus r square; now, theta dot square maximum will be omega$_n$ square, this term is given as 1 by 4th m r square theta phi dot minus theta dot is this, and this is the maximum speed and which one verification will be 3 by 4th m. Similarly, PE minus will be equal to half mg and equating what we get, 3 by fourth m and

$omega_n$ square is approximately that the answer, so here is find; that the problem if we try to solve an Newtonian approach is very difficult you have taken the energy approach of course conservation principle have been used.

You could also add the two and differentiate with respect to time and get the differential equation in theta; solving, which we could get the strain natural frequency. On other hand, you assume that all small oscillations are simple harmonic function of time and by then equating the maximum kinetic energy with maximum potential energy we get an identical equation for the natural frequency. So, I think one should practice both approaches. Sometimes Newtonian approach is more convenient, but in many cases we will find that the energy approach also gives the result. Therefore, also keep in mind that all approximate solutions considering the small oscillation and approximation that's why here always we keep this approximate time.