Mathematical Methods in Engineering and Science
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Module - II
The Algebraic Eigenvalue Problem
Lecture - 04
Householder Method, Tridiagonal Matrices

Good morning, in the previous lecture, we studied 1 of the 4 methods to work out suitable similarity transformations for solving the Eigenvalue problem that was through plane rotations. Today in this lecture we consider the second method to work out suitable similarity transformation. This is also related to geometrical ideas and this particular method is based on reflection. So, today we will study householder transformation and tridiagonal matrices.

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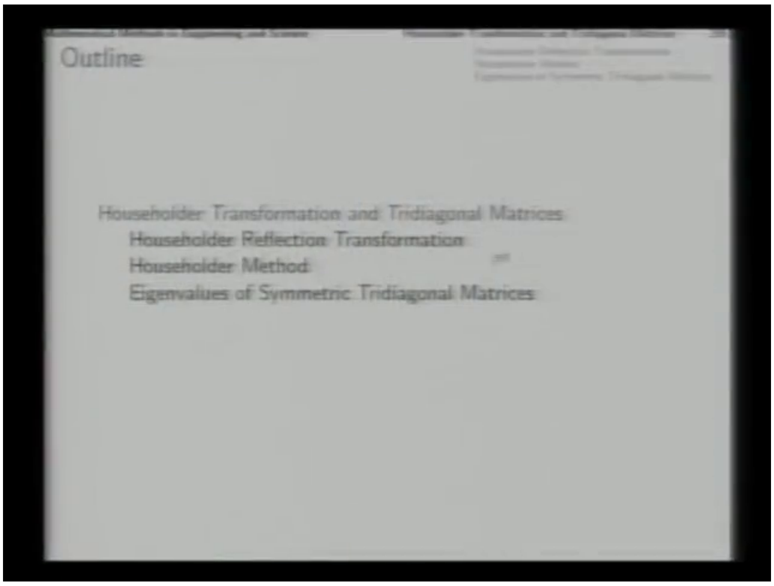

First we consider the householder reflection transformation in a geometric sense, and then we work out how to find the householder method for tridiagonalising a given symmetric matrix and next we see what to do with that resulting tridiagonal matrix symmetric tridiagonal matrix.

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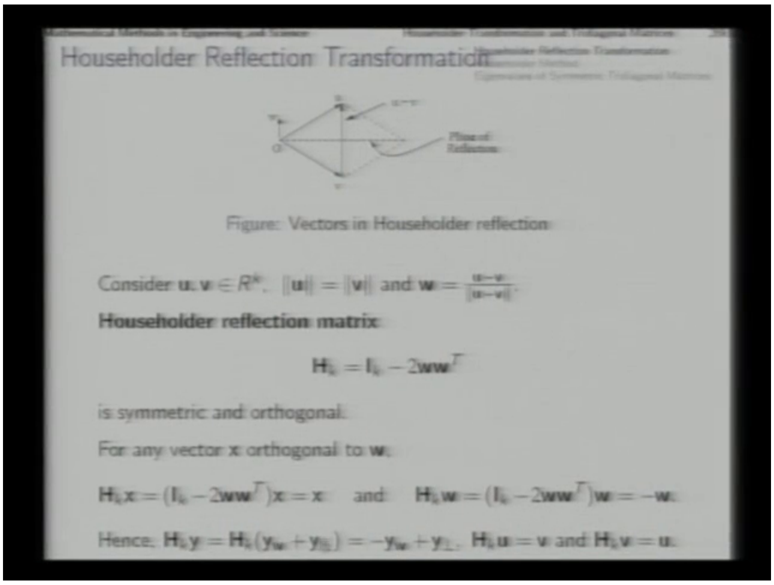

So, first consider this in a k dimensional space; consider 2 vectors u and v in this k dimensional space. Both having the same magnitude if u and v have the same magnitude like this then we proceed to find this particular vector w, which is the unit vector along the direction of difference of u and v that u is this v is this this vector is a difference u minus b and we divided it with its magnitude to find the unit vector w in this direction.

Now, this small vector w unit vector in this direction is perpendicular or a orthogonal to this plane or hyper plane, that actually bisects the angle between these 2 rays showing the 2 vectors. Now with this w in hand let us construct this matrix Hk which is a k by k matrix formed by subtracting the matrix twice ww transpose from the identity matrix this matrix is called the householder reflection matrix, this has a lot of interesting properties. This is a matrix which is symmetric and orthogonal at the same time. Symmetry is easy to see here because identity is anyway symmetric and ww transpose is symmetric. So, this defines a symmetric to check for orthogonality we can find out whether a Hk transpose Hk is identity.

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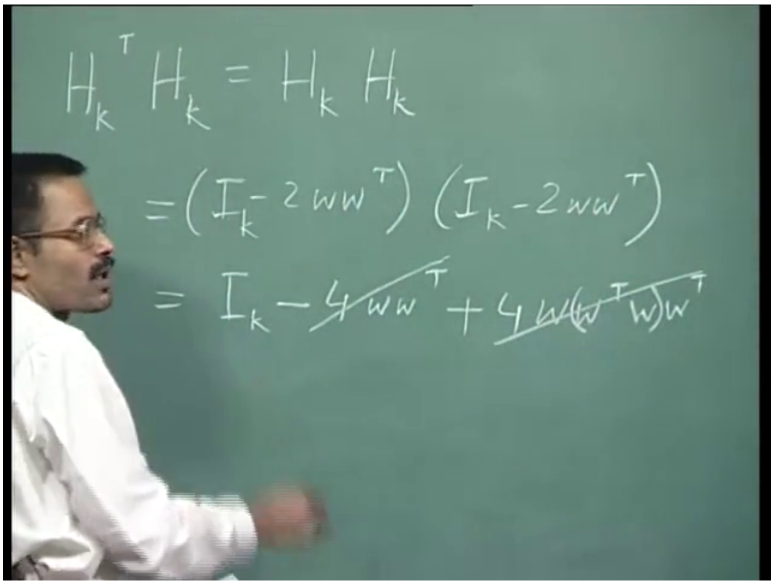

To see that is actually quite simple Hk transpose Hk and symmetry we have already confirmed we have already verified. So, in place of Hk transpose Hk we can simply write Hk. So, Hk into Hk right identity minus twice w w transpose into the same thing. As we open this product we get I into I that is identity minus I into twice ww transpose and again twice ww transpose into identity.

So, total 4 ww transpose minus minus plus 2 into 2 4 and we get ww transpose ww transpose since matrix multiplication is associative. So, whichever order you multiply these 4 it does not matter. So, if you multiply in this first then you will find that w transpose w is unity because w is a unit vector then what remains 4 ww transpose which is as same as this. So, that will mean that Hk transpose Hk is identity that establishes the orthogonality of this householder reflection matrix. Now what does this symmetric and orthogonal matrix do why it is called reflection matrix. To see that consider its action on 2 vectors; 1 around w and the other perpendicular to w or orthogonal to w. Orthogonal to w will mean a vector in this plane which is shown here as plane of reflection.

So, take any vector x which is orthogonal to w; that means, which is in this plane. So, when you apply it apply Hk on x, you find that Ik minus 1 twice ww transpose x what you will get identity into x is x and this 1 as you open you will find first w transpose x and from the very definition of x being orthogonal to w, w transpose x will be 0. So, what will remain identity x into x which is x; that means, a vector orthogonal to w that is in the plane of reflection gets mapped to the same vector itself there is no change.

On the other hand how does w itself get, gets mapped get mapped Hkw as you apply this on w you find that identity into w will give u w, but this fellow will give you twice w, w transpose w is 1. So, you will get w minus twice w that is minus w; that means, w itself when operated upon by Hk gets mapped to its negative and the vector in the plane, plane of reflection gets mapped to itself that is the way a reflection takes place. This plane of reflection operates like a mirror right. So, if there is any other vector which has some component on the plane and some component perpendicular to it, then we can consider it like this applying Hk over y which has 2 components along w which is perpendicular to the plane and perpendicular to w which is along the plane. The component which is along w gets mapped to its negatives and the particular 1 remains as it is right. So, this is typically the action of a mirror reflections. So, this is why this matrix is called a householder reflection matrix. In particular it will map u to v and v to u because they are mirror images of each other with this plane as the mirror or plane of reflection.

Now, this concept and this particular matrix, how do you utilise in reducing the symmetric matrix to a form more suitable for solution of the Eigenvalue problem, in this case we will try to make it tridiagonal how do you use that.

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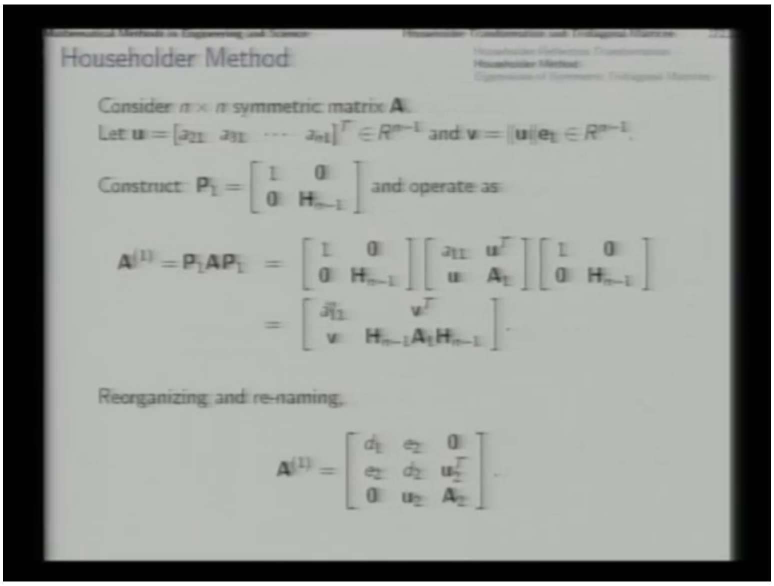

So, that brings us to the point of householder method. Consider an n by n symmetric matrix.

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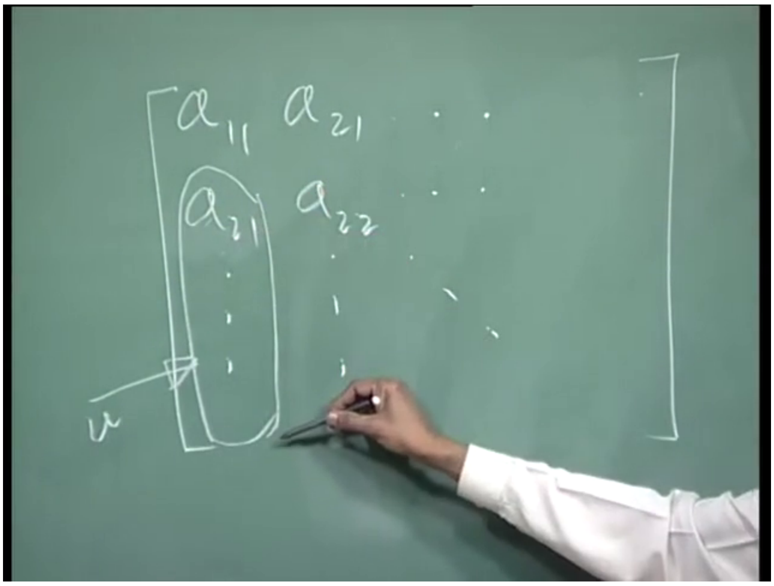

And so on right symmetry is shown here a to 1 a to 1 and so on. Now take u in that context in the reflection context take this u to be the vector a to 1 to an 1 this transpose makes this row vector a column vector through transposition.

Now, if the matrix is m by n, then this vector u is an n minus 1 dimensional vector because that top entry a 11 we have left out and that vector u starts from a to 1; that means, this vector this much is taken as u and then v is taken to be a vector of the same dimension, but it is having a first entry which is the same as norm of u. So, whatever is the norm of this that turns out to be the first entry of v all other entries of v are 0; that means, the top entry of v will be the norm of this vector from a 21 to a n 1 and all the other n minus 2 entries of v will be 0.

So, like that construct the vector v with this u and v we work out w u minus v divided by its magnitude and then we work out the householder matrix. That householder matrix Hk in this case k is n minus 1. So, it will be an n minus 1 by n minus 1 matrix hn minus 1, we call it right, then out of that H n minus which is an n minus 1 by n minus 1 matrix, we develop this larger matrix by inserting a 1 here a 0 row above and 0 column on the left of this H. So, this is P 1 now P 1 is its own transpose because it is symmetric and it is orthogonal also. So, then what we do is that we apply this orthogonal similarity transformation P 1 transpose A P 1 since P 1 transpose is same as P 1. So, we have just written P 1.

Now, what we have here is a 11 sitting here u is this this whole thing is u transpose and this much the trailing n by n minus 1 by n minus 1 matrix, for that we given name call it a 1 whatever it is then as we apply this P 1 a P 1 through the multiplications you will find that this u has become in place of u have got v now and in place of u transpose you have got v transpose, this multiplication you can see immediately we have got P 1 0 right.

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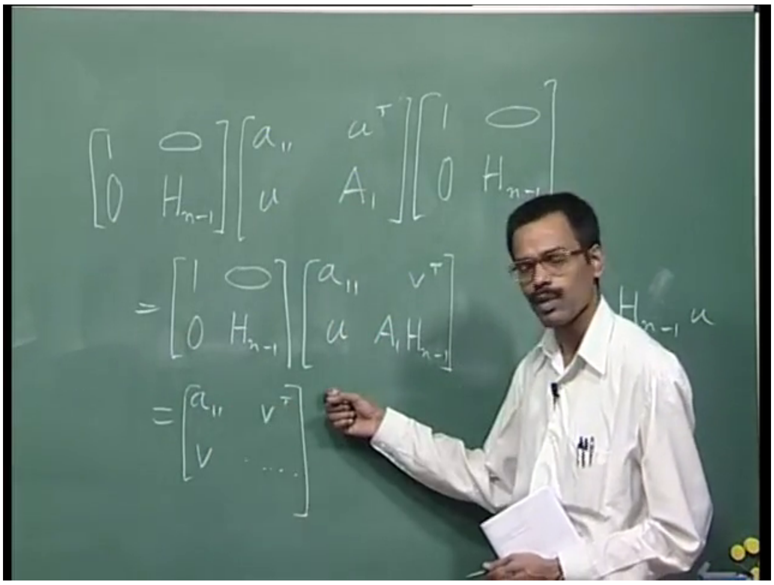

So, 0 row 0 column and H n plus 1 n minus 1 here then a which has a 11, u on this side u transpose on this side and a 1 here, here and again the same thing.

Now, as you conduct the block operations in Eigenvalue problem solution methodologies you will come often across these block operations. So, first keep this as it is and we multiply these 2; a 1 1 scalar into 1 plus u transpose row vector into 0 column vector that is 0. So, you get a 1 1, next a 1 1 into 0 row you get a 0 row plus u transpose into this what is u transpose into this that will be the transpose of this right.

So, u transpose into this will be the transpose of this right and what is this? This matrix is its own transpose. So, this is simply Hn minus 1 u and through the property of the householder reflection matrix that we have seen just now this is nothing, but v. So, therefore, u transpose this will be v transpose. So, you get v transpose here next the lower row block u column vector into 1 that gives you u only, plus a 1 into 0 that is u you get u finally, this big block this is a scalar this is a row vector this is a column vector now we have got the trailing n minus 1 by n minus 1 matrix here.

Column vector u into row vector 0 that is a 0 matrix plus A 1 into H we write it finally, this multiplication 1 into a 1 scalar a 1 plus plus 0 row into u column that is 0. So, you get a 1 1 a 1 next 1 into v transpose row vector v transpose plus 0 into whatever. So, you get v transpose here, here 0 into a 1 1 that is 0 that is a column vector plus Hn minus 1 u that is v we will get v here and here you will get 0 into v transpose that is 0 plus h into a 1 into H right. So, what you have got? You have got in the first column you have got a 1 1 and then the vector V similarly in the first row you have got a 1 1 and then v transpose and what is the structure of v that we started with first entry of v is full size of u and all the other entries of V are 0; that means, below the second entry from the top everything else will be 0. So, that is what you get here right.

So, now, we rename, see in their whole process a 1 1 has remained unchanged a 1 1 has not been operated upon by anything because the first column and first row of P 1 is same as identity. So, a 1 1 has less has been left unchanged now we rename a 1 1 as d 1 and whatever is the first entry of v we name it as a 2 below which everything else is 0 and out of symmetry that same e 2 will be sitting here on the right of which there will be all zeros and this 2 2 diagonal entry we now call d 2. In the next step this block will remain unchanged though in the first step this remained unchained a 1 1 in the second step this much will remain unchanged. What we do in the second step? iIn the first column below the top 2 entries everything else has be become 0, next round in the second column below the top 3 entries we were went to make everything 0 this is the process to make it tridiagonal.

So, what we consider is that below the 2 top entries whatever is the vector sitting we call that u 2 and then construct a similar v 2 which has the same magnitude as u 2 and all, but the first entry all, but the first all the other entries are 0 right and that size of the matrix in a vector u 2 and v 2 is n minus 2 then we construct the next householder.

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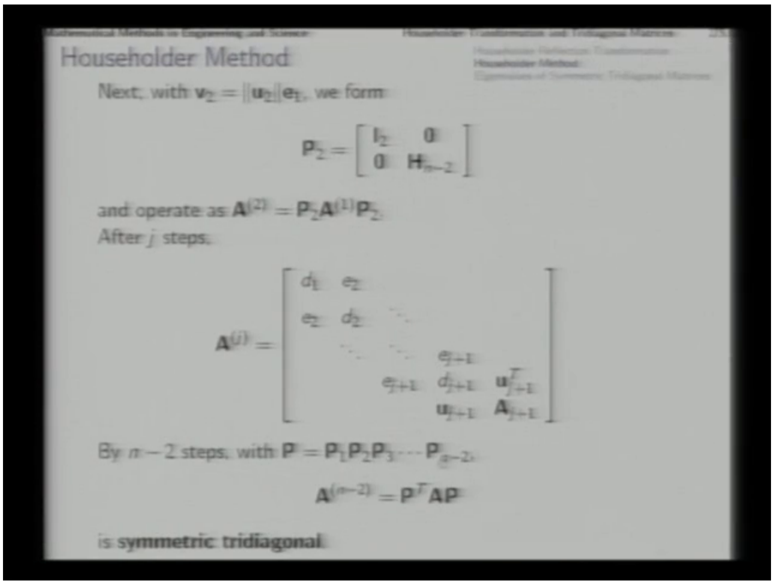

Transformation matrix of size n minus 2 by n minus 2 and enhance it within identity matrix of size 2 by 2 here, equivalent number of zeros here and here to complete the size. Then apply that on the previous result this and this will keep unchanged the leading to by 2 block of a 1 and you will get the next step which will have this much d 1, d 2, d 3, e 2, e 3 in the place correct places and the first 2 columns and the first 2 rows have been made processed up to the extent that below the sub diagonal and above the super diagonal and we have got zeros in those first 2 columns and rows.

Like that we keep on conducting steps with smaller and smaller householder matrices in the trailing part and the leading part we will have the identity matrices of gradually increasing size. After j th such steps till this point it has been converted to tridiagonal and remaining fellows are full and as we go on conducting this kind of steps at the end of n minus 2 steps, we will have this complete transformation.

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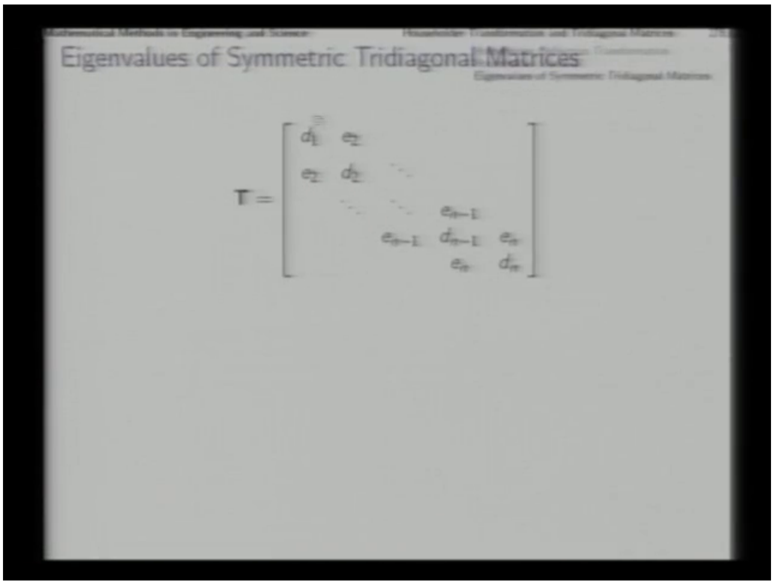

We want to P 1 to P n minus 2 which will result in a completely symmetric tridiagonal matrix right which will look like this.

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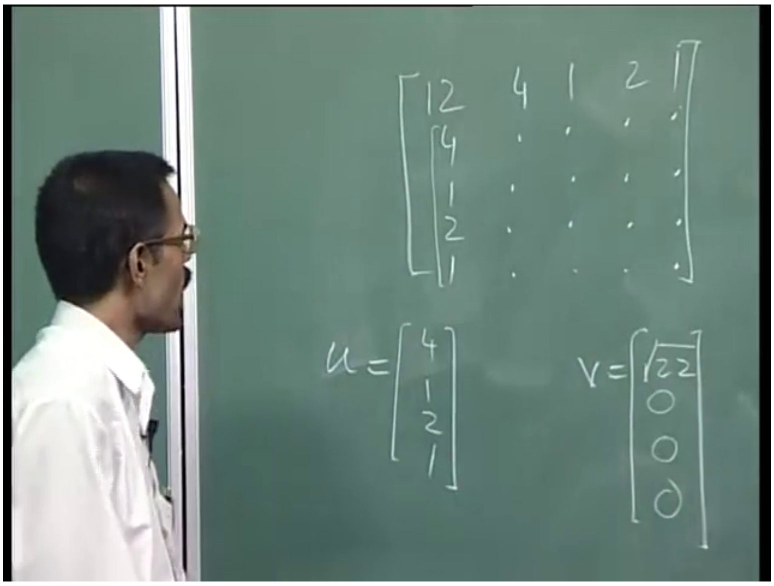

Let us see a quick example this is a 5 by 5 matrix, now this part is what we called a 1 there. Now in order to reduce it to a symmetric tridiagonal form, we would like to have first 3 zeros in these locations right. So, we take u as these vectors 4 1 2 1 and we want v in which the last 3 entries are 0 and what is the first entry? First entry is the size of this magnitude this vector u. So, what is that size?

This square plus this square plus this square plus this square; so, we will have 16, 17, 18 and 4 22. So, root 22. So, this will become our v and with this u and this v it is easy to find out w the difference of u and v and divide it by whatever is its magnitude right.

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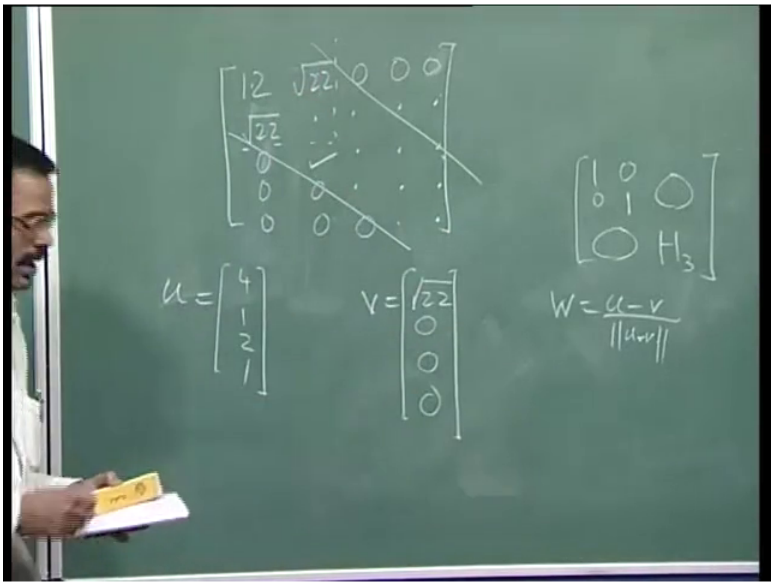

So, with that we find w and then we work out twice w w transpose, subtract it from identity and that matrix is our 4 by 4 householder transformation matrix, that 4 by 4 matrix will be sitting here.

Let us call it H 4 zeroes here zeroes here, when this matrix is multiplied on this side and this side to this matrix, then the transformation that will take place will make these root 22 0 0 0 similarly here, root 22 0 0 0 right then we will find at this much is secured and whatever is here this 3 dimensional vector will be then taken as the next u and then the next v will be taken as something 0 0 that something will be the size of this and then through the similar process in which the householder transformation matrix in this case will be I 2 H 3 0 0 this 0 matrix is of size 3 by 2 this is of size 2 by 3 and so on.

When this is multiplied on this side as well as on this side, you will get something here zeroes here the third step here will make this as 0 and whatever happens on this side will happen on this side also. So, you get a symmetric tridiagonal matrix like this, now the question is that after we have reduced the matrix.

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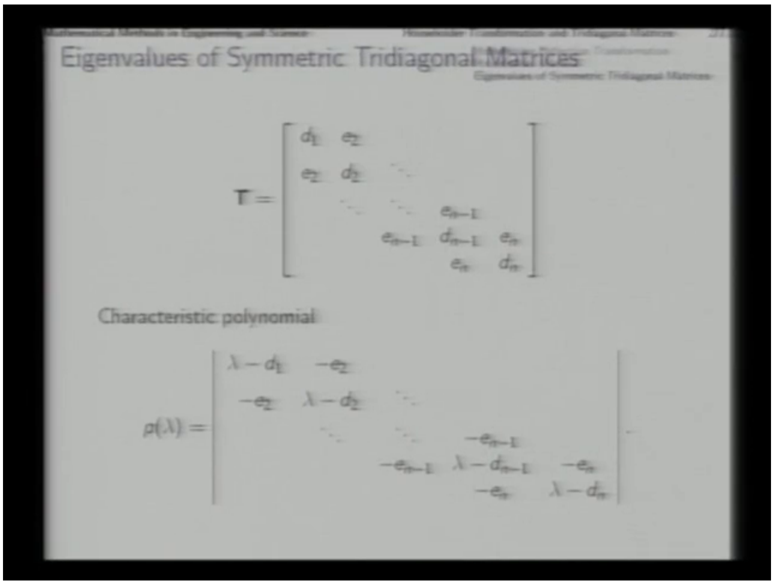

To this symmetric tridiagonal form what do we do with it that is, is the solution of the Eigenvalue problem of a symmetric tridiagonal matrix anyway simpler compared to the original symmetric matrix the answer is yes. There are several ways 1 can handle this kind of symmetric tridiagonal matrices one way we consider now and the other way we will consider in the next lecture. There is a very interesting piece of theory which tells you how to work out the characteristic polynomials of sub matrices of this that is leading 1 by 1 sub matrix leading 2 by 2 sub matrix leading 3 by 3 sub matrix and form a sequence out of these characteristic polynomials and then try to solve the Eigenvalue problem based on those interesting properties.

So, what will be the characteristic polynomial of this? So, for that we have to find out the determinant of lambda I minus P. So, this is this the characteristic polynomial right lambda minus d 1, lambda minus d 2 etcetera sitting in the diagonal places and minus e 2 minus e 3 etcetera sitting in the off diagonal (Refer Time: 23:45) places note that d is indexed from 1 to n and e the sub diagonal super diagonal entries which is 1 less in number, they are indexed starting from 2. It could have been indexed as e 1 to en minus 1 that would be equivalent to this, but in this analysis we have indexed from e 2 to en. So, there is nothing called e 1 in this analysis fine. So, with this characteristic polynomial you find that the characteristic polynomial of leading 1 by 1 part is simply lambda minus d 1 right.

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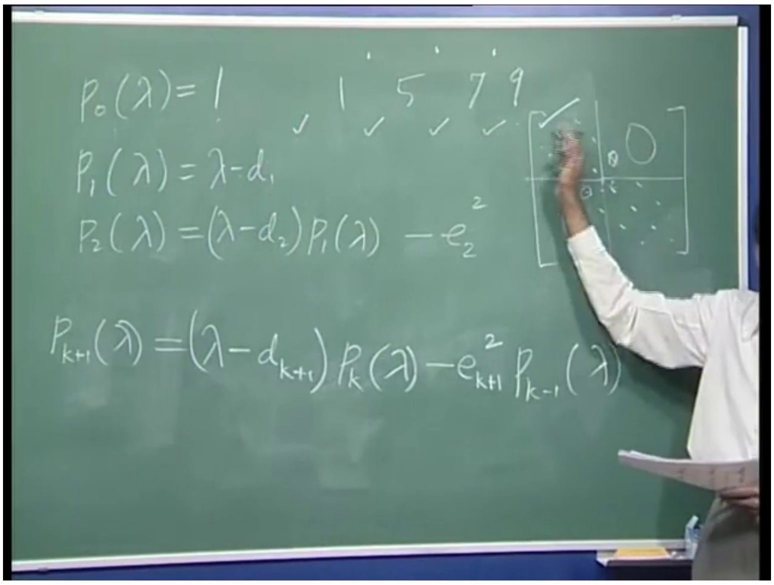

So, we call it P 1, P 1 of lambda that is the characteristic polynomial of the leading 1 by 1 sub matrix from t. So, you call it P 1 that is simply lambda minus d 1, right.

Then for the leading 2 by 2 sub matrix we have got the characteristic polynomial from here lambda minus d 2 into lambda minus d 1 minus e 2 square. In this place lambda minus d 1 can you simply put P 1 my P 1 of lambda you can. So, we write this similarly we can work out P 3 P 4 etcetera, but let us go 1 large step and try to determine P k plus 1 lambda in terms of P k lambda and P k minus 1 lambda. So, that will establish a recursion among all these characteristic polynomials of the leading sub matrices.

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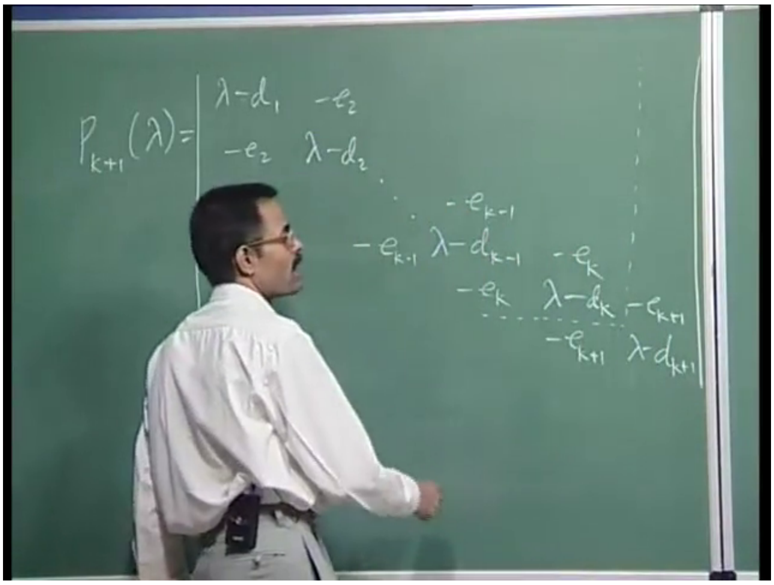

So, as we try to do that let us write down here, we are going to write down here the same matrix appearing there, but not up to all the way to lambda minus d n, but up to lambda minus d k plus 1. When you try to expand this determinant from this column, what do we find? We find it is lambda minus d k plus 1 into this determinant minus this thing into the determinant that we will find by crossing out this row and this column. So, let us do that and all other entries are zeroes right.

So, we get lambda minus dk plus 1 into this determinant which is same as the characteristic polynomial of the sub matrix 1 order less that is lambda minus dk plus 1 into pk lambda minus dk plus 1 into pk. Then plus this thing which is minus ek plus 1 minus ek plus 1 into something we try to find out that something what is that something? That something will be the determinant found by removing this row and this column. So, let us do that remaining thing will no longer be this. So, we will be removing this column and this row, this determinant should be sitting here and see its diagonal entries are lambda minus d 1 lambda minus d 2 lambda minus d 3 etcetera up to lambda minus dk minus 1 and then next because this guy has taken this place actually after removal of this row and in this row other than this entry everything else is 0.

So; that means, the determinant that we are asking for is this into this determinant right and this is minus ek plus 1 which earlier that minus sign that plus we have not made it plus. So, that minus will actually now help and because this minus this is remains minus finally, and ek plus 1 comes once more sorry it is square now and what else is here that is pk minus 1 that is the characteristic polynomial of the matrix of 1 further order less and now what we do for the other than this now this relationship this recursive relationship will define up to P n in terms of the older ones. So, P 3 will get defined in terms of these 2, P 4 will get defined in terms of P 2 and P 3 and so on through this relationship.

At the top we also put a dummy element in this sequence in order to complete the sequence and that is one. So, then we will say this we will have 0 roots no roots this will have 1 root which is d 1 this will have 2 roots which is we can find out what are those 2 roots and so on. So, finally, P n will have n roots, as we construct this sequence then this sequence has some interesting properties these expressions or rather this expression this recursive expression which is here helps us in evaluating these polynomials extremely fast.

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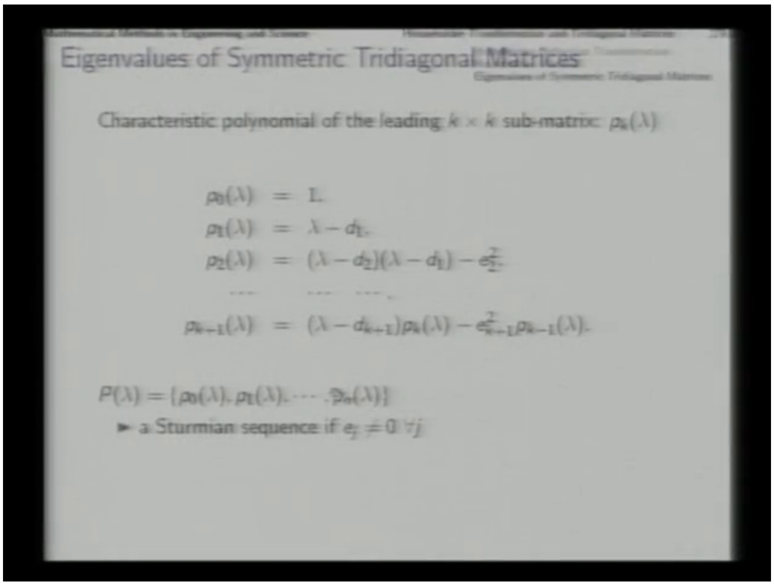

Other than that this sequence of these polynomials of increasing degree, has further properties they in particular they have the property called a sturmian sequence property, that property they have if all the e js all the sub diagonal and super diagonal entries are non-zero. In that case this sequence P 0 P 1 P 2 all these polynomials the sequence of all these polynomials has an interesting property.

Now, our rest of the process will directly depend on that property, but before that we need to ascertain what we should do if there is some ej which is 0, then that is actually for some j j say ej is 0 some of the sub diagonal and super diagonal entries turns up to be 0 that is actually a good news because in that we can skip the matrix.

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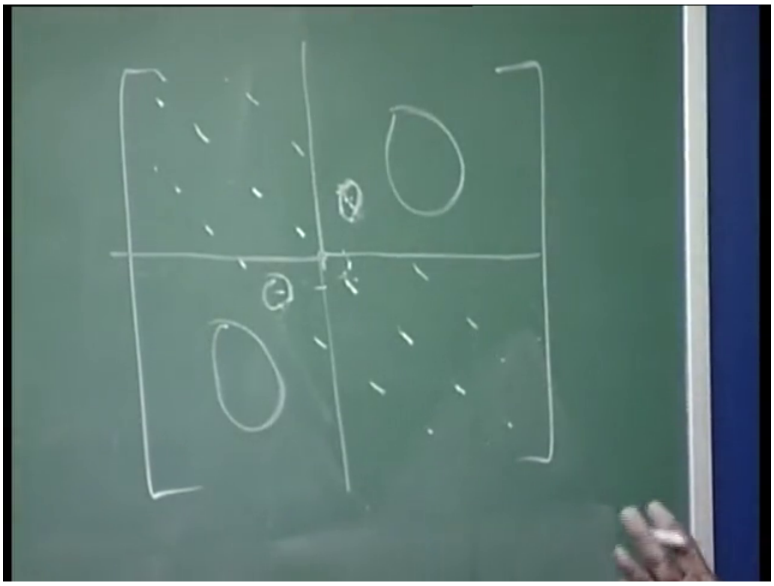

We have d 1 d 2 etcetera up to dn, we have e 2 e 3 etcetera up to en. Now if there is a other things are already zeroes if there is some e which is 0 here as well as here, then this is actually going to obstruct us from using this succeeding formulation for the complete matrix, but these 2 zeroes will actually help in treating the matrix into 2 part because then we will have the complete matrix in the form of a block diagram matrix with these 2 0 sitting here. Earlier if we had these as non-zero, then it was such a huge long matrix large matrix n by n.

Now, these 2 zeroes here will decouple the 2 subspaces completely and we will actually have this as a block diagram matrix this is one block and this is another matrix. So, whenever we have ej equal to 0 at that location we can always split the matrix into smaller block such that we can consider each block separately. So, having some ej as 0 helps us in splitting the matrix into small matrices, until each such as block has non zero ejs all through right. So, we can consider only those cases, which have non 0 entries here for which the rest of the theory holds.

Now, what is that particular property?

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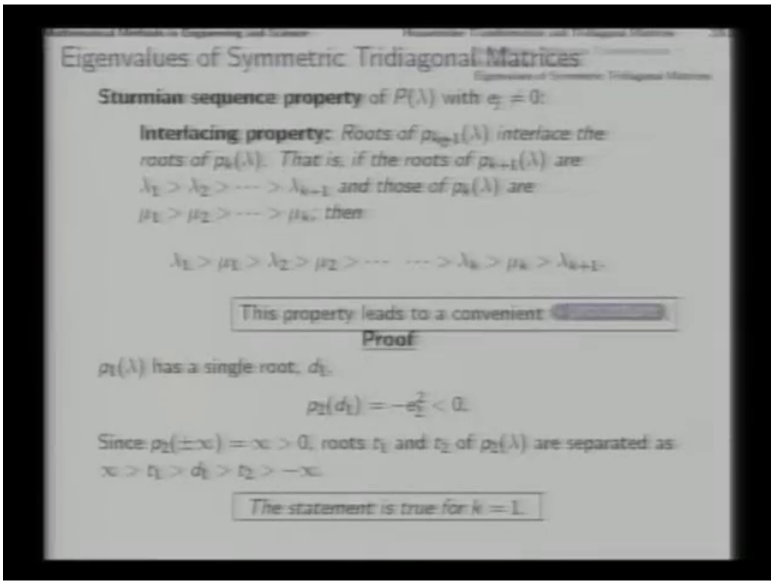

The Sturmian sequence property says that roots of pk plus 1 interlace roots of pk what is that? That means, if you have roots of pk sitting at locations 1 5 7 9, P k plus 1 7 5 7, 9 if these are the roots of pk then the next 1 pk plus 1 which has 1 more root 5 root say P 4 has these as root P 5 which has 5 roots will certainly have 1 root below 1, another root between these 2 another root between these 2 another root between these 2 and the fifth root above 9; that means, the roots of pk plus 1 will interlace the roots of pk which in turn will interlace roots of pk 3 a P 3 epk minus 1 and so on right. So, this is the interlacing property which is shown mathematically like this and this property leads to a convenient procedure for finding the Eigenvalues.

Now I will skip the proof of this particular property, but I will just give you the line of proof and strongly advise you that in the textbook, you go through the proof in this textbook or in these slides which are available in the internet you should go though the proof quite carefully because the proof has a as a has an inherent beauty in it. So, the line of the proof is as follows first we considered the case of k equal to 1 that is if this statement true for a equal to 1 and that is trivially true because there is only 1 root and nothing is there to interlace in the case of 2 then you verify this. So, the statement is true for k in the sense that roots of P 2 interlace the roots of P 1. So, the first entry d 1 is interlaced by the Eigenvalues of the leading 2 by 2 matrix d 1 e 2 e 2 d 2 this you verify that shows that the statement is true for k equal to 1, next you assume that the statement is true for k equal to i.

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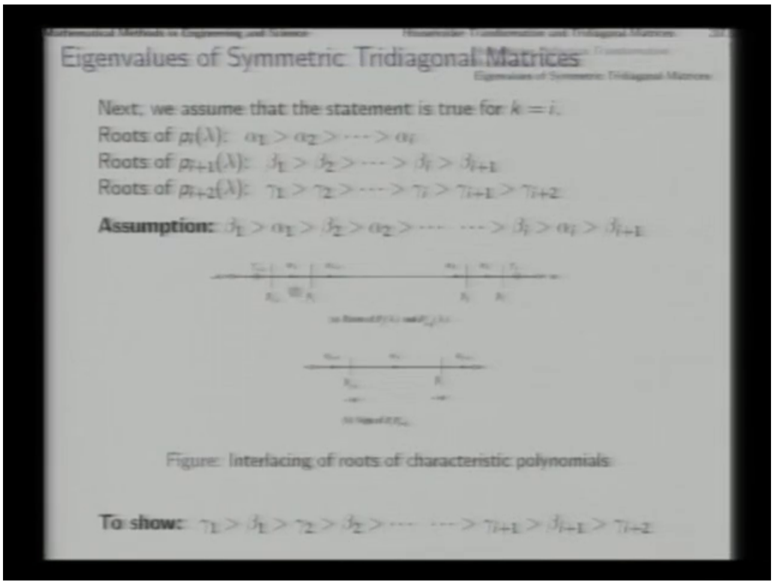

Then you denote the roots of pi as alphas roots of pi plus 1 as betas and roots of pi plus 2 as gammas.

As you assume the statement to be true for k equal to i you assume this. That is the betas interlace the alphas that is the i plus 1 betas will interlace the i alphas and in the number line you can show the alphas which crosses and beta as bars and the picture looks like this. Then you need to show that in turn gammas will then interlace the betas i plus 2 gammas will interlace i plus betas and that you establish based on this (Refer Time: 38:12) consideration and changes of signs in the roots of the succeeding polynomials. So, rest of the proof I will omit here in the class, but I strongly suggest that you go through the proof a little carefully we will go rather to the procedure.

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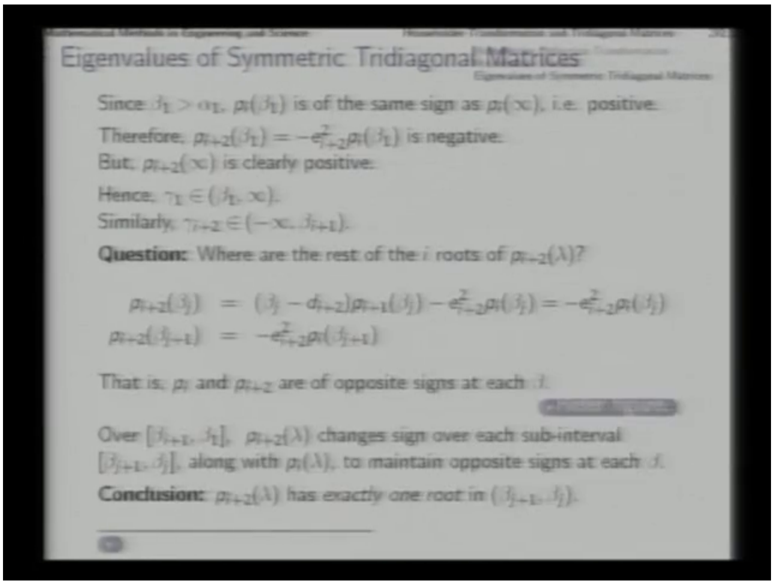

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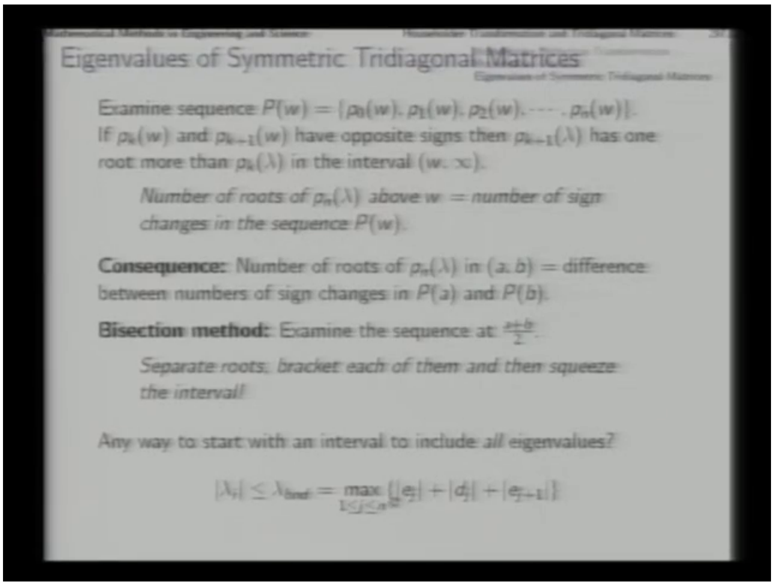

We examine this sequence p 0, p 1, p 2, pn 3 up to pn for different values of w. For a particular value of w if we know that the pk and pk plus 1 and pk minus 1 will have their locations of roots in this manner see one question we are never rising that whether the roots are real or not because that question we are never rising because the matrix is system matrix. So, all the roots are real that is anyway known all the Eigenvalues are real that is anyway known.

So, if pk has this kind of relationship with pk minus 1 and pk minus pk plus 1 their roots then one thing is very clear. If pk w and pk plus 1 w have opposite signs then the number of their roots above w can differ by just 1 why? Because if this sign that is in this (Refer Time: 39:45) suppose w falls here and pk w has a certain sign and the at the same w pk plus 1 has a sign difference from that. That will mean that above that value above that w whatever is the number of roots of this and the number of roots of this can differ at most by 1 because at infinity all the pks will have plus infinity value, infinity minus into infinity minus something into infinity minus something and so on. So, at infinity that polynomial all of these polynomials will evaluate to plus infinity. So, all of them are positive. So, the moment 1 root is encountered the sign changes for each of them.

So, it is impossible that one of them encounters too many roots and the other the next 1one has not encountered any roots because of this interlacing property. So, pk and pk plus 1 2 succeeding 1 2 continuous ones in this sequence having opposite signs will means that the higher 1 has the one root more than the lower one above w. Now we will find that number of roots of pn above w will be number of sign changes in the sequence from this end to that end because as many sign changed if compared to pn pn minus 1 does not change sign; that means, pnn pn minus 1 will have the same number of roots above w and then from pn minus 1 to pn minus 2 if there is a sign change then we will know that for (Refer Time: 41:29) pn minus 2 1 root less and so on. So, in this entire sequence the number of sign changes at w will tell us the number of roots of pn above w. So, P 0 has no root. So, number of changes will tell you at the end how many roots this guy has above w.

Now, if you if we do this operation at w equal to a and then w equal to b then above a how many roots pn has and above b how many roots pn has. The difference of the 2 numbers will tell us how many roots pn has in this interval ab. If at a particular value in this entire examination in this entire investigation the pn turns out to be 0 we know that that value is the root is a root. So, after closing like this that how many roots in the interval ab, we can consider a plus b by 2 and then see out of those roots in the interval ab how many are in the lower half a to a plus b by 2 and how many are in the upper half a plus by 2 to b and so on. So, like this we can repeatedly used bisection to squeeze each of these roots separately and then further we can use bisection itself to go on squeezing the interval till we find the root to our required accuracy or rather than bisection we can find some other equation solving process after locating the roots and separating all the roots ok.

So, with what interval we start do we start from minus infinity to infinity, then it will be very difficult to process the whole thing because bisection will work independently, there is a little trick in starting the process if you want to solve for all the Eigenvalues and that tells you this all the lambda their magnitudes are bounded by this quantity, which is the maximum over all rows of the entries of the rows ej plus dj plus ej plus 1 take all their magnitudes and whatever is the maximum of the sum over all the rows no lambda no Eigenvalue of the matrix can have a magnitude which is larger than that.

So, if you take the initial interval from minus lambda b and d to lambda b and d then all the identities are bound to follow in that and then you can go on applying bisection in order to separate the roots and separate Eigenvalues and once you have separated them then solving for them, you can apply either bisection itself or some other equation solving process or root finding process so that gives you this algorithm.

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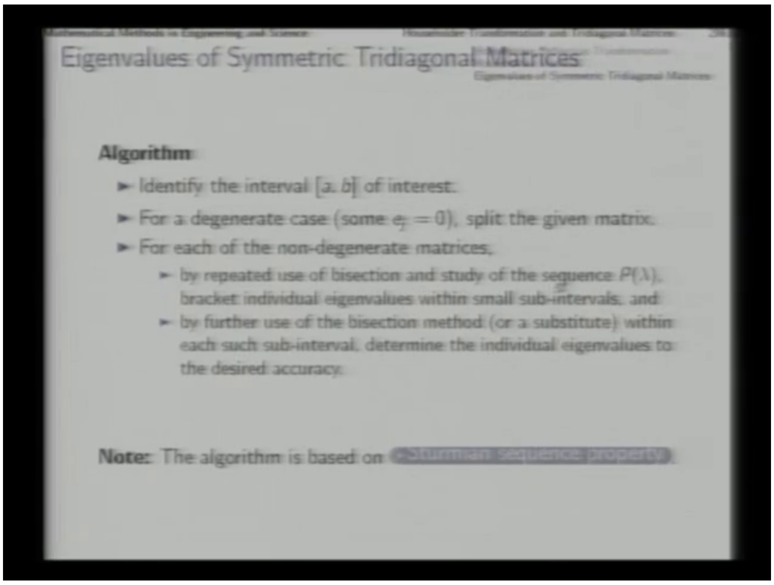

First identify the interval ab of interest now interval ab of interest can be either the entire interval minus lambda b and d to lambda b and d if you are interested in finding all the roots all the Eigenvalues or sometimes the your problem may suggest that you are interested in an Eigenvalue in a given domain only in a given interval only you are not bothered with rest of the Eigenvalues which may fall outside this interval. In that case you take that interval at a b otherwise you take the larger interval in which you are sure that all Eigenvalue will lie. Now for a degenerate case in which some sub diagonal or super diagonal entry of the symmetric tridiagonal matrix is 0 you split the given matrix and operate separately with the different blocks. For each of the remaining non degenerate blogs or matrices you just do 2 things by repeated use of bisection and study of the sequence P lambda you bracket or separate individual Eigenvalues within small subintervals and then in these bracketed subintervals by further use of bisection itself or some substitute some other root finding method within each subinterval determine the individual Eigenvalues and when the interval becomes extremely small say the interval size becomes equal to point 0 0 0 0 1 then; that means, you actually found the (Refer Time: 46:02) Eigenvalue. So, there is no further need to go into that.

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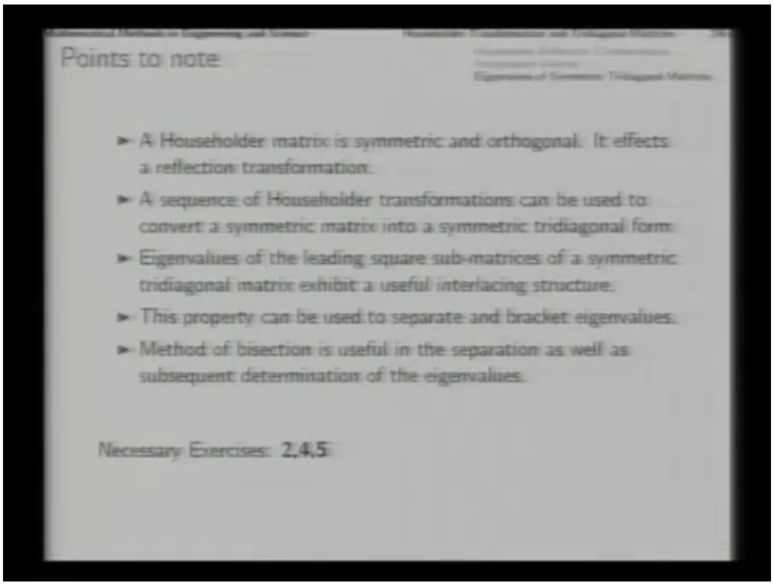

So, in this lesson what are the important points that we should keep focus on first point is that the householder matrix is symmetric and orthogonal and it effects a reflection transformation. Second is a sequence householder transformations can be used to convert a given symmetric matrix into symmetric tridiagonal form, and then the Eigenvalues of the leading squares matrices form a sturmian sequence, which has interlacing structure in its in their roots and this property can be used to separate and bracket Eigenvalues and further solve in a systematic manner. So, we have a little time in hand.

So, let us consider a quick example at least half way after which you can proceed on that example.

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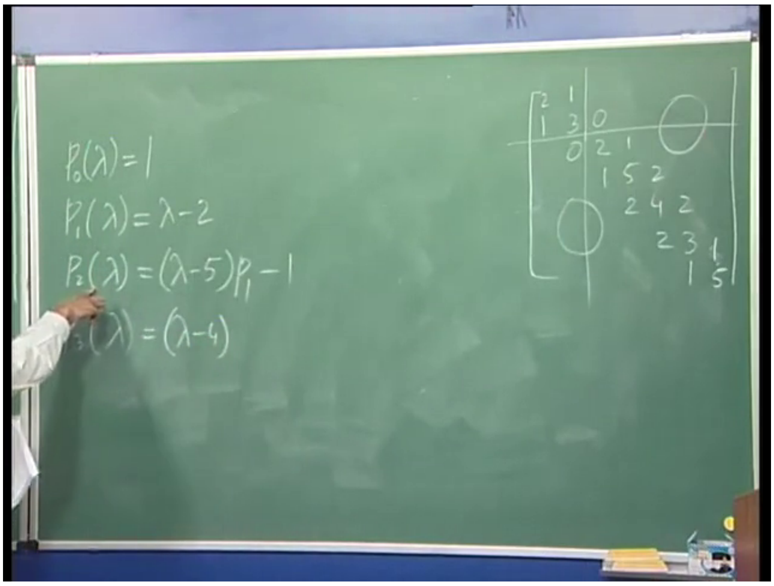

Suppose you have got this matrix 2 3 2 5 4 3 5 7 by 7 matrix; these are the diagonal entries and the off diagonal entries are 1 0 1 2 2 1 right sub diagonal entries are also same and all other entries are 0.

So, this is a symmetric tri diagonal matrix possibly obtained after a series of householder transformations. So, this is the matrix which we are going to solve for Eigenvalues. Now these 2 zeros will allow us to split the matrix in this manner. So, there is a 2 by 2 component and there is a 5 by 5 component, this is actually nothing this you can solve from the diagonals of this you can solve from the (Refer Time: 48:20) definition itself because that will involve only the solution of a quadratic. This otherwise would involve the solution of a quintic equation which is more difficult. So, for this we apply this methodology based on the Sturmian sequence property. So, for that we construct these polynomials first one is trivial second will be lambda minus d 1 that is P 1.

Next P 2 will be lambda minus d 2 that is lambda minus 5 into P 1 minus e 1 square that is 1. Next we will have P 3 which will be lambda minus d 3, 3 into P 2 into P 2 minus e 3 square and what is e 3 here is e 3 here is 2.

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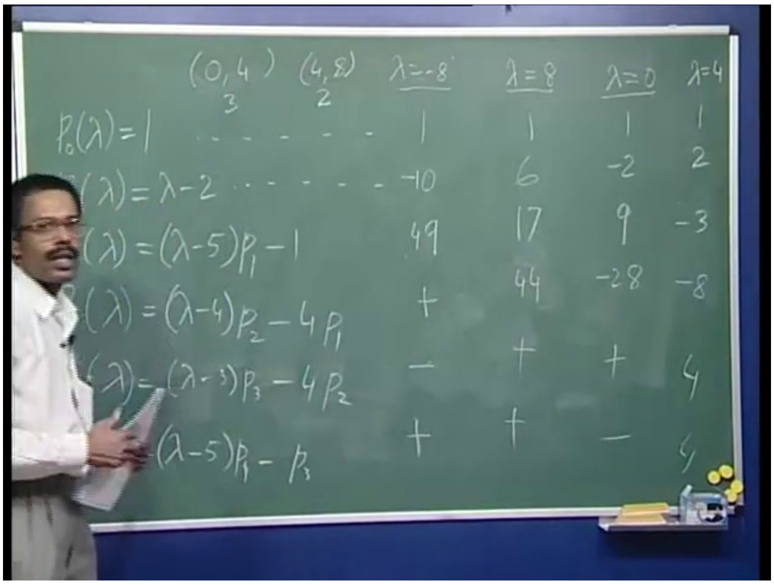

So, e 3 square into P 1 right then P 4 will be lambda minus b 4 into P 3 minus e 4 square e 4 is 2 into P 2 finally, P 5 will be lambda minus d 5 into P 4 minus e 5 square which is 1 into P 3. These things we will now try to evaluate for different lambdas to locate the roots of this polynomials. So, for now consider the interval which you need to consider you would have noticed the intervals as one the rows that is 1 5 2 2 4 2 these are the biggest rows. So, sum of that turns out to be 8 that means, no Eigenvalue of this matrix can have magnitude higher than 8. So, you consider the interval minus 8 to plus 8

So, at lambda equal to minus 8 you try to evaluate d this is one for all of them. This is minus ten this is minus 8 into minus 5 that is minus 13 into this that is minus 10 already in hand. So, what do you get you get minus 13 sorry minus 5 into minus 10, minus 5 into minus 10 that is plus 50 minus 1 we will get 49. Then you come here minus 8 minus 4 that is minus 12 right minus 12 into P 2 what you have already got minus 4 into this.

So, you will find that this turns out to be positive then this turns out to be negative this turns out to be positive and then you will find that 1 2 3 4 5 sign changes are there that will show this I suggest that you would verify and check that these turn out to be positive negative positive and so on and then we will find that there are 5 sign changes from top to bottom; that means, above minus 8 P 5 will have 5 roots; that means, all the 5 roots are above minus 8. Then you consider the case of lambda equal to 8 this is 1 and as you put 8 here you will get 8 minus 2 that is 6 positive ok.

Then you put 8 here 3 into P 1 that is P into 6 18 minus 1. So, you get 17, then you come here 8 minus 4 that is 4 4 P 2 4 into 17, 68 minus 4 into 68 minus 24, you will get 44 still positive like this you will find then in this case all of them turn put to be positive that will mean what is the number of roots of P 5 above 8 above the value 8, number of roots is the same as number of sign changes here no sign change here. So, no root above 8 up to this we have verified that all the roots are actually above minus 8, 5 root is above minus 8 and no root above plus 8. So, we have verified that bound that is all the roots actually lie within minus 8 and 8 right. Now applying bisection you try to find out the number of roots above 0 above 0 how many. So, this is 1 as you put 0 here you get minus 2, as you get put 0 here you get minus 5 into this that is plus ten minus 1 that is plus 9.

Then you come here and you find minus 4 into 9 that is minus 36 minus 4 into minus 2 that is minus 36 plus 8. So, minus 28; like this as you continued you will find that this turns out to be positive and this tunes out to be negative that will show the number of sign changes at lambda equal to 0 for this polynomials this sequence of polynomials is 1 2 3 4 5. So, all 5 roots above all 0. So, all positive roots. So, this gives you a little further information that all the roots are within the interval 0 to 8 in particular this is a positive definite matrix, because all the Eigenvalues are positive. Now what you will do for bisection you will evaluate the polynomials the sequence of polynomials at lambda equal to 4. So, as you will evaluate at lambda equal to 4 you will find that you get 1 2 and then here minus 2 minus 1 that is minus 3, and then at lambda equal to 4 this is 0 minus 4 into P 1 that is minus 8.

Then here 1 into minus 8 that is minus 8 minus minus plus 12; that means 4 finally, here minus 4 minus minus plus 8 is minus 4 plus 8 that is plus 4. So, how many sign changes here 1 sign change here 2 sign changes here. So, above 4 you will have 2 Eigenvalues and below 4 you will have 3. So, you have started bracketing 3 in this interval and 1 in this interval that is 3 in this interval and 2 in this interval right. So, 2 sign changes here at lambda equal to four; that means, above 4, P 5 will have 2 roots 2 sign changes right. So, in this there will be 2 roots in this there will be 3 roots next you will go on splitting this next you will evaluate for finding Eigenvalues in this time interval, you will evaluate at 2 and then possibly at 1 or 3 and so on similarly here like this you go on subdividing the interval till you have separated each of the intervals containing exactly 1 root of P 5 and further continuation in the same process we will squeeze the root for you.

So, I suggest that you continue this process till you find the Eigenvalue is with a accuracy of point 1, that will give you enough practice us and you will find that the method for quite comfortably there was a small error in the calculation in the board work.

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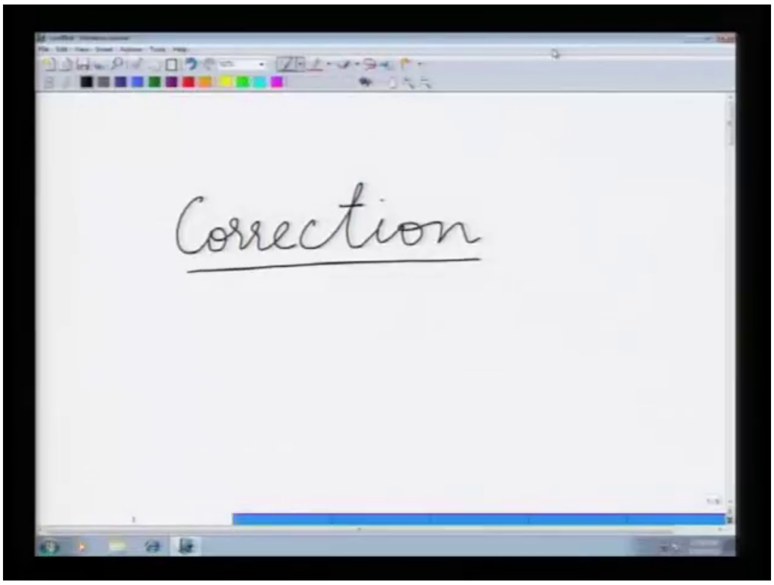

So, please note this correction. Here what you saw in the board was this;

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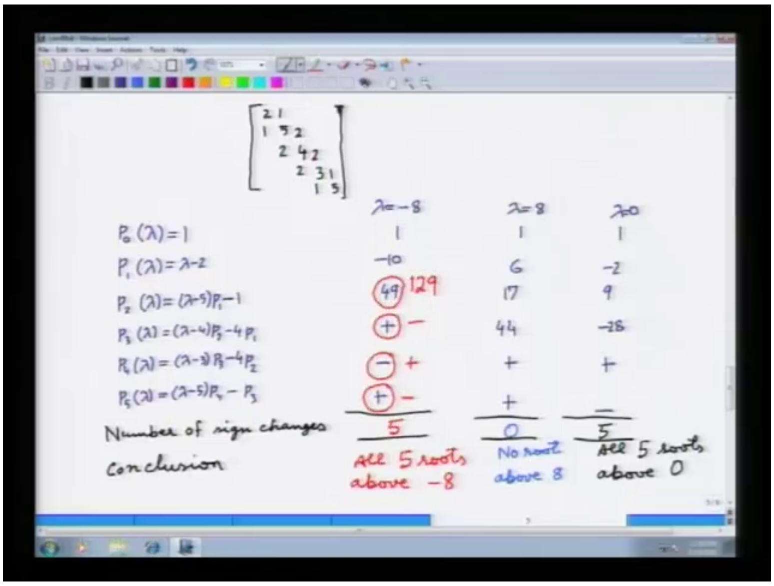

We were analysing the Eigenvalue problem of this problem and this this is what appeared on the board and in this and there is a correction this 49 for P 2 at lambda equal to minus

8 was not right, the correct calculation shows that it should be 129, the result of which was that the next 3 signs were also mistaken and the next 3 signs will be this way minus plus and minus and with this as we will notice that for lambda equal to minus 8 there are 5 sign changes and; that means, that all 5 roots are above minus 8 and then for lambda equal to 8 there is no sign change and that shows that no root is above 8 and in between through bisection then you will evaluate at lambda equal to 0 in which case you will find that all 5 roots are above 0.

So, the first 2 columns in this data for lambda equal to minus 8 and lambda equal to 8, you basically get the verification of the bounds of minus 8 and 8 for all the Eigenvalues and the third column lambda equal to for lambda equal to 0, shows that all the 5 Eigenvalues are positive which means the matrix is positive definite other than this everything else is all right in board work.

Thank you.