**Dynamics of Machines**

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**Module No. # 08**

**Lecture No. # 02**

We have seen in our previous lecture that study of oscillatory motion or what we call commonly a vibration of mechanical system is very important; primarily because, the characteristics or the behavior or the response of the system to dynamic loading can be very different from that when the loading is static. A successful design will require, for a study of the system, from oscillation point of view or vibration of the system, what is being designed. We also have seen that vibrations can be of various types depending on various parameters. So, what we propose to do now? We suppose to start a detail analysis of vibrations of mechanical or similar systems and to begin with, we take up the simplest of all; that is the free oscillation of simple single degree freedom system.

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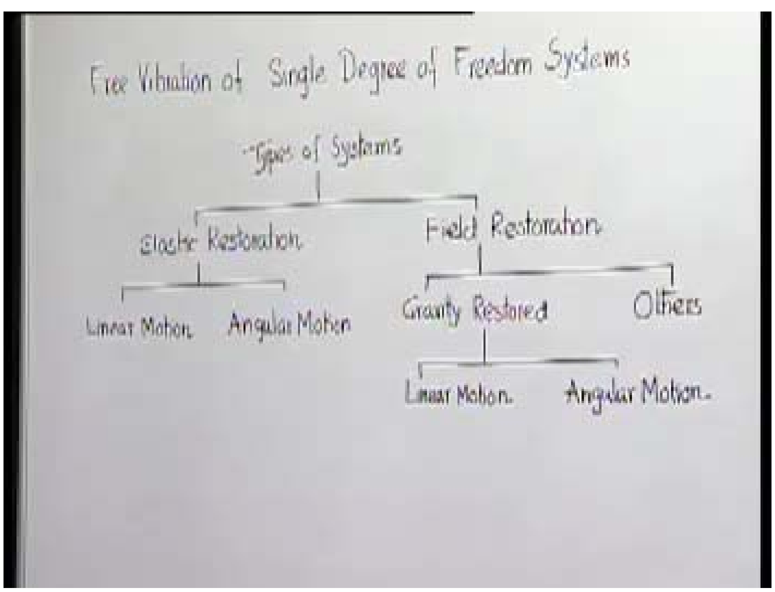

Before we start the detail modeling and analysis, let us see that, there can be different types of system. Based on the nature of restoration - that is what brings it back to its equilibrium or another stable equilibrium position - it can be divided into two groups, one type of system where the restoration is through elastic members or spring. The other can be field restoration which means, there is no elastic member or there is no strain energy stored in the system, which brings it back to the original equilibrium position, but it is a field which acts on mass or the inertia that is responsible for restoration. The most common type of field, what we encounter every day is the gravity, other is others: electromagnetic field, electrostatic field, magnetic field; which we did not go into the details. In both cases, we will find that the oscillation can be again classified into two groups based on the geometric nature of the motion; that is linear motion which means, that the mass moves in a straight line or angular motion, same may be the case here.

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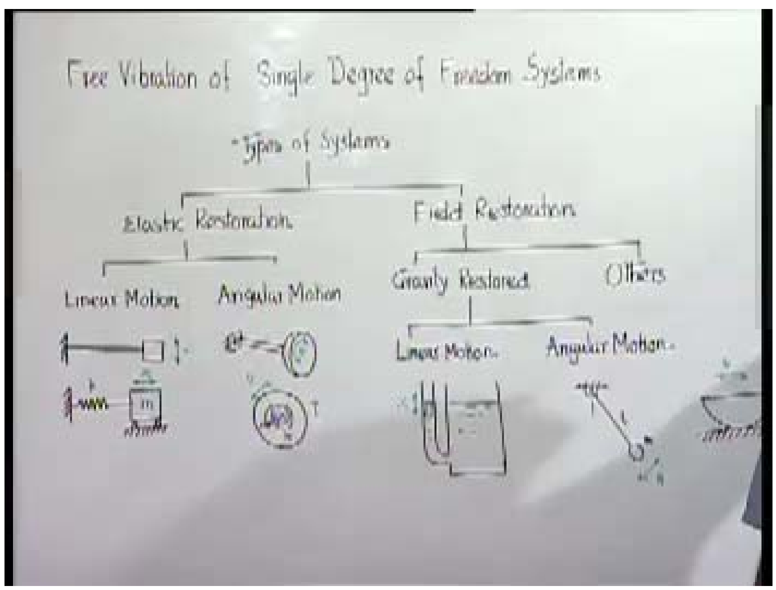


What we will do? We will see the popularly encountered system of these types. Say, one example could be cantilever thin beam carrying a mass at its end or the parametrically lump parametric module of a system. In this case, mass is moving like this; in this case, in both cases the motion is rectilinear (Refer Slide Time: 05:31). In case of angular motion is something like that, a disc mounted at the end of a shaft, other end of the shaft being rigidly mounted or simple model, we can have is torsional spring and an inertia here of course, we know that the oscillation will be angular, same thing here oscillation will be angular.

Here if it is represented by x; we will represent by theta. In this case, the inertial property is represented by mass; here the inertial property is represented by moment of inertia. The stiffness in this case is represented by k, which gives force per unit deformation. Here the stiffness will be represented by capital T which will be torque or unit angle of rotation.

Again, you will find, if you take for example, a tube, here is some kind of fluid. If we depress the fluid level little bit, we will find that it will oscillate after we leave it. This is example of the linear motion which is gravity restored (Refer Slide Time: 07:05). The angular motion which is gravity restored is very commonly known and almost everyone is familiar with that; here the motion is angular. There are other types of gravity restore system which can be considered to be… Suppose, we take half cylinder lying on a rough horizontal plane, if we disturb from the equilibrium position, we know that it will rock. These are the elasticity restored system; these are the gravity restored system. There are many other examples, we did not consider now.

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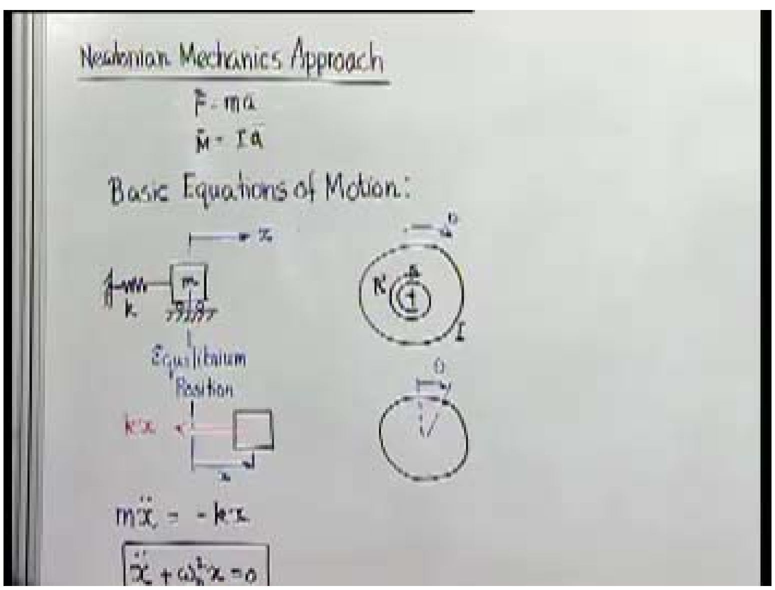


Solving these problems, let us first take up one approach which is familiar to all of us, that is Newtonian approach; that is Newtonian mechanics, which means that we will use this relationship: force is equal to mass into acceleration or torque is equal to moment of inertia into angular acceleration, both are the commonly known forms of Newton's Laws and we use this laws and try to solve these problems.

What we will do, we will try to find out first the basic equation of motion, say this one. Now, one thing we have to keep in mind that we have to consider the system to be in stable equilibrium. This is the simplest possible system we have taken now. As mention that, this is the equilibrium position, what does it mean? It means, if it is left alone it will remain in its rest position. If we disturbed little bit, it will try to come back to this position because this spring is there. Therefore obviously, under this situation total force action on this mass will be 0; that is in this particular case, that is spring force will be 0 in the horizontal direction, which means that this spring is in the natural length.

Now, we displace it little bit, first thing is that how do we measure the displacement? Now, almost as a rule unless otherwise is a very special reason to do so. We should always measure the displacement or change of position from the equilibrium position of the system. Therefore, if this be the equilibrium position then the displacement will be this much; there will be only. Now, in vertical direction the reaction by the ground and the weight is continuously balancing each other because, we are now interested only in the horizontal motion and here the only horizontal force which is acting on the system is the spring force. The spring force is given by the stiffness into its elongation from its natural length, which is nothing but x.

The acceleration of this is identical in this direction and the Newton's law says that mass into its acceleration must be equal to force in this direction. How much is the force in that direction? It is minus k x, minus because the force is acting in the negative direction. So, the basic equation, what you get is x two dot plus omega$_n$ square x equal to 0, where in this case omega$_n$ squared is k. If you take up the other systems - this one; here the moment of the inertia is I and torsional stiffness of this spring is capital K and this is the equilibrium position.

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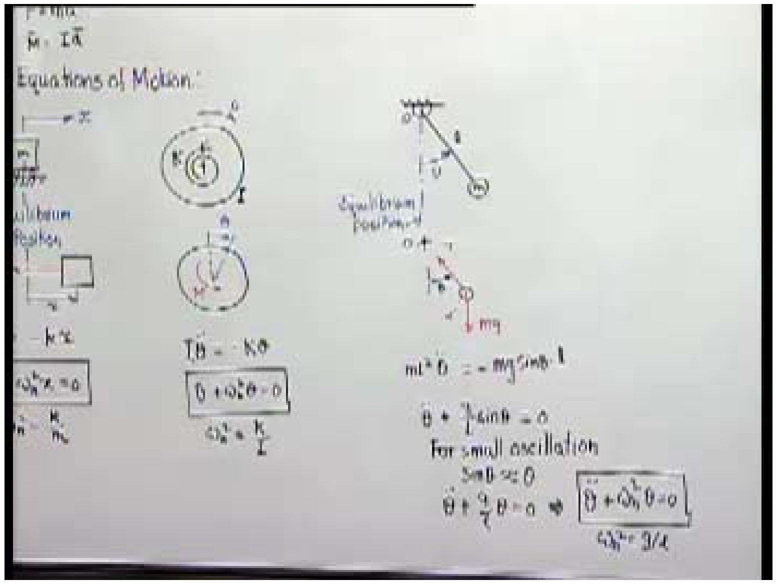


When it rotates, suppose we measure the displacement in this direction from here and this is the equilibrium position. So, when the angular displacement is theta, as it torsional spring is now to form it will apply the moment in this direction is M. The equation of motion that means, I moment of inertia into angular acceleration; that is, theta two dot must be equal to total moment acting in that direction, which is nothing but minus k theta. Here again, we find the basic equation theta two dot plus $\text{omega}_n$ square theta equal to 0 where $\text{omega}_n$ square is equal to capital K by I. So, the form of the equation both cases is exactly the same.

Let us consider the angular case. We do only one - that is the pendulum others, we will take up later. Here, the equilibrium position obviously, this is the equilibrium position and we have to displace it by an angle theta in this direction. When you take this mass the force acting on this will be gravity and the tension in this direction that is the hinge.

Now, what will be the equation of motion for this? We can do it in two ways; we find out the total moment of the external forces about this point O and then we can apply this moment of inertia of the system about this into the theta two dot must be equal to the total moment acting in that direction. So, what is the moment of inertia about this point? It will be m l square and the angular acceleration in the algebraically positive direction is this;

this must be equal to the total torque acting in this direction (Refer Slide Time: 16:50). Total torque about this point is actually clockwise, that means in the opposite direction and it is equal to minus mg sine theta into l; mg sin theta is nothing but the component of this in the normal direction and this angle being theta; this is mg sine theta and this into the arm length l is the total moment acting in this direction and putting a negative because it is minus.

Here again, we get the equation in the basic form. We find something interesting here that the forms are somewhat different and moreover this is a non-linear equation. However, for a small oscillation, which is always the case unless otherwise mention, we can write sin theta is approximately equal to theta and theta two dot plus g by l theta is equal to 0 or theta two dot plus omega$_n$ square is equal to 0 (Refer Slide Time: 19:10). We find in gravity base system also the basic equation is of the same form.

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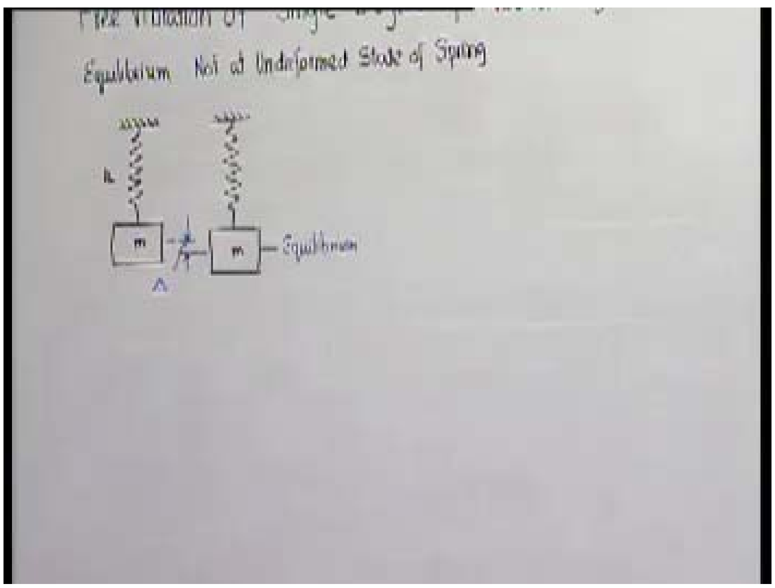


Let us take up another case where the equilibrium is not in the undeformed condition of this spring. What happens in such cases? We are discussing this particular situation where the system is at its equilibrium position, but corresponding to that position the elastic members are already deformed. Now, such system comes quite often and in many cases there is confusion that whether the gravity force should be taken into account or not; that is why I am separately discussing to remove any confusion which may be otherwise may be created. Let us take a simple spring mass system but now, it is suspended rather than being put on a horizontal table; it is now suspended. What will happen next, if we suspend this; the spring will stress little bit and the mass will come here, because due to gravity there will be force acting on this spring and this spring will be stressed so the equilibrium position - this is the position which corresponds to un-deformed spring length - but this is the equilibrium position when it gets deformed by an amount delta (Refer Slide Time: 21:35).

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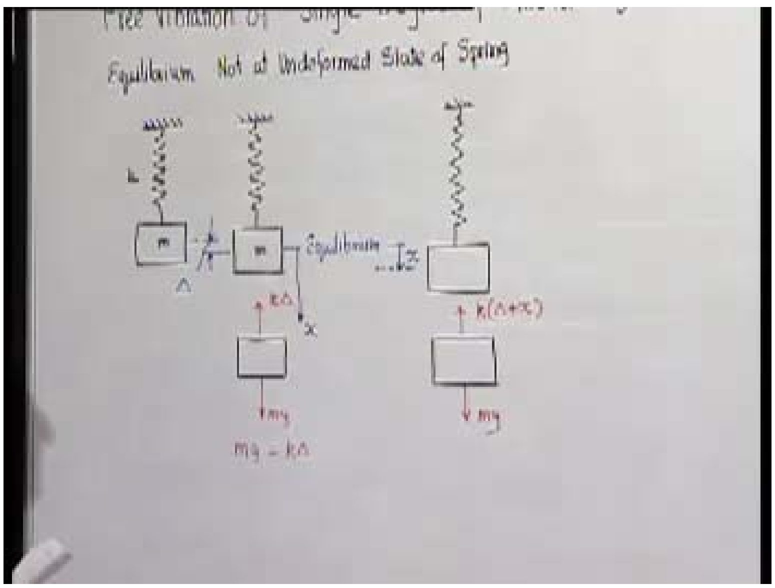


Now, what is the equilibrium situation? What does it signify? The total vertical force on this mass, one is the weight mg and other the spring force which is k into deformation of the spring delta. Since it is equilibrium, mg must be equal to k delta. We measure the displacement always from the equilibrium position. Now finally, we give some further deformation or displacement to the mass from its equilibrium.

If this is equilibrium position as mention before, we should measure the displacement from this position. This is the instantaneous position and therefore this must be equal to this, which is measured from the equilibrium position (Refer Slide Time: 22:45).Now, let us see, what is total force acting on this; one is the gravity and other is the spring force which is k into delta plus x.

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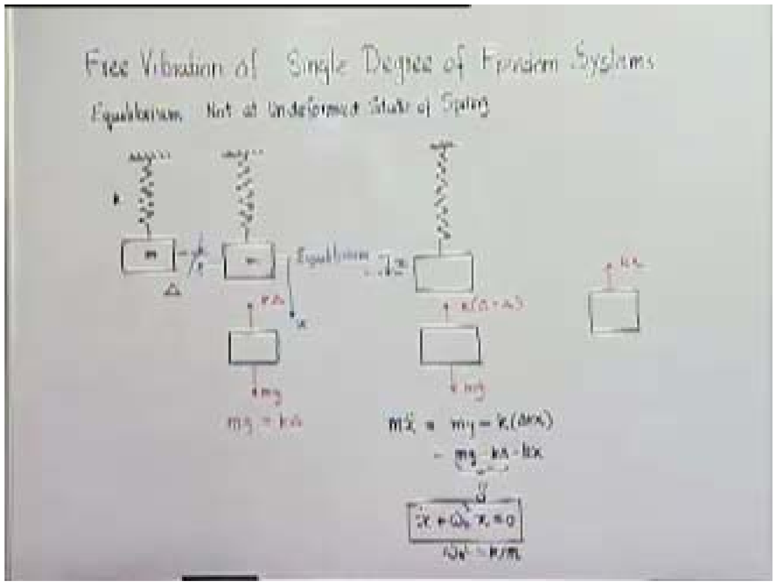


Now the total deformation from its un-deformed condition or position, that means delta plus x. The equilibrium in this direction will be m x two dot, is the total force acceleration of the body into the mass in this direction and that must be equal to total force in this positive x direction; that is mg minus this or when you expand(Refer Slide Time: 24:13). Now, as you can see this is equal to 0. Finally, we again get back the same form of equation. So this is a very important point we should always keep in mind that even if the system is under equilibrium under the action of the gravitational field and we measure the displacement of the configuration or the mass from it equilibrium position there is no need to account for the gravity at later stage.

The gravitational pull which was already balanced by the spring force while it was in its equilibrium position is always there and gets nullified by each other. When we write down the equation of motion, there is no need to put this, what you could do? You could just simply ignore that existence of gravity. We will consider the case that there is only the displacement of the spring from its equilibrium position of the mass, which is kx and we will not consider that the total displacement stretch its delta plus x, that is will be k into delta plus x. We can forget about gravity everything and that way we should get the correct solution. This is discussed in great detail because; this is a source of the problem and difficult for many students. Now, let us come to the question of solving because if we find that all the single degree freedom system in some form or the other we get the same equation of motion. We will now solve this equation of; so the equation is same, so therefore you have to find out the solution.

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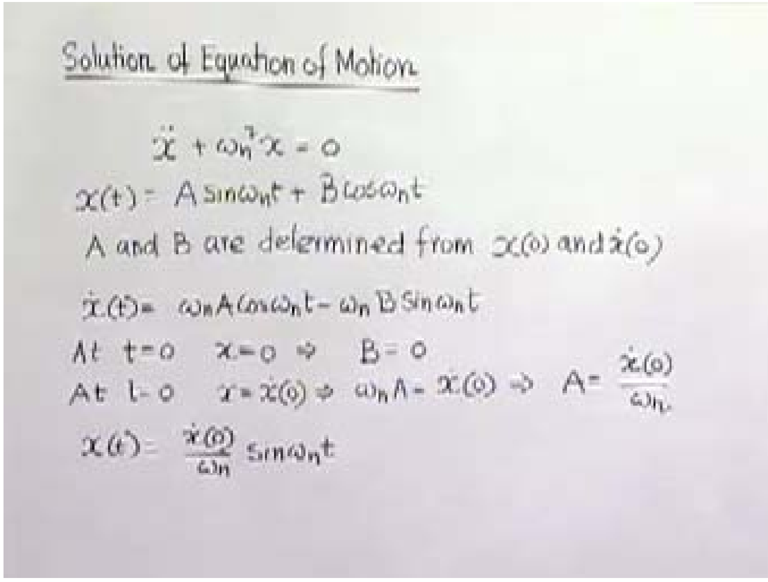


The equation which we will solve is of this form x two dot plus omega$_n$ square x equal to 0. You have all solved this equation, which is a very standard one and we know that the solution is a harmonic function of time. We all know that A and B are constants and this constant are determined from the initial condition. The values of the position and velocity at t is equal to 0; they have to be specified otherwise the problem is not completely stated and once there are given, we can find out the values of A and B. How do you do it? We find out x dot t is equal to (Refer Slide Time: 28:32). So, at t is equal to 0, x equal to 0. If we put t is equal to 0 here, this goes out; only this B remains; so that tells us that B is equal to 0.

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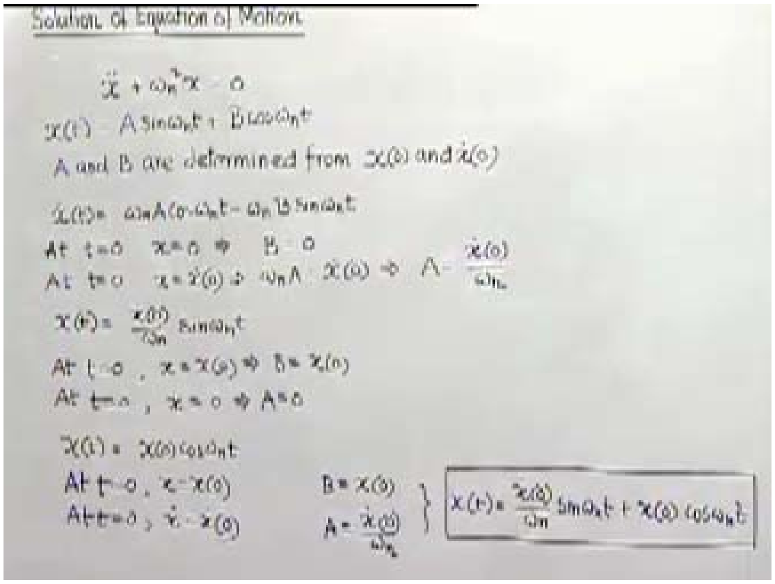

Similarly, at t is equal to 0, x is equal to x dot of 0; we cannot say both are 0, then what happens? We get the substitute t is equal to 0, so omega$_n$ A is equal to x 0 dot; so we get omega$_n$ A is equal to x of 0. That gives as A equal to x dot of 0 by omega$_n$. The form of the solution in this case will be (Refer Slide Time: 30:06). If we do the other way that will give initially some displacement, but no velocity; then what happens? At t is equal to 0, x dot is equal to 0. Now, if x dot is equal to 0; if you put t is equal to 0, what remains omega$_n$ A; now omega$_n$ A is equal to 0, that means A is equal to 0; if A is equal to 0 then if you put t is equal to 0, this goes and only B remains; B is equal to x of 0 so the solution in that case will be (Refer Slide Time: 30:18).

So, depending on the initial condition, we can have the solution the general case will be (Refer Slide Time: 31:44). Solution to this is quite straight forward, you can solve it that means, we will get at t is equal to x; x is equal to two x 0, so B is equal to x 0; this is 0 only B remains that is x 0. Similarly, this will become this; this is 0; this is omega$_n$ A; A is equal to x of 0 divided by omega$_n$ and the solution is finally (Refer Slide Time: 32:50). This will be the most general solution (Refer Slide Time: 32:26). However, we can express the solution with two constants as A and B to be determined for initial condition in another way.

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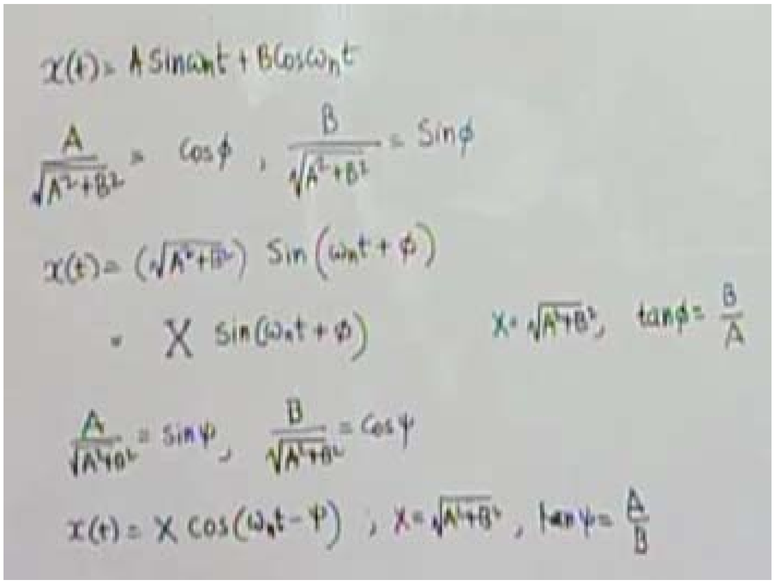


Say, we take the solution and let us substitute A by squared root of A square plus B square is equal to cosine pi (Refer Slide Time: 33:57). What you can get x t is equal to. Now, you can first multiply by square root of A square plus B square and divide this by A square plus B square; divide this by A square root of A square plus B square which are nothing but cosine pi, sine pi.

Therefore, it becomes sine of omega$_n$t plus pi. We can write this quantity as one quantity capital X sine of omega$_n$ t plus phi. Here, you know that x is nothing but; tan phi is equal to B by A (Refer Slide Time: 34:59). We could do it other way also all kinds of things are possible, we could write A by square root of A square plus B square is equal to sine psi, for example. Then x of t will be equal to X cosine of omega t plus psi where x is the same and tan psi equal to A by B. Getting sine, cosine or such combinations there are basically saying as we can guess than all simple harmonic functions of time.

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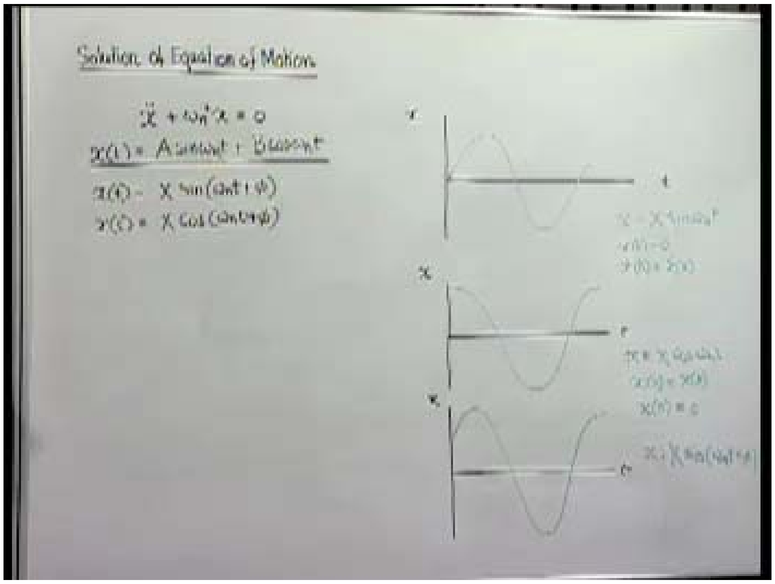


Generally, we can write this solution in any form and start from there. Therefore, we can write the solution in this form x of t is equal to X sine of omega$_n$ t plus phi or X cos of omega$_n$t plus phi or any of this forms can be used, here actually in the same thing. I think once we get this. Therefore, we find our solution in graphical form; will look like this, we plot x versus t (Refer Slide Time: 36:53). If it is a sine function, simply then obviously that means when we get it already we have seen is that this is the situation when x 0 is equal to 0, x dot 0 is equal to x dot 0. This is that means, since we start it without any initial displacement, but higher by a velocity.

Similarly, it is possible to have a solution like this, where x will be equal to Xi cosine omega$_n$ t (Refer Slide Time: 37:58). This is the case, we get when we initially start with some displacement but no velocity and then you get cosine function. If both are present that means, initially we have given some displacement and velocity.

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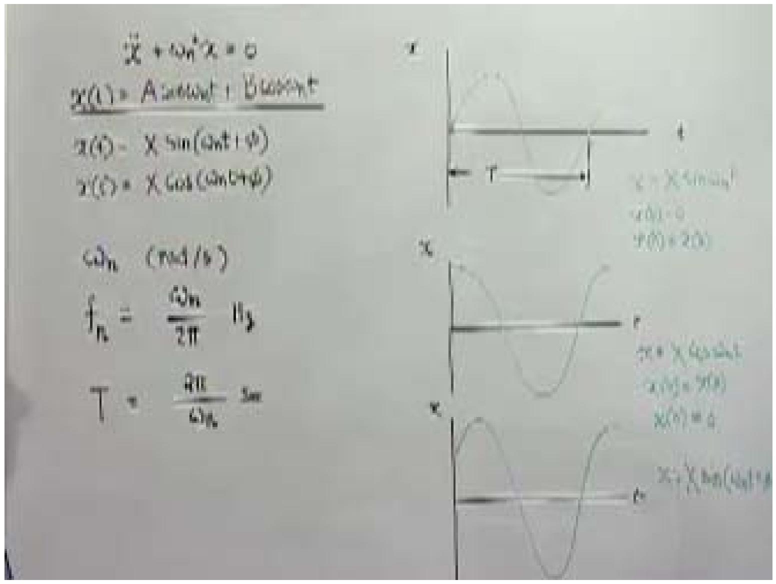

Therefore, what we find that say a harmonic - the solution to this is simple harmonic function of time either sine cosine or a combinations of that - where the circular frequency in radian per second; this is the natural circular frequency; the actual frequency is in cycles per unit time. Now to complete one cycle, the omega or the angle of the angular displacement of the vector representing this will be 2pi; that means, from here to here you will find that it requires 2pi radians for each cycle. If there are so many radians per second, the number of cycle per second which we call as Hz, is omega$_n$ by 2pi.

Now, if you want to find out the time period then, what is the duration of the one cycle? Time period will be obviously, that if 2pi is the angle to cover one cycle and omega$_n$ be the speed at which it is covering angle. So obviously, this will be the time, these are the quantities. Therefore, we have found very important thing that all single degree freedom systems without any damping, if disturbed from the equilibrium position executes oscillatory motion; where the displacement or small oscillation can be represented by simple harmonic function of time and it vibrates through a particular frequency of the time period, which is the characteristics of that particular system and its value depends on the parameters. We will solve problems to find out that how this omega$_n$ or $f_n$ or T are found out that will solve some problems.

The importance of this study - one thing to be kept in mind is that you have noticed that not the determination of x. It is not a system property; it will depend on how much displacement you initially gave and how much energy you have put into it of the system to initiate vibration.

On the other hand, this frequency which is oscillating that is a property of (()). Our general objective of studying free vibration problem of system is to find out this natural frequency of system (Refer Slide Time: 42:12). We will see in the subsequent lectures, why it is so important for us to know in natural frequency of system which perhaps, we are designing that we will see later, but it will be enough for present to know determination of natural frequency of the system is a very important task for the designs.

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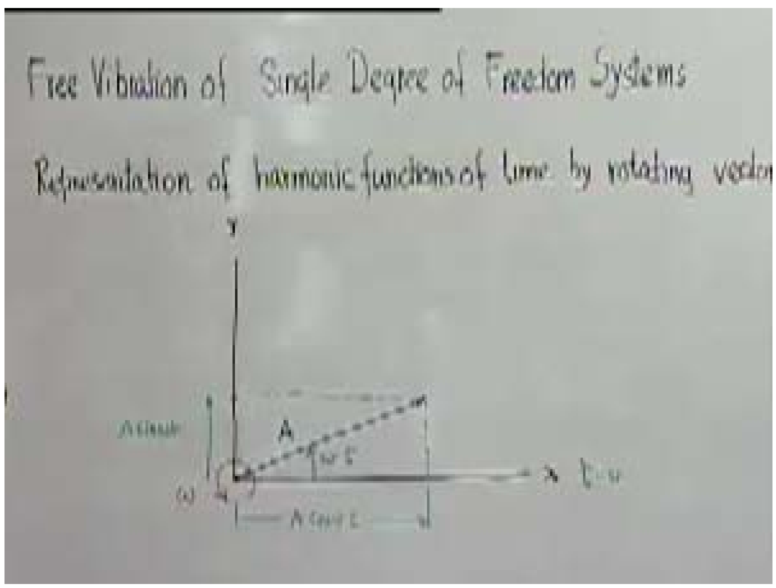


Now, I think before taking up some other thing at this stage will be desirable, so we have some discussion on the representation of harmonic quantities or harmonic functions of time by rotating vectors. We will make use of this in subsequent lectures that is why we are discussing, so that at later stage we don't go as our mainstream of discussions and take up this, which is not exactly related to vibration. Here, we want to represent harmonic function of time as graphical, very convenient way of doing this, if you take a vector A and it is rotating with a constant angular speed omega and we have started measuring time at this; this line represent t is equal to 0 (Refer Slide Time: 44:47).

If that be the case, how much is this angle? omega$_n$ into t, because when it was here, time was 0 the amount of angle it has covered will be nothing, but the angular speed and the time. Now, let us find out projection; if you take this projection; how much is this; this is obviously A cosine omega t and sorry omega$_n$ and y component at this instant is A sin omega$_t$.

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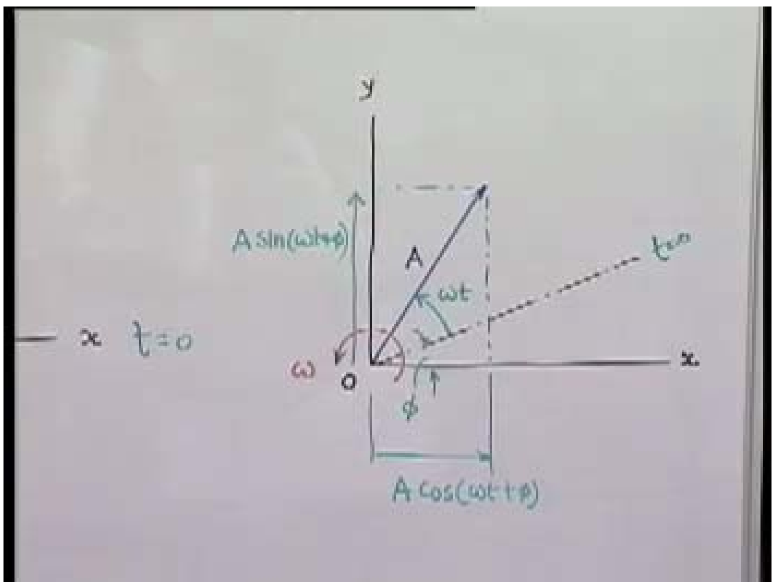


Now, if it is rotating we will find this component; y component will vary as the sine function of time and x component will vary as a cosine function of time. The same thing you can do in more general way. Let us consider this free vector of magnitude A is rotating at a constant angular speed of omega and let us also pickup this position which corresponds to time t is equal to 0. That means, when vector A was here then only we started our clock. Therefore, the angle it has travelled is omega$_t$ and let this angle be phi. Now the x component and y component; it is x component, which is A cosine omega t plus phi and this component is A sine of omega t plus phi.

The general representation of a harmonic function of time can be done by taking the projections of a rotating vector; here the angular speed of this rotating vector whose projections are varying as harmonic functions of time, that corresponds to the circular frequency in radian per second. Each component and the phase difference phi we can always have by suitably taking our location of the vector, with respect to the x and y direction along which we are taking the projections that angle whatever select that gives as these phase phi.

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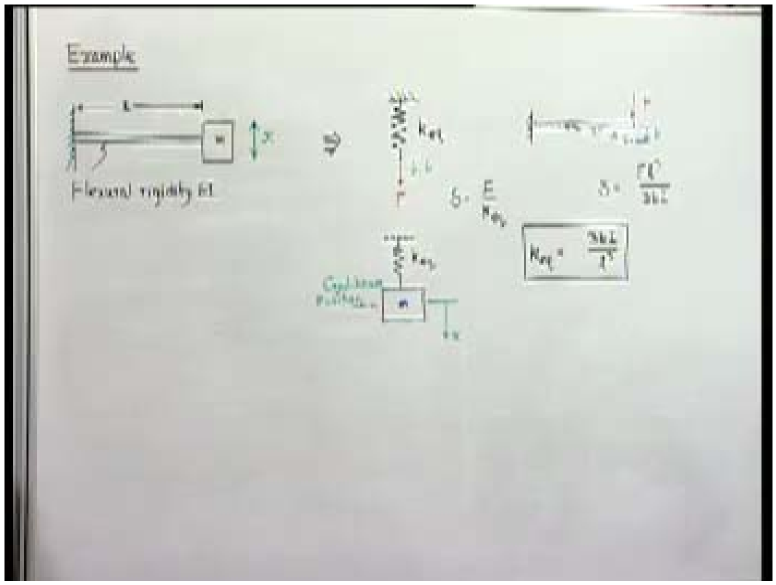


Now as I mention, let us take a particular example. We will take a very simple example to begin with; the problem is like this, we have a cantilever beam being of length l and the flexural rigidity; that means the product of its second moment area of the cross section and the modules of elasticity of the material EI. At the free end of the cantilever beam we have placed a block of mass m. We also mention that the mass of the cantilever beam at this stage we can ignore compare to mass of block at the end. Therefore, we will treat a cantilever beam as a simple spring element and if does not process any inertia. We know in this case motion is going to be up and down motion of this (Refer Slide Time: 50:45).

We can actually treat this problem the lump parameter module, we can make the cantilever beam n which can move up and down, but it is subjected to some constant force which is proportional to the deflection and multiplied by the stiffness represented by this. Therefore, we can represent it by the equivalent mass-less spring with stiffness, simply $k_{equivalent}$. How much is $k_{equivalent}$? How to find it out? We take the same actual beam and apply a load, say load F then what happens? The beam deflects and this deflection; let me call as small delta (Refer Slide Time: 52:10).

If we apply this same force here, it should deflected by same amount. Here the deflection delta is equal to how much? F by $k_{equivalent}$ by definition of the spring (()). How much is the deflection of this, here we know delta is equal to - you know the flexural rigidity of this is given therefore, it is going to be - l cube by 3EI. It is from beam theory for simple cantilever beam the deflection at the free end is given by this. If this has to be replaced by this, the deflection of this end because of same force must be same. Therefore, we will get $k_{equivalent}$ is equal to 3EI by l cube.

This is module link and we have now replaced the actual system by a lump parameter module (Refer Slide Time: 54:16). Now, you have seen the equation of motion, this is the equilibrium position and we measure x from here. We should remember now about previous analysis and recognize fact that we don't have to consider gravity anymore because you are measuring the displacement from the equilibrium, which was under the action of gravity.

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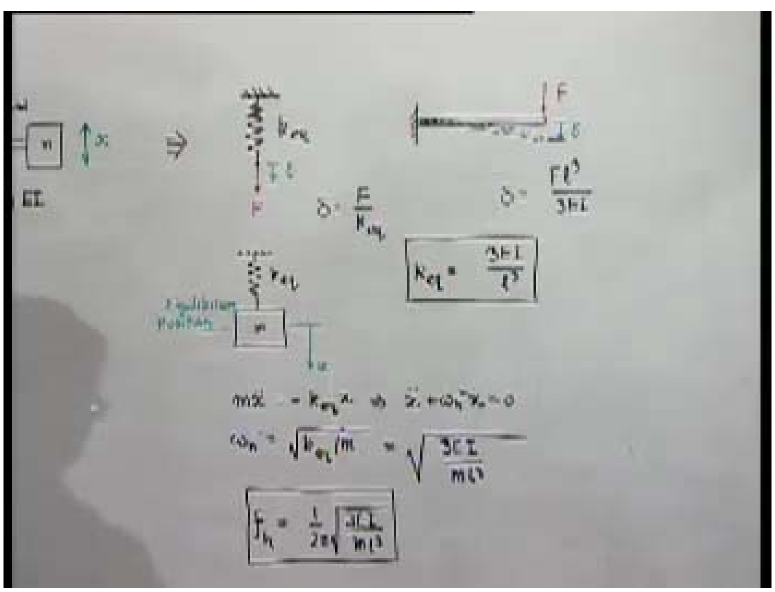

What is the equation of motion? You all remember that m x two dot is equal to minus $k_{equivalent}$ x or $omega_n$ square is simply that will give you x two dot plus $omega_n$ square x equal to 0, where $omega_n$ square is square root of $k_{equivalent}$. Now $k_{equivalent}$ , we have already found out, so this is nothing but square root of 3EI by m l cube. Frequency is equal to 1 by 2pi square root of 3EI divided by m l cube. For a cantilever beam the natural frequency of oscillation when it carries a block of mass m at its free end is given by this quantity.

If the material is harder, that is E is higher, frequency will be higher. If the system cross section is thicker I is more, it will again become higher. Otherwise if the length is more, than it slows down and it increases or decreases at a very (( )) position. If l is large frequency will decrease and it is inversely proportional to the l to the power 3 by 2. On, the other hand if l is increased again the frequency will decrease as squared root l. The most sensitive factor in controlling the frequency of a cantilever beam is explained.

This is a fairly simple example, but I wanted to tell this step which normally one has to take sometimes we do it very quickly without going in detail to the intermediate step like this, but first thing is developing the mathematical module, lump parameter module, then framing the equation of motion and solving this equation. I think in the next lecture, we will take up more complicated problems particularly we will also like to take situations where the restoration is by gravity. We will also try to figure out, if there are other approaches of solving such problems particularly in case, where there is no damping or no dissipation. We have noticed that we are solving the simplest possible that there is no damping. In such case, the total mechanical energy is conserved; either you can use that to make use of that particular condition of conservation of total energy in solving some problem. We will look into that in the next lecture.