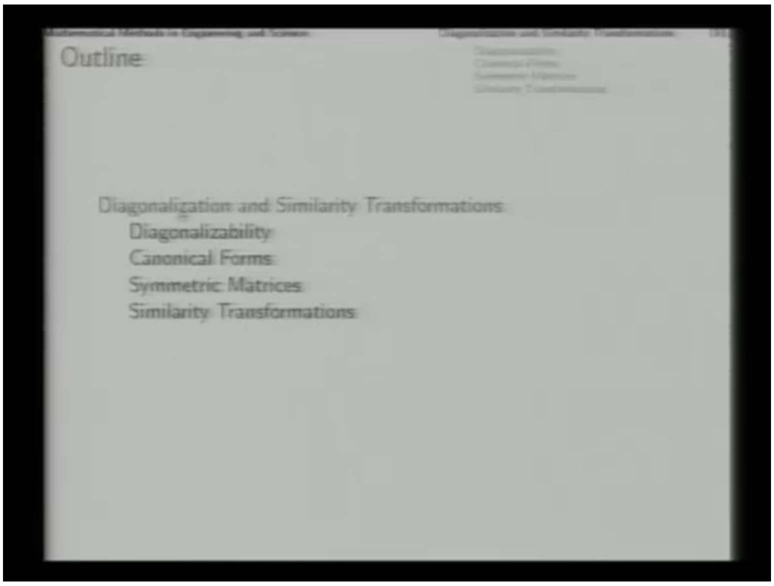
Mathematical Methods in Engineering and Science
Prof. Bhaskar Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

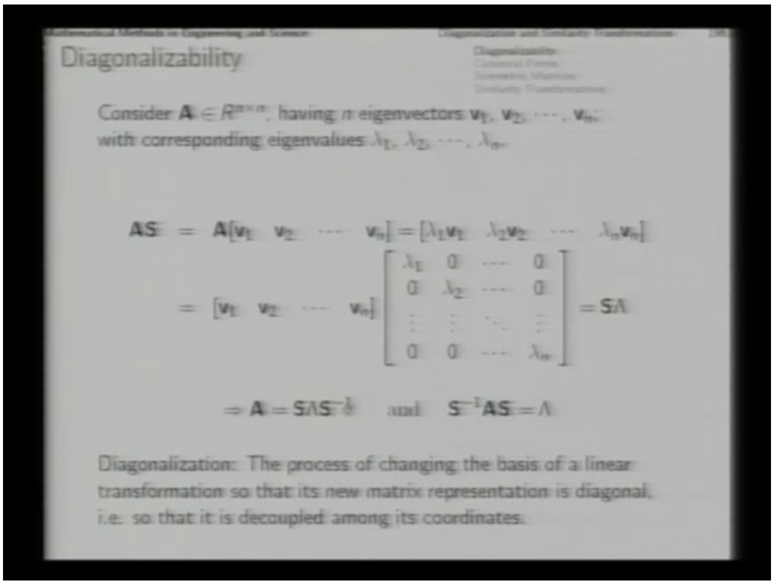
Module – II
The Algebraic Eigenvalue Problem
Lecture – 02
Canonical Forms, Symmetric Matrices

(Refer Slide Time: 00:21)



Welcome in this lecture, we will study Diagonalization and Similarity Transformation. First, we will discuss the issue of Diagonalizability which I initialed in the previous lecture and then we will go through two very important topics in the algebraic Eigen value problem; one is the canonical forms and the other is; the special advantages that we can take of; the symmetry of a matrix, first diagonalizability.

(Refer Slide Time: 00:54)



Consider an n by n matrix; which has n linearly independent; Eigen vectors; v 1 to v n, with corresponding Eigen values; lambda 1 to lambda n. Some of these may be repeated for that matter; that is lambda 2 and lambda 3 can be equal; need not be all different. These are all linearly independent, then only we will talk about n different Eigen vectors. If we have the matrix, which has n linearly independent Eigen vectors; then consider this.

If we pack; all these n Eigen vectors into 1 n by n matrix; with these vectors as columns then we have this n by n matrix here which we are denoting by S. And then we examine the product; A S, now what will be the first column of this product? A into the first column of this matrix S; that is A v 1, since v 1 is an Eigen vector with Eigen value lambda 1; so, A v 1 is; lambda 1; v 1. Similarly, the second column of the product will be A v 2; which is lambda 2; v 2 and so on; till this.

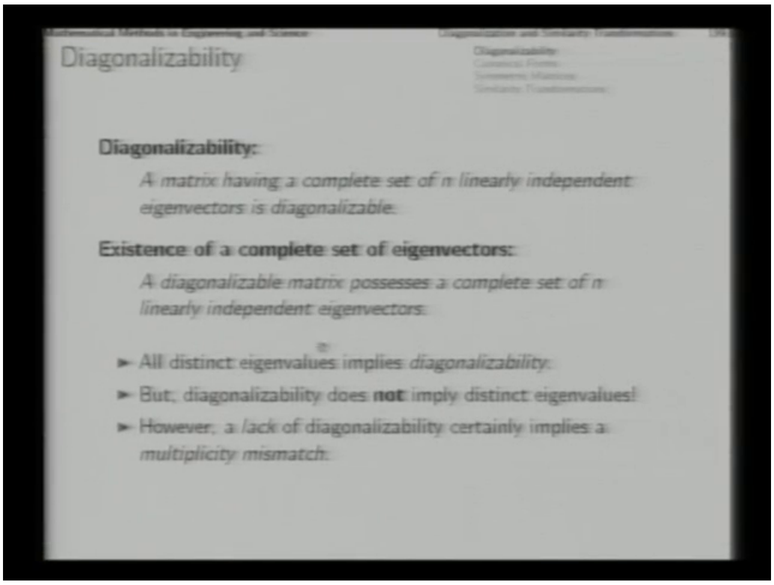
Now, we came that the same product we would get; if we multiple the matrix S from the right side; with a matrix, with a diagonal matrix having the Eigen values as the diagonal entries. Let us verify this a little carefully; what will be the first column of this product? It will be this matrix multiplied by the first column of this; that will be v 1 into lambda 1, which is this, plus v 2 into 0 plus v 3 into 0 and so, on. That is we find; that the first column of this product is also lambda 1; v 1.

Similarly, the second column of this product will be v 1 into 0; thus v 2 into lambda 2, which is this, plus v 3 into 0 and so on. That means, this second column is also same; similarly we will find that all the columns, till the nth column; will be found identical to this. And this diagonal matrix with the Eigen values of A; sitting in the diagonal position we will call as lambda and then this gives us S into lambda; that means, we have got A S; equal to S lambda.

Now, if we post multiply this equality with S inverse; then from this side, S get canceled here we get S lambda; S inverse that is this; that means, that A can be expressed as S lambda, S inverse per lambda is the diagonal matrix with Eigen values sitting at the diagonal locations. Similarly, if we pre multiply both sides of this relationship with S inverse, then we get S inverse A S is equal to lambda; that means, that the matrix A; in the new basis S, gets diagonalized. And this process of changing the basis of a linear transformation; so, that its new matrix representation is diagonal; is called diagonalization.

In this process, the transformation; the mapping gets decoupled among its coordinates. What was the necessary condition for this to happen? The necessary condition was just this; that is the matrix has n linearly independent Eigen vectors or we can say the matrix processors a full set of n linearly independent Eigen vectors. So, this was the only requirement for this diagonalization to be possible.

(Refer Slide Time: 05:09)



So, we can say this about diagonalizability of a matrix; a matrix having a complete set of n linearly independent Eigen vectors is diagonalizable, that is the n by n matrix can be diagonalized, if it has a full set of n linearly independent Eigen vectors.

The converse is also true that is; a diagonalizable matrix processes a complete set of n linearly independent Eigen vectors. If you want to prove this, then you will say that we already know that there exists a basis S; in which the matrix representation will be diagonal. And in that case what you do? If we multiple with S and then you say that we already have this relationship; A S is equal to S lambda; that means, this and this will turn out to be equal and from that you can figure out that A v 1 is lambda 1; v 1, A v 2 is equal to lambda 2 v 2. So, which will mean that all these linearly independent vectors v 1, v 2, v 3, v 4 etcetera are indeed; Eigen vectors, that will inform convince you about the existence of n linearly independent Eigen vectors.

So, this statement and its converse both are true that is; if a matrix possess a full set of n linearly independent Eigenvectors, then it is diagonalizable with that same Eigenvector sitting as the columns of the similarity transformation matrix, giving the new basis. On the other side, if the matrix is diagonalizable; then you can claim that it does have n linearly independent Eigenvectors, a full set. Now, a few important reminders, a few important points which are actually quite simple, but they are sometimes confused. So, note these small simple statements and do not get confused.

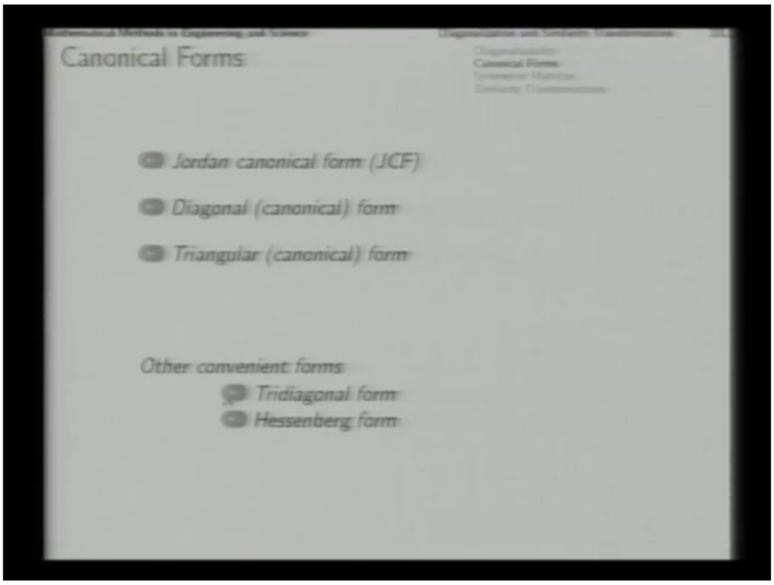
One is all distinct Eigen values will directly imply diagonalizability; because if all Eigen value are distinct then for every Eigen value, there must be an Eigen vector that is necessary for the definition of Eigen value itself. That means, an Eigen value must have one Eigen vector associated with it, if not more. So, if all Eigen values are distinct; that means, n distinct Eigen value are there and each Eigen value will have associated with itself one Eigen vector. So that means, n Eigen vectors are guaranteed and that implies diagonalizability.

But on the other side, diagonalizability does not imply distinct Eigen value because it is possible that the matrix is diagonalizable even with repeated Eigen values. In that case, a repeated Eigen value will have that many Eigenvectors associated with it as its algebraic multiplicity. So, from diagonalizability; we cannot conclude distinct Eigen values, but from distinct Eigen values, diagonalizability can be directly concluded.

However, if the matrix is not diagonalizable then from the first statement itself, we will know that there is certainly some multiplicity mismatch. And for the multiplicity; to have mismatch, the algebraic multiplicity must be greater than 1. So, these points we need to remember; when we deal with matrices. Now, we note that diagonalizability is not possible for all matrices; and that gives raise to two questions.

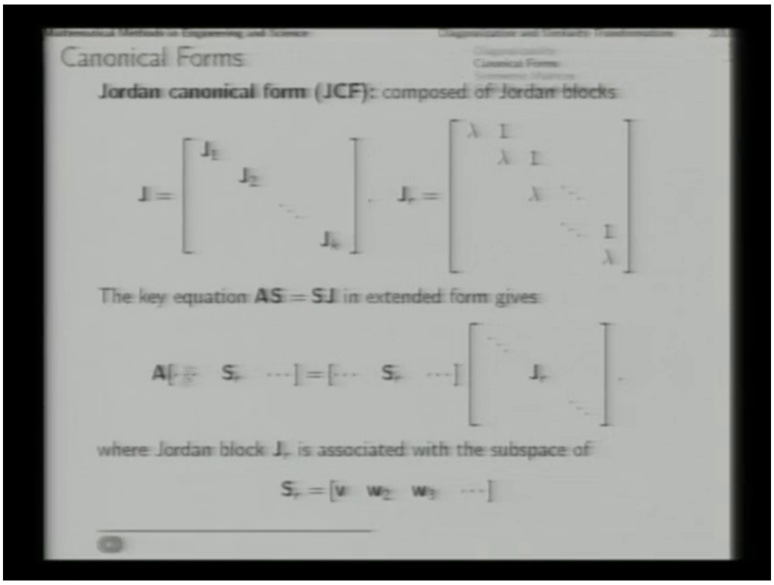
What simplified representation is possible for all matrices? And for what kind of matrices, diagonalizability is guaranteed to be possible? The first is the issue of canonical forms and the second is the questions of symmetric matrices; these are the two important topics that we study in this lesson.

(Refer Slide Time: 09:33)



First canonical forms; there are three forms to which a matrix can be reduced or linear transformation can be expressed through change of basis. They are known as the canonical forms; Jordan canonical form, Diagonal canonical form and Triangular canonical form; first the Jordan canonical form, which is always possible.

(Refer Slide Time: 09:54)

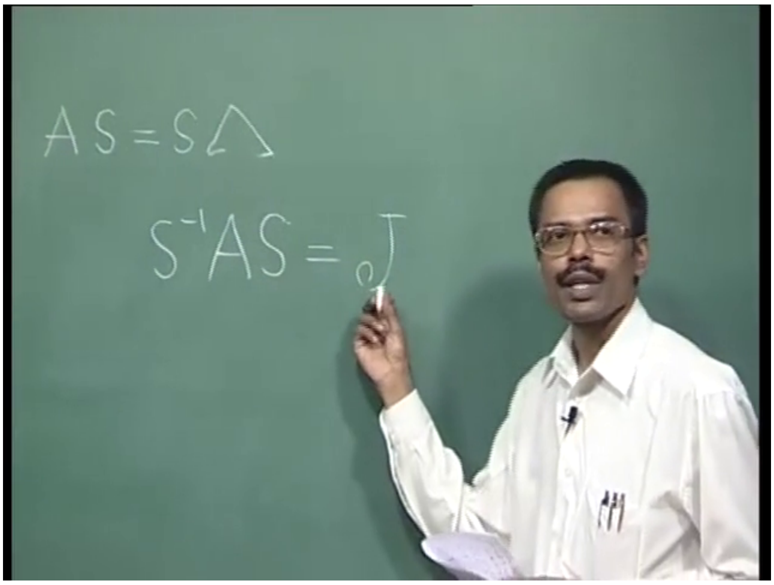


A Jordan canonical form is the simplest possible form, simplest possible matrix representation of a linear transformation; which is possible for all transformations, all matrices. And the form is like this; which is composed of diagonal blocks, these are blocks having small square matrices. About these diagonal blocks and below these diagonal blocks; all the other entries of the matrix are 0.

And these diagonal blocks are also specified in their shape, along that diagonal in such a block say the r S block; J r we have the Eigen values; same Eigen value that is one Jordan block is associated with a single Eigen value, though for a particular Eigen value more such blocks are possible. Now, one such Jordan block looks like this in which along the diagonal entries, the corresponding Eigen value will be there. And just on the super diagonal there will be ones; everything above the super diagonal and everything below the diagonal will be 0.

So, this is the typical shape of a Jordan block; such blocks sitting as diagonal blocks in this block diagonal matrix, will make this Jordan canonical form.

(Refer Slide Time: 11:32)



Now unlike the previous case for diagonalization, which we were studying earlier; where we looked for a suitable matrix S such that, we could say this in which this lambda is diagonal. Here, we will look for this J sitting in place of this lambda. So, the reduction will not be always possible to this diagonal form, but it will be possible always to this Jordan canonical form.

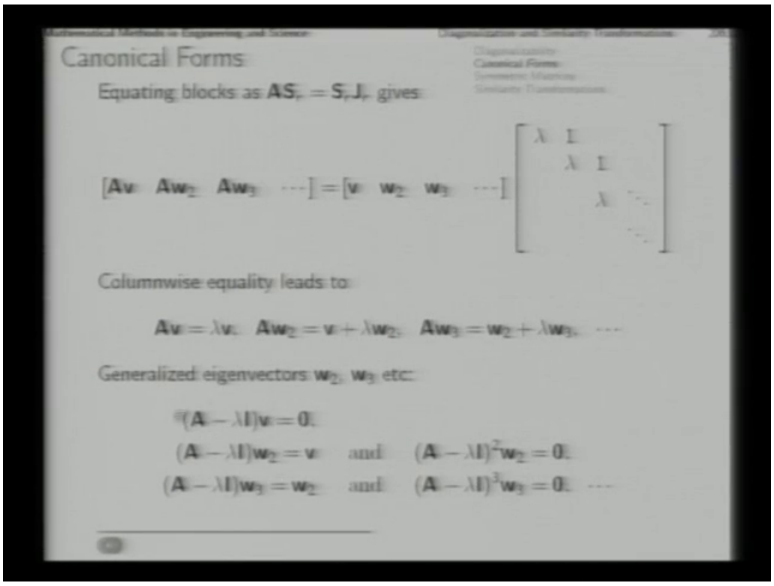
So, the associated similar transformation S, will have bunches of its columns; if J 1 is 3 by 3; then the first bunch of 3 columns in the matrix S will be called S 1; which will have three columns. Similarly, if J 2 is 5 by 5, then corresponding bunch of vectors here will be denoted as S 2; which will have 5 columns and so on. So, that way if there are A Jordan blocks here, then there will be A such bunches with appropriate vectors clubbed together.

And then we will have A into this entire matrix S is equal to the same matrix S multiplied with this Jordan canonical form of the matrix A; like this. Now here if we try to see what is there in this bunch of vectors S r; then we will have a form like this, in which the number of column vectors in S r will be the same as the number of columns or rows in this square block J r.

First of these entries; first of these columns will be an Eigen vector of the matrix corresponding to Eigen value lambda. And next, we will have other vector w 2, w 3, w 4 etcetera; the requisite number to fill up the number of columns which are called generalized Eigen vectors. They are not Eigen vectors, but they are in some sense resembling to the Eigen vectors and they are called generalized Eigen vectors. Now, what are those vectors? We will find out; for that what we do is that; we consider the product A; S r, that will be a bunch of columns taken from the correct location, from this product.

The same will be on this side S r into J r; why? Because on this side, we will try to look for the rth block of columns; then we will have S 1 multiplied 0 block; plus S 2 multiplied with another 0 block and so on, till we reach here; when we get S r multiplied with this non zero J r block; plus again S r plus 1 into 0 block and so on. So, the non zero component of this will be only S r; J r; so A; S r will be S r, J r.

(Refer Slide Time: 14:33)



So, block by block; if we equate like this, then we find A S r equal to S r; J r. Then using the expression for S r from here with columns v; w 2, w 3 etcetera, we can write like this A v, A W 2, A W 3 etcetera in the product for A S r. On this side, S r is this and J r shape is this; that will give us v multiplied with lambda as the first column. In the second column, there will be two terms; v multiplied with 1 and w 2 multiplied lambda.

So, that way if we equate column by column and the first column equality will give us A v equal to lambda v. Second column equality will give us A w 2 is equal to 1 into v; plus lambda into w 2; that is this. Third column will give us A w 3 equal to 0 into v; 1 into w

2 and lambda into w 3; that is this and so on. So, from here the first one is already familiar to us; this is somewhere we determine the Eigen values.

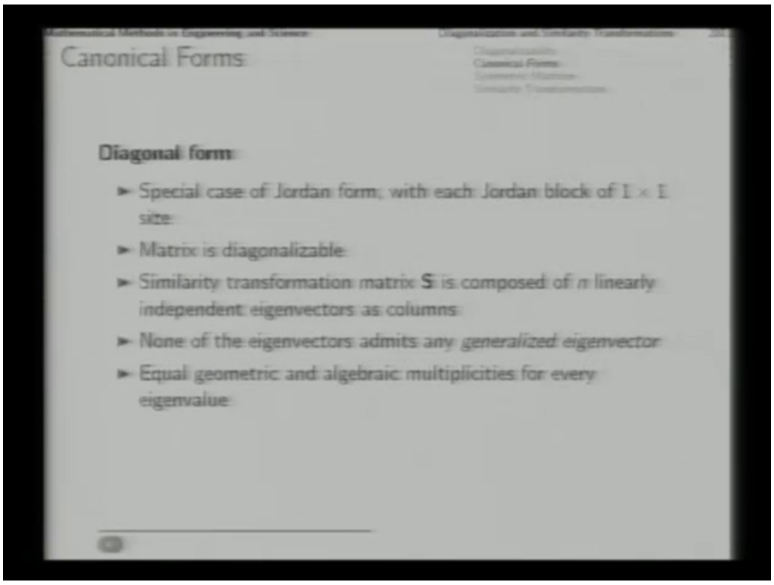
Then once the Eigen vector; so Eigen vector we determine from this equation. Once the Eigen vector v is determined; the next generalized Eigen vector that is a first generalized Eigen vector is immediately after the Eigen vector; that will be found as a minus lambda I into w 2 equal to v.

Once w 2 is found, we can find out w 3 like this; from this relationship and so, on as there are number of columns in S r. So, compared to that number; one less generalized Eigen vectors will be found because the first slot is taken by the Eigen vector itself. After that; if we try to find one extra generalized Eigen vector, we will find that the system of equations that come out; will be in consistent. So, that exactly that many generalized Eigen vector, we will find as many are really required to fill the block.

Now, with these Eigen vector and generalized Eigen vector sitting in columns; we will have the full matrix S, which in this kind of S transformation; this is change will not transform A to diagonal form in general, but it will reduce it to the Jordon canonical form. And this canonical form is always possible for all matrices; all square matrices. The second canonical form is the diagonal canonical form which we have already seen, but now we will have another quick look at it; to see, what is its relation to the Jordon canonical form?

In the Jordon canonical form; which is like this if all Jordon blocks are of 1 by 1 size; then what happens? If this is not such a big matrix, but if it is a 1 by 1 matrix; that means, it will be just a lambda and there will be no place for that super diagonal 1 and that is the case of diagonal canonical form; it is a associated points.

(Refer Slide Time: 18:00)

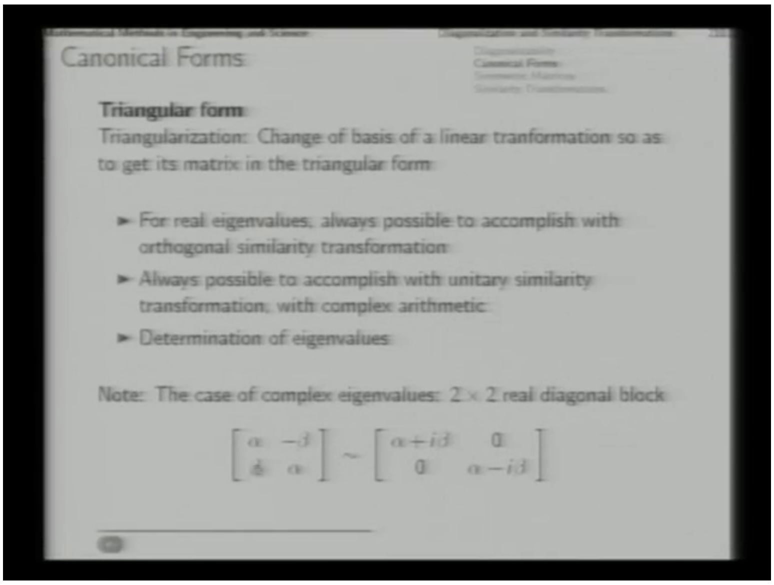


We can quickly note; first is that the diagonal form is a special case of the Jordon form, with each Jordon block of 1 by 1 size. This will immediately mean that with the absence of those super diagonal ones, you have only diagonal entries in that canonical form and; that means, that the matrix is diagonalizable. And that will also mean; that each S r bunch will have a single vector, the Eigen vector itself sitting there.

So, that will mean that the similarity transformation matrix S is composed of all Eigen vectors; that is n linearly independent Eigen vectors as columns; that means, all linearly independent Eigen vectors exist in this case and the Jordon block size of 1 by 1 size or the matrix is diagnosable. And in that case, if you try to find generalized Eigen vector corresponding to an Eigen vector; already found, then you will find that none of the Eigen vectors will admit any generalized Eigen vector; the correspond equation will turn out to be inconsistent.

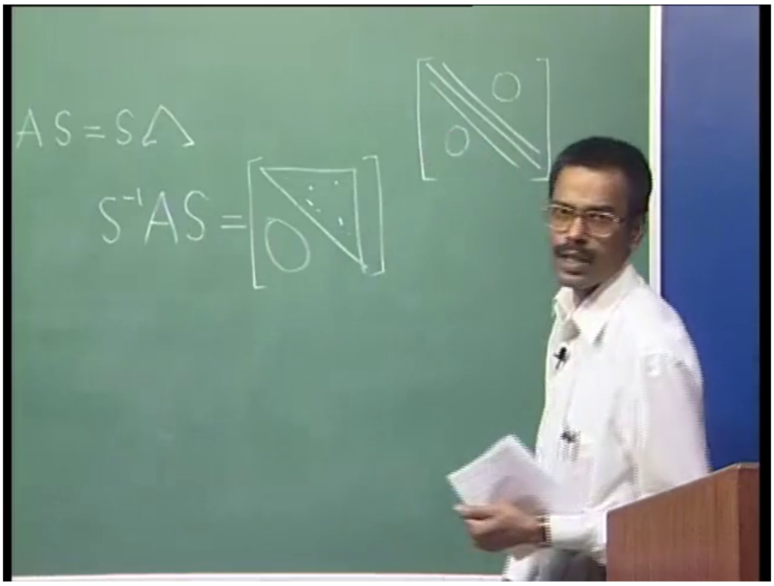
This will also mean that for every Eigen value, the geometric and algebraic multiplicities are same. There is a third canonical form; which is of a very important practical significance and that is triangular canonical form. We have already come across triangular matrices in our study of systems of linear equations. Now, here when we say triangular canonical form of a matrix, then we are actually referring to the triangular form of the linear transformation. That means that we are talking about converting a given matrix to the triangular form through similarity transformations.

(Refer Slide Time: 19:55)



Now, special significance of triangular form arises from the triangularization which is always possible. What is triangularization? Triangularization of a matrix or of a linear transformation; is basically the change of basis of a linear transformation in such a manner; that its matrix is in the triangular form.

(Refer Slide Time: 20:29)



That is apply some suitable similarity transformation on the matrix such that the form of the resulting matrix in the new basis turns out to be like this; with all 0's here, non zero entries only on the diagonal and above; below that everything else is here this is the triangular form.

Now, the practical significance of it is that this is; always possible and this is possible always through orthogonal similarity transformation. The Jordon canonical form is always possible, but for Jordon canonical form; the S that you need is not necessarily orthogonal. Triangularization, you can always conduct with orthogonal S and change of basis to orthogonal transformations has a lot of practical advantage, the advantage is both analytical as well as computational.

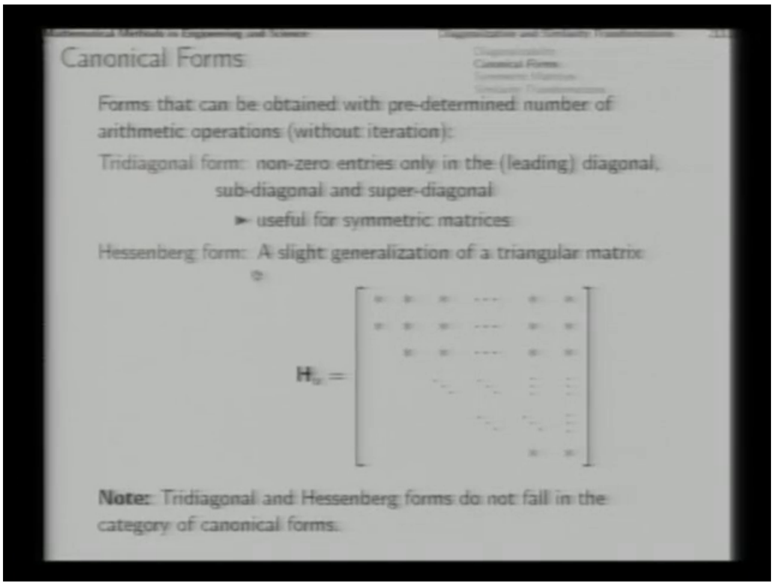
So, we find that the triangular form is always possible; in particular if the Eigen values are all real, then it is always though orthogonal similar transformation. Even if Eigen values are not real, even if you have complex Eigen values for the real matrix; even then you can take recourse to complex arithmetic in your calculations. And you will be always able to triangularize the matrix with unitary similarity transformation. Whatever holds for orthogonal similarity transformation in the case of real Eigen values, in case of complex Eigen values; the same will be through with unitary similarity transformation that is S in that case it will not be necessarily orthogonal, but it will be unitary; which is a complex analogue of orthogonal matrices.

Now, if you insist on working with real arithmetic only and orthogonal similarity transformations only; even through there are complex Eigen values, then you can almost do the triangularization. Except that for a pair of complex Eigen values, you may be left with real diagonal blocks of 2 by 2 size like this. Because this is actually equivalent to the triangularized version; which is this and you will never be able to reduce this matrix till this point, unless you allow complex arithmetic in your calculations.

But then with this kind of diagonal block sitting, you will be able to recognize that you have a pair of complex Eigen values there. Other than that the rest of it, you can do even if there are complex Eigen values. Now, if you can reduce; so, orthogonal similarity transformation a matrix to this shape; you have not completely solved the Eigen value problem; in the sense that you have not been able to determine the Eigen vectors completely, but Eigen values are all there along the diagonal.

So, that way if you are first interested particularly in the determination of Eigen values; then with much less amount of computation, you can reduce it to triangular form and get the Eigen values. And once Eigen values are determined, there are some methods which help you in finding the Eigen vectors with less cost. Now other than these three canonical forms, there are two other forms which are actually not canonical forms, but which have some important advantages, when we talk about computational methods, for solving the Eigen value problem. One is the Tridiagonal form and the other is the Hessenberg form.

(Refer Slide Time: 24:17)



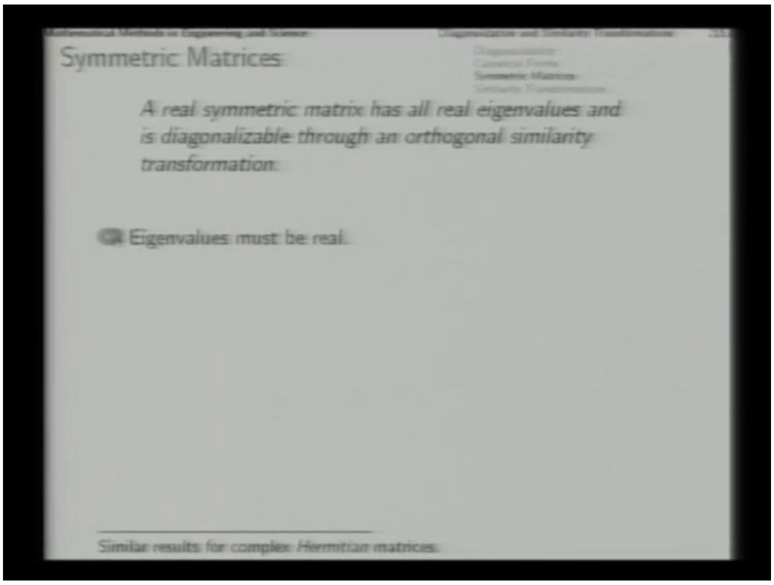
These forms are advantages in the sense that; that can be obtained through a pre determined number of arithmetic operations. That is reduction to these two forms is not iterative, they can be done with a constant number of calculations. The reduction to other forms; that is Jordon canonical form or diagonal form or triangular form; they might need iterations and there is a question of convergence.

But these two forms; though not canonical forms, but they are useful important forms of matrices to which we will try to reduce the matrix, to similarity transformations. And these two forms are of advantage because the reduction to a tridiagonal form or hessenberg form can be accomplished with a free determined number of arithmetic operations; a straight forward operation applied only once; not iterative.

So, tridiagonal form as the name suggests has non zero entries; only around 3 diagonals. The main diagonal, the super diagonal and the sub diagonal and everything else is 0. So, this is the tridiagonal form; the reduction to this form for any matrix is; that is for those matrices which we apply this; is a matter of fix number of calculation, depends only on the size of the matrix and there is no iteration involved, there is no question of convergence.

Similar situation is there with Hessenberg form; in which other than the upper triangular matrix, one sub diagonal is extra. To this stage, reduction can be done in a fix number of arithmetic operations and after that to apply further similar transformations so as to reduce these entries to 0; that may take a lot of iterations. So, Hessenberg form is used typically for handling non symmetric matrices; on the other hand for symmetric matrices, we will typically try to handle it through tridiagonal forms.

(Refer Slide Time: 26:34)

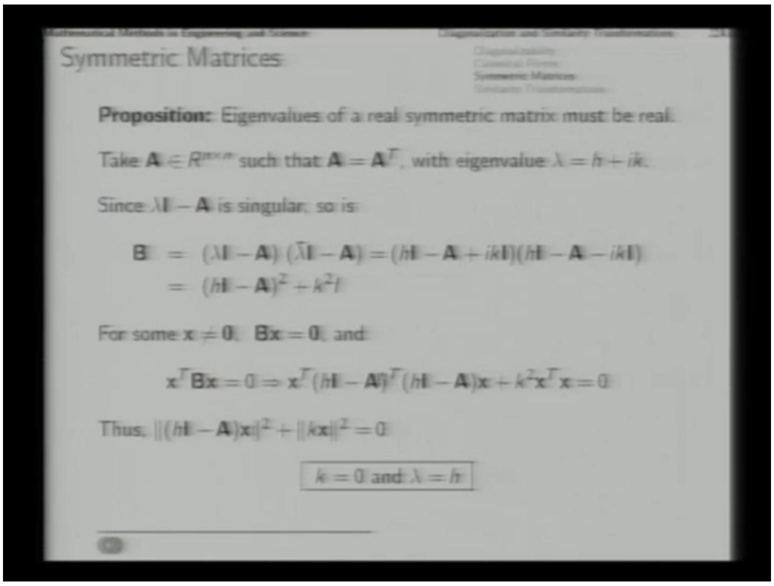


Next we come to the most important topic of this lesson and that is the topic of Eigen value problems of symmetric matrices. Central to this issue; is this very important result and that is; a real symmetric matrix has all real Eigen values and it is diagonalizable through an orthogonal similarity transformation. A similar result is there for complex permission matrices, for which this is actually a special case.

Since in our course; we will be mostly concerned with real matrices; that is why in all the discussions, I am trying to concentrate on the real versions of the theorems rather than going into the complex version. But the complex permission matrices version is also very similar that would read as a Hermitian matrix as all real Eigen values and is diagonalizable through a unitary similarity transformations and no other change. The steps through which you establish that result, is also similar to the one that we are going to go through right now.

Now, this one sentence; actually has got built into it is several smaller statements. First is a real symmetric matrix has all real Eigen values; that is first issue is that Eigen values must be all real; how do we convince ourself of that truth of this statement?

(Refer Slide Time: 28:13)



So, let us consider this; pixels as the independent propositions, that is Eigen values of a real symmetric matrix must be real.

For this what we do? We assume that there is a matrix A; n by n matrix A, which is symmetric; that is for which A is equal to A transpose and its Eigen value is; one of its Eigen values is lambda which is h plus i k. Now, what we need to prove? We need to prove that the Eigen value must be real; that means, this lambda must be real; that means, its imaginary part A should be 0; that means, this is the hypothesis and this is; what we want to establish, what we want to show.

So, what you do? You say that since A has an Eigen value, which is lambda that will immediately mean that lambda I minus A is singular; lambda I minus A into v equal to 0; that means, it has null space in which there is vector v and so on. So, lambda I minus A is singular; if lambda I minus A is singular, in this matrix is singular; then any other matrix multiplied to it, the product will also be singular.

So, this product is also singular that is B is singular. Now, note what we have put here to multiply with lambda I minus A? Lambda bar I minus A, because we want to establish something in terms of real quantities, we want to kill whatever complex imaginary stuff is here and therefore, we bring in the complex conjugate.

Now, use lambda is equal to h plus i k. So, if we insert lambda equal to h plus i k here and expand this, so lambda is h plus i k, lambda bar will be h minus i k. So, this gives us h I plus i k identity; minus A and here we will have lambda bar i; that means, h I minus i k I; minus A.

Now, you note this h I minus A into h I minus A; that is h I minus A whole square; i k identity and minus i k identity, that will give you minus i square; small i square, k square identity; i square, small i square is minus 1. So, that minus i square is 1; so, you get only this. Now, you found that since A has an Eigen value lambda; so, lambda A I minus A is singular and therefore, B is also singular; which is product of this with something else, this also singular.

If B is singular; then it has a null space, which has at least one vector in it; that is if B is singular, then there must be a vector to which B multiplies and gives 0. So, let us consider that vector as x; some non zero vector x will be there, to which B will multiply and give us 0. And then, if we pre-multiply both side with x transpose; that also will be 0; that will be a scalar 0. So, you get x transpose; B x equal to 0, in this relationship; in place of B; co insert this.

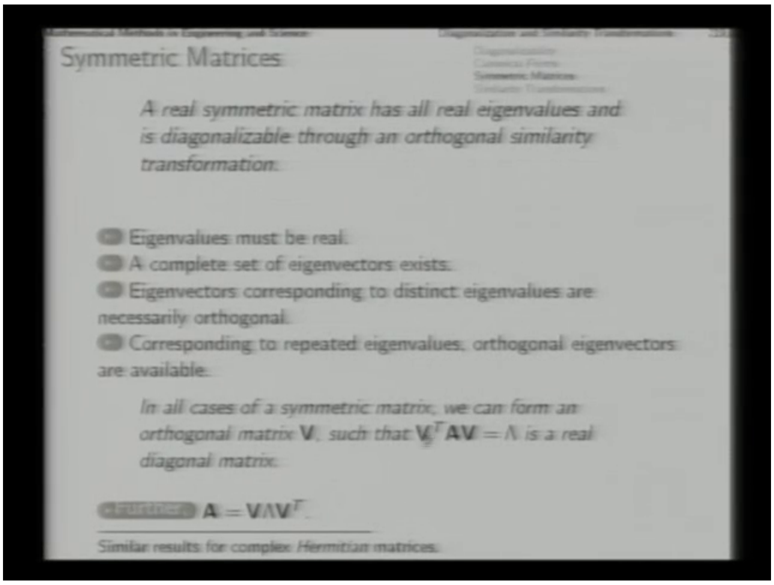
And here; at this insertion point, we have used just symmetry of A; x transpose, this whole thing into x. So, the second part is very easy here; A square is scalar and rest is x transpose identity into x that is this. The first one is; x transpose h I minus A into h I minus A x. The second h I minus A; we have left as it is, the first h I minus A; we have replaced with its transpose.

This is valid because A is symmetric because identity is anyway symmetric, if A is also symmetric then h I minus A and its transpose is same. Now, note from here to here; we have h I minus A, multiplied with x and from here to here, we have exactly h transpose. That means, we have got this fellows transpose multiplied with this fellow; that means, the non square; that is this part is non square of h I minus A, and this part is non square of k; so, we have got this equal to 0.

Now, you see norm is a positive quantity, norm square is certainly a positive quantity. Now, you have got the summation of two positive quantities or rather non negative quantities; two non negative quantities is equal to 0. Then since neither of them is going to be negative; so, for the sum to be 0; each of them must be individually 0. And that means, x is a non 0 vector; so k must be 0 and we come to the conclusion of our proof; k is 0, which will mean that lambda is h.

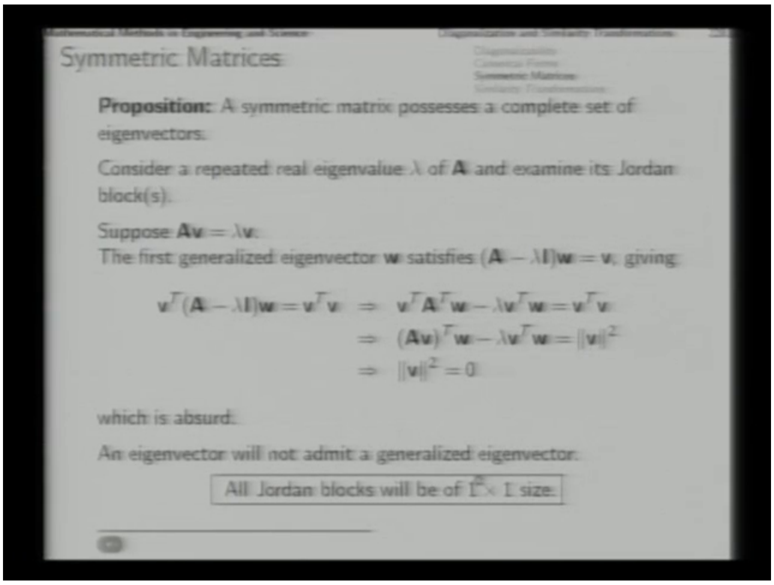
So, what does that show? That shows that only due to the symmetry of the matrix, which has been used at this step; we come to the conclusion that k must be 0; that means, lambda is (Refer Time: 33:44).

(Refer Slide Time: 33:49)



Now, the first point of the proof is established; other than this real Eigen value issue, what else is there in this statement? This statement it is the matrix is diagonalizable; that means, it has a full set of n Eigen vectors; a complete set of Eigen vectors exists for a symmetric matrix, we consider this statement separately.

(Refer Slide Time: 34:15)



A symmetric matrix possess a complete set of Eigen vectors; for that what we do; is that we consider a repeated real Eigen value lambda of A. Because, if all Eigen values are distinct then we already know that each distinct Eigen value will be associated with one Eigen vector, which will immediately tell us that n Eigen values will give; n distinct Eigen vectors.

So, there is nothing to examine there; so, what we do is that; we consider a repeated Eigen value, which might be at some problem. So, we consider repeated real Eigen value of the matrix and examine its Jordon blocks; what we want to establish? We want to establish that all Jordon blocks will be of 1 by 1 size. There will be no place to write that super diagonal one; so, if all Jordon blocks are of 1 by 1 size, then that will be a diagonal matrix; so, this is what we want to establish.

So, what we do is that corresponding to that Eigen value lambda, suppose Eigen vector is v; then we will have A v equal to lambda v. And we try to find out; we try to determine the first generalized Eigen vector w; which must satisfy this relationship. Now, if it has to satisfy this relationship; then by pre multiplying both sides of this relationship with v transpose; that is by taking dot product or inner product of this with v, we will get v transpose, this whole thing is equal to v transpose v. Now, we open this; v transpose A w; here in place of A; let us write A transpose; that is valid because A symmetric, this is the place where we are utilizing the symmetry of the matrix.

V transpose A w; as written as v transpose; A transpose w; minus lambda v transpose identity w; lambda v transpose w, that is equal to this. Now, the right side v transpose v is norm v square; that is obvious. Here v transpose, A transpose is the transpose of A v; we have got A v transpose w, and here it is written as it is, but then since v is Eigen vector corresponding to lambda Eigen value; that will mean that A v is the same as lambda v; and what is this?

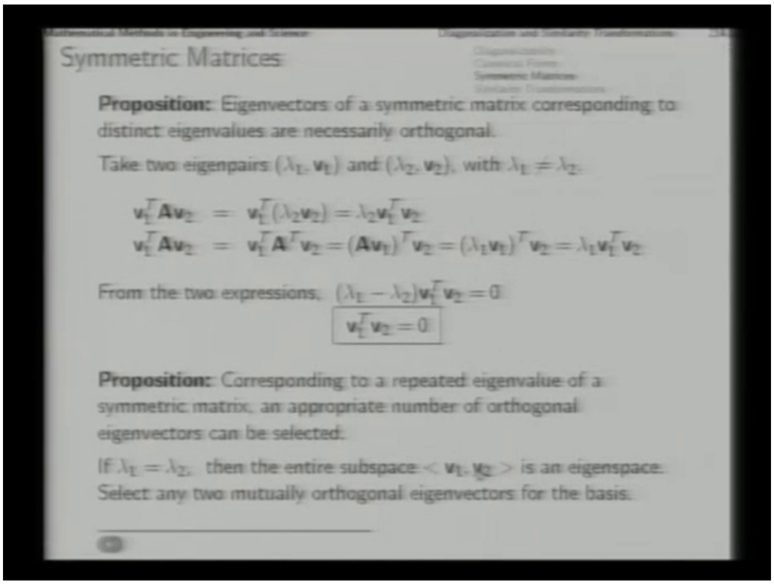
This is lambda v transpose w minus lambda v transpose w; which is 0 and; that means, v norm square is 0; what does that mean? That means, v is a 0 vector, but that cannot be the case because v is the Eigen vector for which linear independence is a must to qualify as Eigen vector. So, in which the direction is the only information.

So, you cannot have an Eigen vector which is 0; so, this is absurd. That means, it is not only absurd; it basically means that this gives rights to an inconsistency; that means, there will be no w, no generalized Eigen vector which will satisfy this; that means, that the Eigen vector will not admit any generalized Eigen vector, that will mean that all Jordon blocks are of 1 by 1 size; which means that it is diagonalizable. So, we have established two points from that statement; as a real symmetric matrix has all real Eigen values and it is diagonalizable.

But till now the matrix S related to the diagonalization, the similar transfer matrix and be anything. The further statement says that it is diagonalizable through an orthogonal similarity transformation; that it in the diagonalization possess, we can use a matrix S which is orthogonal. That means, the matrix S which houses the Eigen vectors should have all mutually orthogonal columns, do Eigen vectors have to be mutually orthogonal?

We say that in two parts; first we say that Eigen vectors corresponding to distinct Eigen values; are necessarily orthogonal.

(Refer Slide Time: 38:54)



So, here the proposition is that Eigen vectors of a symmetric matrix, corresponding to distinct Eigen values, unequal Eigen values are always orthogonal. They must be orthogonal; to show this, we take two Eigen pairs; two Eigen values; lambda 1, lambda 2 and their corresponding Eigen vectors v 1 and v 2; with the statement that lambda 1 and lambda 2 are not equal. And we want to show that v 1 and v 2 are orthogonal; they must be orthogonal.

So, for that we take a very simple means to establish this; we ejaculate v 1 transpose A v 2 in two different ways. In the first case, we simple take A v 2 as lambda 2; v 2 because lambda 2 and v 2 are the Eigen value; Eigen vector pair. So, in place of A v 2; we write lambda 2 v 2; lambda 2 being a scalar comes out and we get lambda 2 into v 1 transpose v 2.

In the second case, in place of A; we use A transpose; that is the place where we use symmetry. And in that case v 1 transpose; A transpose is the transpose of A v 1, but A v 1 from here is lambda 1 into v 1. And that will tell us that this turns out to be lambda 1 into v 1 transpose v. Now note; that the same expression evaluated in two different ways without utilizing symmetry and utilizing symmetry gives us two different expression, so we subtract.

On this side, subtraction will give us 0; on this side what it will give it will give? Lambda 1 minus lambda 2; v 1 transpose, v 2; that is 0; we have already taken the assumption that lambda 1 and lambda 2 are not equal; that means this factor cannot be 0. So, only way this can happen is at this factor must be 0; that means, v 1 and v 2 are necessary the orthogonal.

So, this is the case for distinct Eigen values; what will be the situation for equal Eigen values? That is an Eigen value appearing twice, giving us two Eigen vectors; do they also have to be necessarily orthogonal? Not necessary, but then see this. What we want to see here? We want to establish; that corresponding to a repeated Eigen value of a symmetric matrix and appropriate number of orthogonal Eigen vectors can be selected; what is idea behind it?

If lambda 1 and lambda 2 are same; unlike this case then the entire sub space v 1 and v 2 is an Eigen space. So, if there are two vectors; which are Eigen vectors corresponding to the same Eigen value, then it is not necessary that they are orthogonal, but the entire plane found by them is an Eigen space. That means, any vector in that plane is an Eigen vector; so, if in a plane; we have infinite possible Eigen vectors available, then two mutually orthogonal Eigen vectors, we can always pick up.

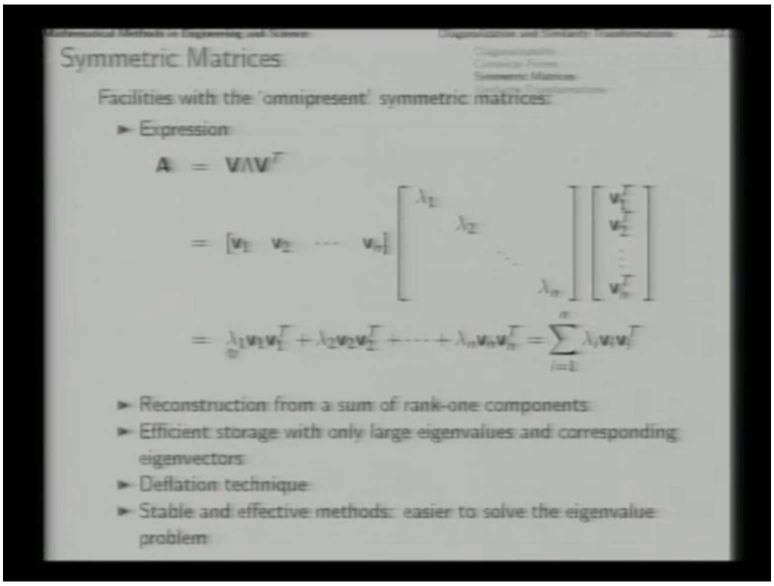
So; that means, we can select any two mutually orthogonal Eigen vectors called using in the basis; that is for filling up the appropriate columns of this matrix S; that is corresponding to repeated Eigen values, orthogonal Eigen vectors are available. The Eigen vectors that we pick up, do not have to be orthogonal, but if we want we can always get orthogonal Eigen vectors.

So, that is why; it says is diagonalizable through an orthogonal similarity transformation that is; it is possible to work out an orthogonal similarity transformation matrix with which we can diagonalize the symmetric matrix. Further, what we get? So, you see that in all cases of a symmetric matrix, we can form an orthogonal matrix v; such that in place of v inverse, in this case we were writing S inverse.

Now, since we are talking about orthogonal matrix; v in place of S; so, for orthogonal matrix v inverse is same as v transpose; which is lot easier, v transpose A v is lambda; which is a real diagonal matrix.

Further try to pre multiply this equation with v and post multiply with v transpose. Then you get this, the matrix A can be represented in this manner; v lambda v transpose, where v is an orthogonal matrix and lambda is the diagonal form of A; that diagonal form is always possible and that is always possible through an orthogonal similarity transformation v.

(Refer Slide Time: 43:59)



This gives us a lot of facilities and this is greatly helpful because symmetric matrices appear in many many locations, in the analysis, in Applied Science and Engineering and it helps that; so, nice properties. So, interesting and useful properties of symmetric matrices are there and symmetric matrices appear in most of the application again and again.

So, most of our problems are comparatively easy and the enormous amount of facilities that this representation gives us is here. First of all this expression A equal to V lambda, V transpose can be written in expanded form like this. If you try to multiply, this three matrices and open it in the form of an expression; you will find that you get lambda 1; v 1, v 1 transpose.

Because you see, if you first multiply these two; then you will get what? You get lambda 1 into v 1 transpose plus 0 into this, plus 0 into that and so, on. So, here in those you will find lambda 1 v 1 transpose, lambda 2 v transpose, lambda 3; v 3 transpose and so on. With which when you multiply this; then we will get v 1 into lambda 1; v 1 transpose plus v 2 into lambda 2; v 2 transpose and so, on.

So, finally, you will get this summation; lambda 1; v 1 transpose, v 1; v 1 transpose plus lambda 2 v 2; v 2 transpose and so, on; you can express it like this. This give raise to a further lot of possibilities; one is that if a particular matrix; huge matrix 4000 by 4000 matrix has Eigen values, which are organized in descending values, descending absolute values; the first 10 are large and compared to them, the next ones are extremely small.

Then what you can do? For the storage of that 4000 by 4000 matrix, you can simply store the first 10 Eigen values; lambda 1 to lambda 10 and throw away lambda 11 to lambda 4000 and the corresponding first 10 Eigen vectors, you store other Eigen vectors also you can throw away.

Later when you need to reconstruct the matrix; these 10 Eigen vectors with their corresponding Eigen values, will be able to reconstruct the matrix completely by a summation of not 4000 items like this, but just 10. Because the contribution of rest of them will be extremely small, this is one advantage. Efficient storage with only large Eigen values and corresponding Eigen vectors; rest of the things, you do not have to store.

In deflation technique, we have already seen the application of this expression that is; and this works only for symmetric matrix. If the matrix is symmetric, then there is a representation like this and after we have determined v 1; then from A, if we subtract this; then what remains has the same Eigen structure as A; except that, its Eigen value corresponding to Eigen vector v 1 transfer to be 0; rather than lambda 1.

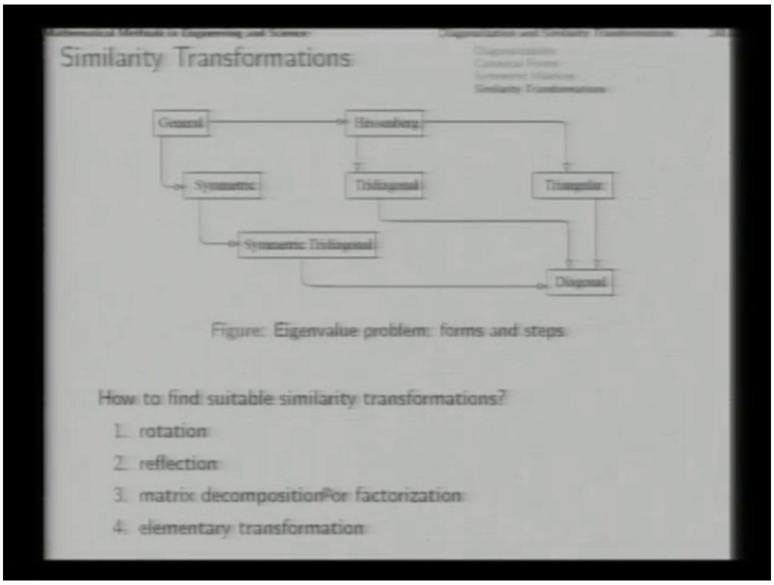
The rest of the Eigen values and Eigen vectors are unchanged; that is the deflation technique which helps us in finding if you top Eigen values and corresponding Eigen vectors. Apart from that, the orthogonal diagonalizability of the matrix in the case of symmetric matrices; helps us in working out practical algorithms, which are stable in which the numerical errors do not grow very fast; as iteration proceed.

Therefore, whenever there is a choice between applying a general similarity transformation and applying an orthogonal similarity transformation, computationally we always preferred to apply orthogonal similarity transformation. And in the case of symmetry matrices, orthogonal similarity transformation alone suffice to reduce the matrix completely to diagonal form.

In the case of non symmetric matrices first of all diagonalizability is not always guaranteed; even when the matrix is diagonalizable; reduction to diagonal form is not always possible through orthogonal similarity transformations. In fact, that is not possible; so you have to take the help of similarity transformations, which are not orthogonal in the case of non symmetric matrices. In the case of symmetric matrices, you can conduct the entire operation with only orthogonal similarity transformation.

This is why solution of symmetric matrices; Eigen value problem is a lot easier compared to general non symmetric ones. Now, the complete picture of different forms; some the raw form and some the final desired form that we have.

(Refer Slide Time: 49:24)



We can see here and any block here in this symmetric diagram, which is on the right or which is below is typically easier to handle for the Eigen value problem, compared to corresponding other blocks which are on the left or above. So, in that understanding; we find that compared to a general matrix, all other matrices; general means which may be non symmetric; all other forms are easier to handle.

And the diagonal matrix is the one in which the Eigen value problem is actually; already solved. So; that means, if we have a matrix in one of these forms on this side; then any algorithm, any part algorithm which helps us to move from this end to the right side or below south east side, then that is one contribution to the solution of the algebraic Eigen value problem.

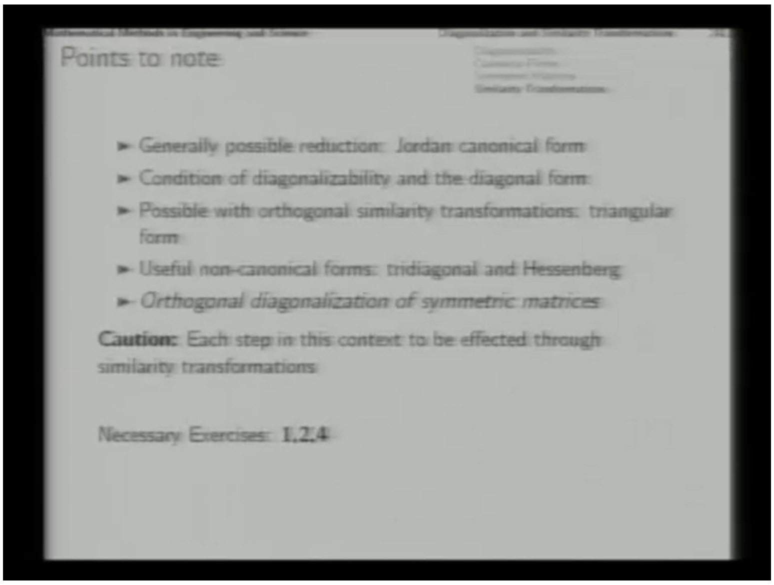
That is from the general matrix, we might try to reduce it to either the Hessenberg form or the symmetric form. If we have a symmetric matrix, then we try to reduce it to Symmetric Tridiogonal form, if we have it already in Hessenberg form, then we try to reduce it to tridiagonal form, which is comparatively easier. Or we will try to reduce it to triangular form, which is more often the case. From the triangular, or tridiagonal or symmetry tridiagonal; another round of reduction, will take us to diagonal form; if a diagonal form exists for that matrix. In this case, it will certainly exist and any moment along the arrows, will mean that we have accomplished one more stage in the solution process of this Eigen value problem.

And all these steps, all these reductions must be carried through similarity transformations only; that is we must multiply the matrix A on the right side with one matrix. And with the left side with its inverse, then only it is a similarity transformation and that will mean that it is basically the expression of the same linear transformation; in new basis; the basis S.

So, through similarity transformation only; we must do all these transformations; straight forward deduction like (Refer Time: 51:52) only; from one side will damage the Eigen structure. So, the similarity transformations should be applied like this and through this any step is preferable, if it helps in the direction like this; like this or like this or any way like this. So, the question arises how to find suitable similarity transformations which help us in moving from this direction to this direction in general? That is reduction of the problem; how to find suitable similarity transformations?

There are four standard ways of working out suitable similarity transformations, they are based on rotation, reflection, matrix decomposition or factorization and elementary transformation; these four methods of finding suitable similarity transformation, we will study in the coming lectures or coming lessons.

(Refer Slide Time: 52:58)



For the time being, to summarize the important issues that we have seen in this discussion is that; generally possible reduction which is possible for all matrices is up to Jordon canonical form. Condition of diagonalizability and the diagonal form, we have studied and we have studied the form; which is triangular form, which is possible with orthogonal similarity transformation.

Note here, that in the previous chapter in the book; there is an exercise which gives you the steps necessary to show this important result, to establish this important result; that any super matrix can be reduced to a triangular form with only orthogonal similarity transformations. This is an important result and I will strongly advice that this particular exercise, which gives you the steps to establish this important result, you must go through.

The other useful non canonical forms are tridiagonal and Hessenberg forms; that we have come across briefly. And the most important lesson of this particular chapter, of this particular lecture is that orthogonal diagonalizational of symmetric matrices is always possible. And all these reductions must be carried through similarity transformations only. So, I would also advice you to go through some of the exercise of this chapter before proceeding further to the next lecture.

Thank you.