Mathematical Methods in Engineering and Science
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Module - II
The Algebraic Eigenvalue Problem
Lecture - 01
The Algebraic Eigenvalue Problem

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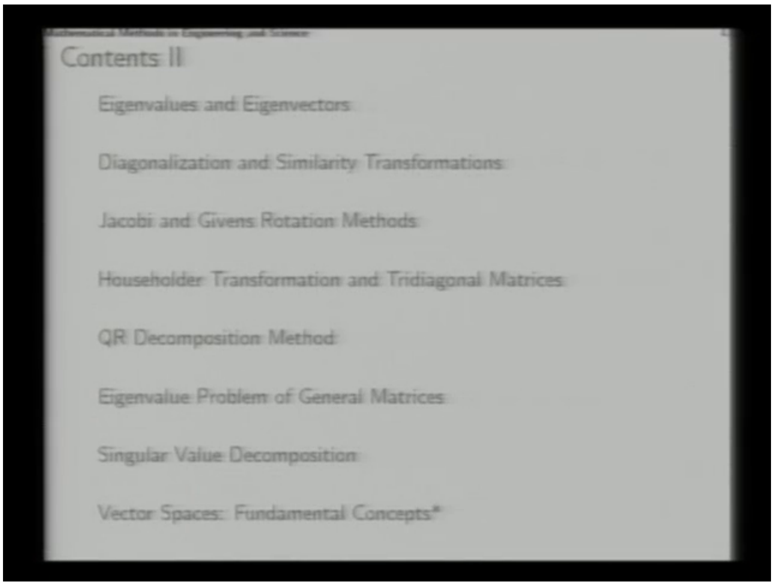

Good morning, as discussed in the previous lecture today in this lecture we will start from this lesson which is the first of a few lessons on the module of algebraic Eigenvalue problem. I will again remind you that in order to follow the lectures in this segment, it is very important that the subject matter of this segment this module is thoroughly emerged in your understanding and therefore, it is very important that at this stage you should have completed most of the exercises of this segment, because some of the background necessary for the following lectures is actually developed through the exercises in the book in the text book that I have referred to you.

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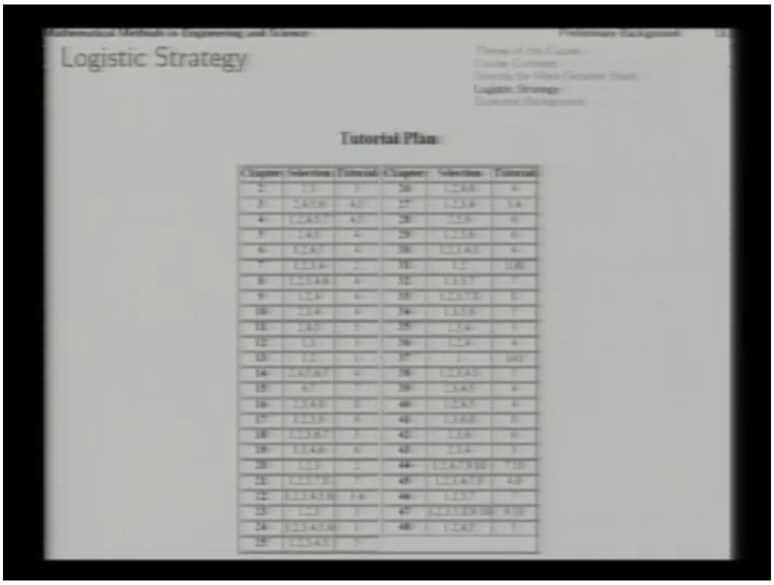

In this tutorial plan the problems of the book listed here you must have completed by now and that will help you in following the lectures, in the coming module which is chapters 8 to 14.

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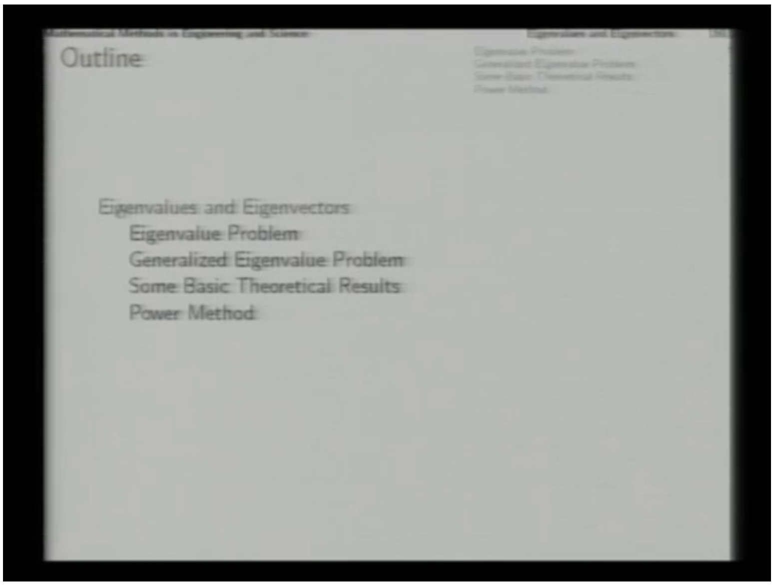

Now, in this lecture we will be studying Eigenvalues and Eigenvectors, in which we will talk about the Eigenvalue problem as an introduction and then generalized Eigenvalue problem which will also expose you to the to one of the practical problems from which Eigenvalue problems emerged. Then we will discuss some basic theoretical results which will be utilized later for sophisticated methods of solving the Eigenvalue problem and then towards the end briefly; we will discuss a quick and easy method of solving the problem which is power method.

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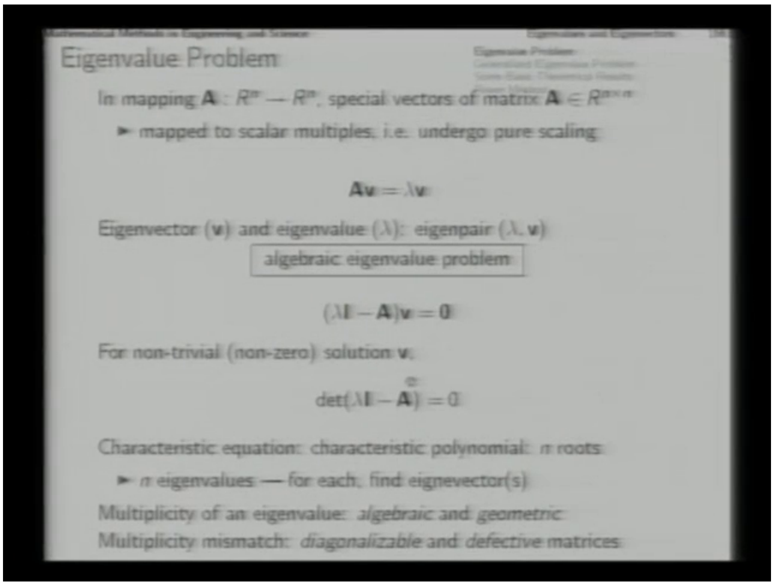

To begin with I again draw your attention to the mapping a which is from R n to itself that is from n dimensional space to itself; that means, it is a the corresponding matrix is n by n;k it is a square matrix. Now when we multiply a vector to a matrix to a vector the vector gets mapped to another vector in the same space in this case, but then in this mapping there are 2 effects produced on the vector. One is a magnification which may be less than 1 which means in that case the actual vector will get reduced in size the other then magnification the effect is turning it rotation it.

Now, this is the general way in which effecter can get mapped through multiplication with a matrix now some of the vectors for every matrix are special there special in the sense that they undergo only magnification or scaling and do not rotate under multiplication with a particular matrix. These vectors are in some sense the own vectors of that matrix or some special vectors for that particular matrix and these vectors are called Eigenvectors, the word Eigen in German means special or (Refer Time: 03:47).

So, as if these vectors belong to this particular matrix. So, if you multiply the vector a matrix a to one such special vector its own vector then the result the mapping is nothing other than a pure scaling. So, in that case we call this vector v as an Eigenvector and the scale factor lambda is called the Eigenvalue or the characteristic value together lambda and v Eigenvalue and Eigenvector are quite often refer to as the Eigenpair. They form a pair now determination of all the lambdas and corresponding vs there is Eigenvalues and Eigenvectors for a given matrix is called the algebraic Eigenvalue problem. Now how we can find the values lambda and the corresponding vectors v from only this much the process the underlying concept is actually very simple.

You can take this lambda v on this side, though you cannot write as a minus lambda into v because a is a matrix and lambda is a scalar so, but what you can is that this v you can write as identity into v and then take lambda I and a together in this manner take taking a v on this side then you get lambda I v minus a v lambda is a matrix, and a is also a matrix. Then you will have this system of linear equations now you will note that this system of linear equations is n equations in this vector or n variables and these equations are homogenous equations that is the right hand side is 0. Then you know that for a homogenous system of equations for the existence of non-trivial or non-zero solution, the coefficient matrix must be singular that is the coefficient matrix must have a null space and we will be actually a member of the null space of this matrix lambda I minus a.

So, for singularity of this matrix you must set its determinant equal to 0. Now you find that we have reached a stage where from a large number of unknowns we will certainly reduced to 1 unknown. In this particular equation you had 1 scalar unknown lambda and 1 vector unknown v which was n plus 1 total number of unknowns. Now the condition that the coefficient matrix is singular tells you that determinant of the coefficient matrix is 0 now you have got a single question in a single unknown. In addition you know that this side is a polynomial in the unknown lambda polynomial of degree n. So, then the question boils down to finding the roots of that polynomial to begin with or to find the solution of this polynomial equation.

And we know that it will have n roots including multiplicities right. So, the polynomial on this side is called the characteristic polynomial of the matrix a and therefore, the corresponding equation this equation is called the characteristic equation and its solutions are the Eigenvalues. So, characteristic equation or characteristic polynomial we will give you n roots of this n th degree polynomial these are the an Eigenvalues and for each of them you will try to find the corresponding Eigenvectors that (Refer Time: 07:38) very difficult because as you insert those Eigenvalues 1 by 1 for every Eigenvalue sitting here you will get a homogenous system of equations, in which the coefficient matrix is completely known all that you need to do is to find the null space of that known matrix lambda I minus A which we have studied earlier.

Now, we have been just talking about the number of Eigenvalues total number of Eigenvalues from this with certainly be n, but that may be repeated for example, suppose we have got a 3 by 3 matrix, for which the Eigenvalues may turn out to be 2 2 and 4 that is possible. So, here the Eigenvalue 2 is said to have and algebraic multiplicity of 2 because it is operating twice in this polynomial. So, this polynomial will be lambda minus 2 whole square to appearing twice into lambda minus 4, 4 appearing only once now we also talk of geometric multiplicity that is when we take this Eigenvalue lambda and try to insert it here and try to find v.

Now, in this particular example if the Eigenvalues are 2 2 and 4, then as you insert lambda equal to 2 here and you try to find the corresponding Eigenvector v, you would expect that there may be up to 2 such vectors; one Eigenvector belonging to lambda equal to 2 in the first instance and the second one belonging to lambda equal to 2 in the second instance you may succeed in finding 2 such Eigenvectors or you may not that depends upon the particular matrix a; that means, that if the algebraic multiplicity of a particular Eigenvalue is more than one, then that may give you one Eigenvector corresponding to it or 2 Eigenvectors or 3 Eigenvectors up to the number which is the algebraic multiplicity.

That means in a larger matrix if suppose an Eigenvalue say this Eigenvalue lambda equal to 2 in a 7 by 7 matrix appears 5 times. So, the Eigenvalues are 2 2 2 2 2 something else and something further one this structure. In that case corresponding to 2 Eigenvalue 2 when you try to find out the Eigenvector, you may find you only one Eigenvector you might find 2 or 3 or up to 5 more than 5 you cannot get that number corresponding to that particular Eigenvalue how many Eigenvectors you could find out that number is called the geometric multiplicity now note this one is algebraic this one is geometric.

Algebraic multiplicity is appearing from this polynomial, how many factors lambda minus a particular lambda is appearing in this polynomial how many times that is appearing that is coming from an algebraic source and that is why it is called algebraic multiplicity. On the other hand the number of corresponding Eigenvectors will span a sub space in the space R n of the dimension, which is equal to the number of linearly independent Eigenvectors that you can find corresponding to that Eigenvalue and this description of this subspace that you are talking about that is a geometric entity that is why that number is called the geometric multiplicity of that Eigenvalue.

Now, note that when you are talking about finding Eigenvectors different Eigenvectors then in that context linearly dependent Eigenvectors are not considered different; that means, if you find one vector as an Eigenvector then it is obvious, the twice of that will be certainly an Eigenvector. So, that is not counted as different from the first now similarly if you have already found 2 Eigenvectors corresponding to a particular Eigenvalue, then a linear combination of these 2 will certainly b an Eigenvector with respect to all corresponding to that same Eigenvalue that is not considered anything different.

So; that means, when we hunt for Eigenvectors we look for a linearly independent Eigenvectors. Now when it happens that for a particular Eigenvalue, the algebraic multiplicity and geometric multiplicity have a mismatch between them there is a algebraic multiplicity is higher geometric multiplicity is lower, in that case we call that matrix as defective. In what sense it is defective, what is the defect and what to do in such a situation that will discuss in detail in the coming lectures when it is. So, that algebraic multiplicity and geometric multiplicities for every Eigenvalue is same in that case we can do certain interesting things very easily.

We can diagonalize the matrix; that means, we can change the basis for representation of this mapping in such a way that the resulting matrix representation for the same mapping the same linear transformation transfer to be diagonal; that means, the directions get completely decoupled. So, such matrices are called diagonalizable; to recognize a diagonalizable matrix the direct straight forward thing is to check the algebraic and geometric multiplicity (Refer Time: 13:27) Eigenvalue. If they match all of them then that matrix is diagonalizable if a single Eigenvalue has a multiplicity mismatch between algebraic multiplicity and geometric multiplicity then that is not diagonalizable.

In that case the Eigenvectors cannot be decoupled the space cannot be decoupled in terms of individual Eigenvalues in the same way as diagonalizable matrices. So, actually the diagonalizability is that way not a property of a matrix as such it certainly is a property of a matrix, but it is actually the property of a much more fundamental thing underlying the matrix there is the linear transformation. So, diagonalizability is actually the property of the linear transformation for which the matrix is just one representation. Now considering these things apart does this outline try tend to suggest that Eigenvalue problem solution method is complete.

It may look so, because finding the determinant of a matrix in terms of lambda is something which we can think of there is setting there equal to 0 and getting a polynomial equation is something which is which does not somewhere it dangerous, and then solving a polynomial equation also is something with which we are acquainted after finding the lambda putting that here and for every lambda finding the corresponding Eigenvectors that also as a part problem is not very difficult problem. But does it mean that all the discussion in Eigenvalue problem gets completed here answer is no the reason is that when the degree of the polynomial equation goes very high in that situation solving this polynomial equation is actually not very easy.

In fact, for solving a polynomial equation one of the very popular one of the very used methods says that try to solve the polynomial equation through the methods of Eigenvalue problem. So, therefore, for solving an Eigenvalue problem the polynomial equation solving as a sub problem is not a very attractive for position, because as the degree of this polynomial goes high it will be very difficult to computationally solves this problem therefore, people look for other ways of packing this Eigenvalue problem directly without first making a recourse to this polynomial equation solving problems and in that attempt mathematician have developed a method of interesting tools to handle matrices and express them in canonical formations and take a lot of advantage from these theoretical developments into several fields of applied mathematics, and these interesting developments will be studying in the coming lectures including this one.

So, in order to make the ground for that, I will need to develop some basic theoretical results first. Even before that it will be a good idea to see a practical problem from which Eigenvalue problem appears.

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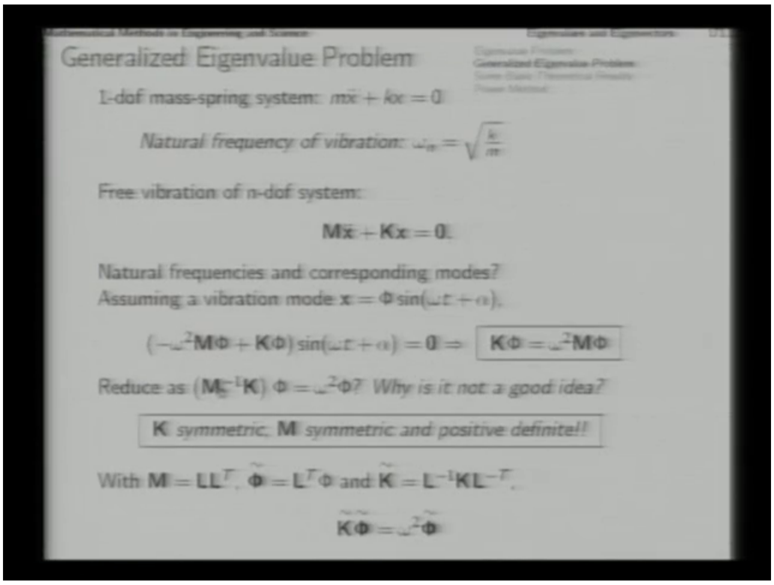

There are many such practical problems in almost all branches of science and engineering, where Eigenvalue problems certainly turn up one such problem is the system of mechanical system with free vibration. For example, if you considered the 1 degree of freedom mass spring system for which the dynamic equation is just this where m is the mass and k is the stiffness of the spring, and then you try to write the assumed solution of this equation in the form.

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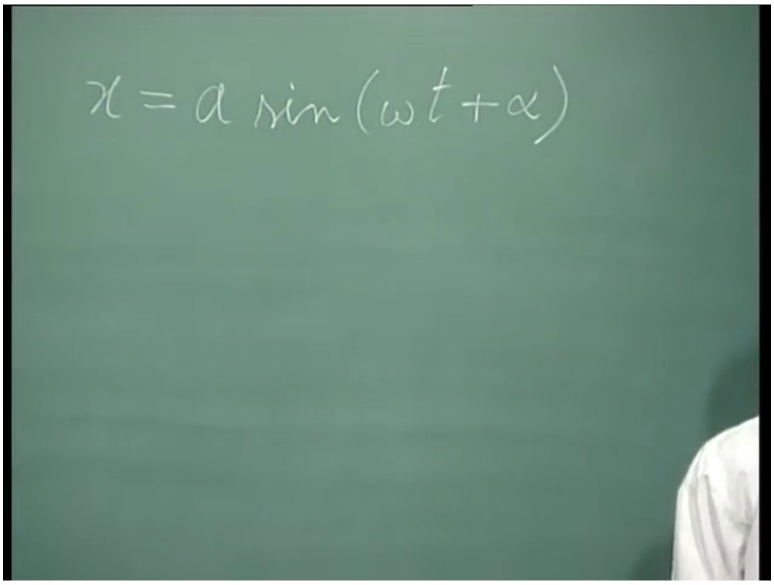

Because you know what kind of a solution this will have? This will have a sinusoidal solution and. So, you try to write it like this and then you differentiate it twice and insert in this right.

So, you know that twice differentiation of this we will produce minus omega square sign is a constant and from that very easily you work out the natural frequency of vibration, in which this mass spring system we will undergo natural vibration. Now when you try to formulate and solve the same problem for a multi degree of freedom system we do not get such a nice simple scalar equation, but we get a matrix vector equation in this manner. So, free vibration of an n degree of freedom system will be governed by this equation where m is the inertia matrix, k is the stiffness matrix, x is the vector representing the coordinates of the system and its double dot is certainly the acceleration corresponding to that.

Now, in this problem when you ask this question what are the natural frequencies in which this particular mechanical system can execute natural vibration and correspondingly what are the vectors x along which those vibrations will take place for example, in a 3 degree of freedom system it might happen that x 1 x 2 x 3 give you a particular direction a particular vector along which the vibration takes place in one frequency. There is another second direction in which the system may vibrate in a second frequency similarly a third direction with the third frequency.

So, what are these vibration modes and what are the corresponding frequencies that becomes the problem for a solution in the in this free vibration problem. Now again in analogy with this equation we now try to assume something vector x is equal to an amplitude vector into a term like this. So, there we assume a vibration mode first in this manner the vibration mode x is a constant vector phi into sin omega t plus alpha again we differentiate it twice with respect to time and insert that x double dot here and that will tell us that this whole thing is equal to 0, because sin omega t plus alpha after twice differentiation we will produce a factor of minus omega square.

So, that minus omega square gets multiplied here we have got this, then the same argument we use what we produce in this case that is for this to be equal to 0 for all time this part has to be 0 because this one will not be 0 always. So, this has to be 0 when we do that then we get the corresponding equation k phi equal to omega square m phi. Now this resembles the Eigenvalue problem that we discussed just.

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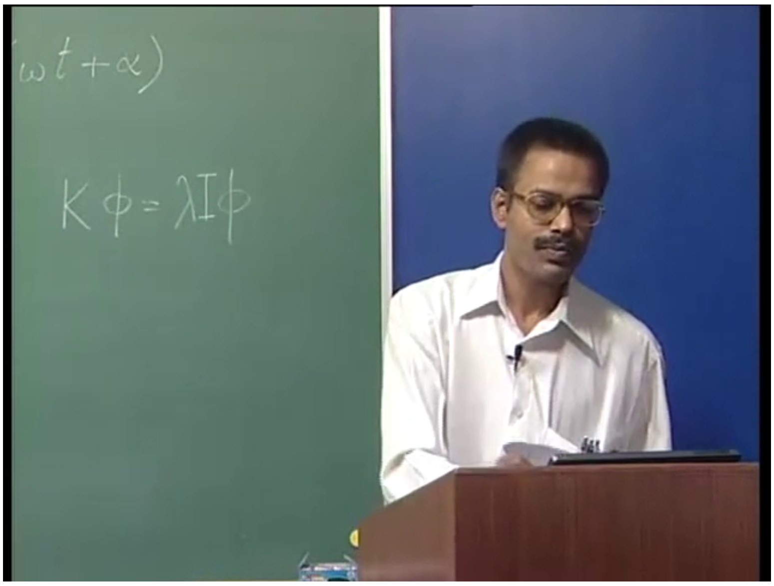

Now, in the earlier case we got a problem of this manner K phi equal to lambda phi, a x equal to lambda a sorry a v equal to lambda v these are kind of problem that we have been discussing just now.

Now, here it is this problem is not exactly the same as this problem, because in this location there is a matrix sitting. Omega square you can identify with this lambda, but here there is a matrix sitting that is why this problem is not called just Eigenvalue problem, but it is called the generalized Eigenvalue problem. As is in the original Eigenvalue problem there was a matrix here which was identity which indeed we inserted when taking it on the other side, right.

Now, in this case in this particular case, it is generalized in the sense that in place of identity matrix now there is a non trivial matrix sitting there now how to solve this problem? Because if you take it on the other side then I mean in place of I if we have m sitting here then as we take it on the other side we will get k minus omega square m that will be the matrix not the straight forward a minus lambda I as we would get in the ordinary Eigenvalue problem. Now how to handle this? One might suggest that if we pre multiply both sides of this equation with M inverse, then immediately we get this problem M inverse a phi equal to omega square phi.

Why not solve this problem because M inverse K we can take as a we know M we know K we can evaluate M inverse k and then it becomes an ordinary Eigenvalue problem indeed it is possible to do that, but then it is not a good idea why doing this is not a good idea? The reason follows from the nature of these matrices that appear in these locations this is not just sum matrix and this is also not just sum matrix, this is an inertia matrix and this is a stiffness matrix such matrices when appearing in practical problem have certain structure.

A stiffness matrix is always symmetric and inertia matrix is always symmetric and positive definite. Now if we evaluate this M inverse K that may lose the symmetry that was originally they are in the original problem. Now it is not a good idea to take a step in the solution of a problem which actually makes the original problem difficult. Later we will study in detail how solution of a symmetric matrix Eigenvalue problem is actually much simpler and much more straightforward compared to a general non symmetric matrix therefore; it would be a bad idea to take a step which will spoil the symmetry of the problem as originally given.

Rather we should try to take a measure which will utilize this particular structure. So, what we do is that we take this symmetric positive definite matrix m and recall that for a symmetric positive definite matrix there exists a (Refer Time: 24:15) composition L L transpose. So, if we conduct the (Refer Time: 24:19) composition of this matrix m in this form L L transpose and then conduct a coordinate transformation the original coordinates phi and now transform to this phi tilde through this L transpose new basis.

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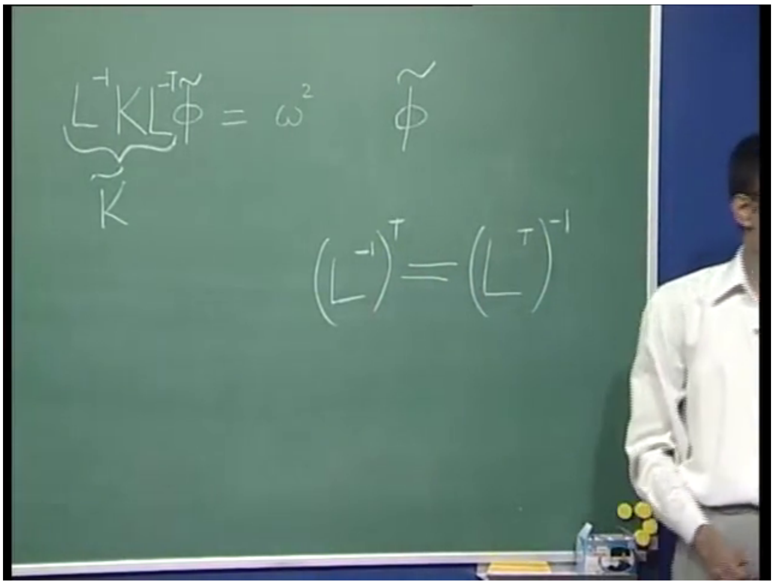

In that case when we insert this here then see how this will look like we have K phi equal to omega square mM phi; first of all in place of this M we will write L L transpose.

The moment we do that we get this L transpose phi which we are going to define as phi tilde right. So, L transpose phi we are defining as phi tilde. Now on this side also we would like to have phi tilde right because we are applying that coordinate transformation. So, if phi tilde is L transpose phi then what is phi in trans of phi tilde that will be found through the pre multiplication of L transpose inverse. Now when we do that we get L transpose inverse phi tilde right. Now we say that we can get rid of this L by pre multiplying both sides with L inverse, as we do that from here L inverse L gives us gives us identity and we have this.

Now, notice that the original generalized Eigenvalue problem like this has been transformed to this problem K tilde call this whole thing as K tilde. Then we have got the new problem as K tilde phi tilde is equal to omega square phi tilde. So, in the new coordinate system in which phi tilde is the vector we have got an ordinary Eigenvalue problem in which this matrix L tilde is actually symmetry because K was originally symmetric on this side we have multiplied it with L inverse along this side we have multiplied it with the transpose of L inverse that will preserve the symmetry.

You can just check that its transpose is itself L inverse k L inverse transpose as you take the transpose of this whole thing you get the same thing back. So, the symmetry is resolved, now note here then when we wrote L inverse transpose or L transpose inverse for this it is not clear whether we have talking about this or we have talking about this whether we have talking about the transpose of L inverse or whether we have talking about the inverse of L transpose there is not clear in this notation. Till this notation is varied because in these 2 cases the result will be same and therefore, this L with minus T here actually means any of the 2 because these 2 are always going to be same.

Now, this is one practical problem from which you get an Eigenvalue problem, there are many other situations in all of science and engineering from which Eigenvalue problems suddenly appear. Now we will start with some of the basic theoretical results of the Eigenvalue problem over which we will build up later methods by which to solve the problem. Apart from that as a byproduct of this process the theoretical results will also providers which tools to handle matrices in nice elegant and canonical ways, which is useful in many areas of a (Refer Time: 28:18) mathematics wherever matrices appear.

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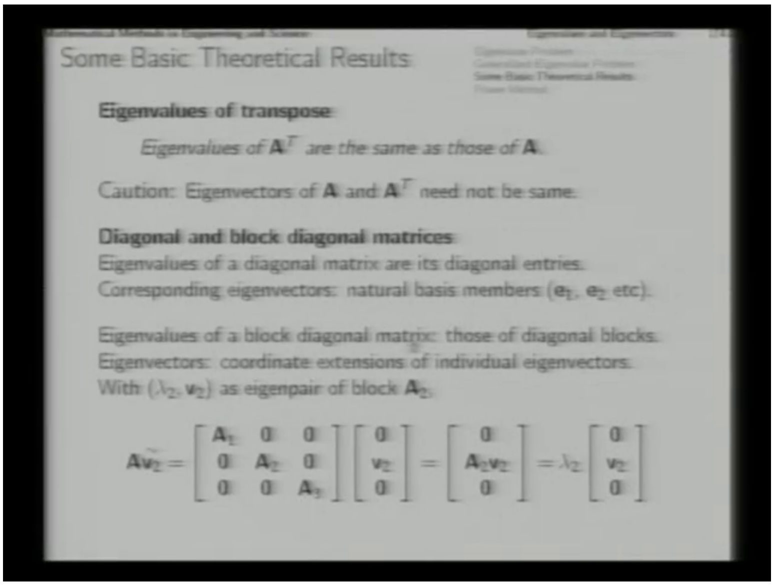

Now, first is the first important result that we should always keep in mind is that Eigenvalues of the transpose of a matrix are the same as though those of the original matrix. This is very easy because we know that determinant of a transpose is a same as the determinant of an original matrix and the characteristic polynomial is found just by the expansion of a determinant. So, these are obviously, the same of course, Eigenvectors need not be same in general they are different. Next important point that we should remember is the situation for a diagonal matrix and a block diagonal matrix.

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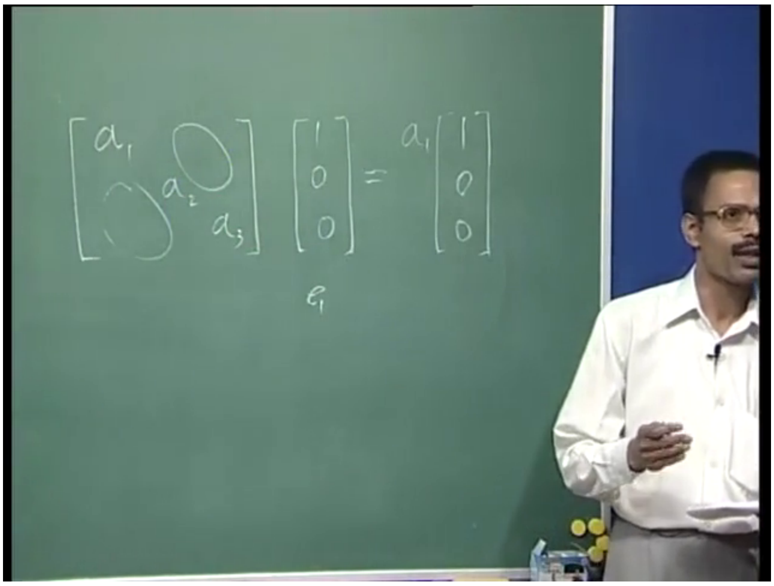

You know what is a diagonal matrix. So, suppose we have got a 3 by 3 matrix in this manner. These are all 0 these area all 0 and this is a diagonal matrix and it is very clear that these diagonal entries are actually the Eigenvalues of this matrix and a corresponding Eigenvectors are the natural basis members for example, if you multiply 1 0 0 with this and; obviously, you will get a 1 0 0 which can be utilized these. So, that shows that you have a v equal to lambda v right v is 1 0 0; that means, a 1 is an Eigenvalue and this vector e 1 the first base natural basis member is the corresponding Eigenvector.

Similarly, a 2 and a 3 will be the other Eigenvalues with corresponding basis members corresponding Eigenvectors as e 2 and e 3 and natural basis members. Now this is obvious; now if you say that this is actually a much larger matrix this log this a 1 is replaced with a matrix a square matrix, this a 2 scalar is replaced with a square matrix and similarly this a 3 then what you get is not a diagonal matrix because this square matrix may have off diagonal entries, there will not be a diagonal matrix, but what you call it is block diagonal matrix which will look like this.

In which this matrix a 1 is filled up quite a bit. Now when you talk of Eigenvalues of a block diagonal matrix then there is a very interesting situation that the match at the

Eigenvalues of this large matrix is the Eigenvalues of a 1 and the Eigenvalues of a 2 and the Eigenvalues of a 3. So, if this is r by r this is s by s this is t by t and everything else outside this blocks is 0 then the r Eigenvalues of this s Eigenvalues of this and p Eigenvalues of this separately obtained, can be all put in a list and this r plus s plus t numbers will be the Eigenvalues of this large matrix, and the corresponding Eigenvectors they are also very easy to find they are just coordinate extensions. For example, if suppose this small matrix a 2 has an Eigenvalue lambda 2 with the corresponding Eigenvector as v 2 then just above v 2 you put as many zeros has required to fit the size of this matrix and below that you put as many zeros as required, to fit the size of this matrix and then as you multiply this you find that this gives you lambda 2 into that same old vector.

That means that the an Eigenvalue of A 2 is the Eigenvalue of A also and the corresponding Eigenvector of A can be found through a coordinate extension over v 2 is as many extra zeros are equal above and below you can put that and then you get the big vector which is an Eigenvector of this matrix, large matrix corresponding to that same Eigenvalue. For diagonal and block diagonal matrices the situation is very simple the matter gets are little complicated when you talk of triangular matrices.

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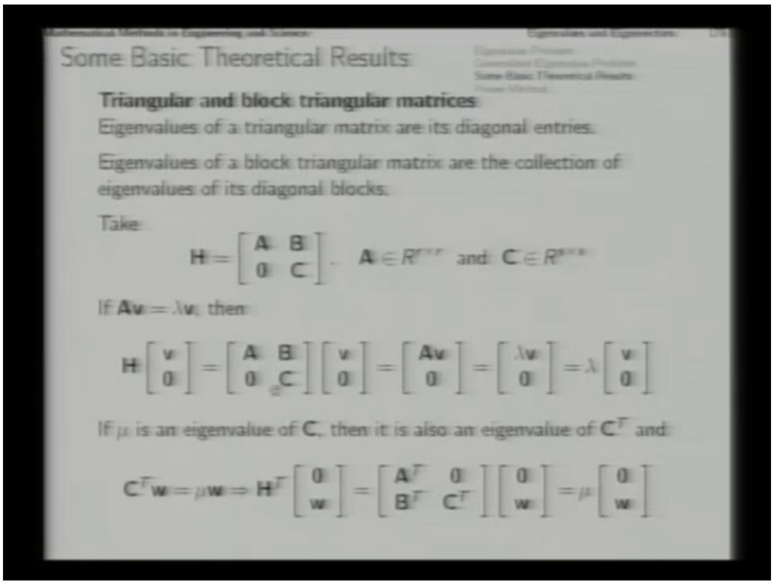

A triangular matrices a triangular matrix will have known 0 entries here, but still the diagonal entries are the Eigenvalues because below that everything as is 0.

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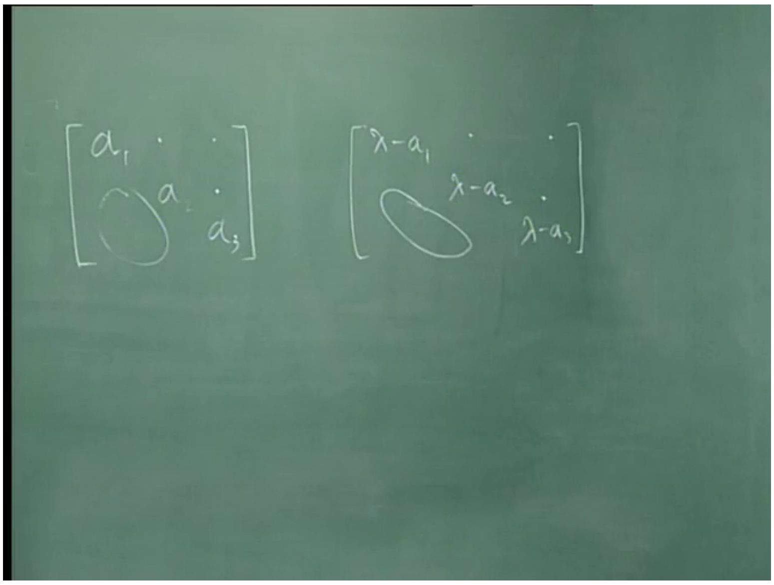

So, when you try to write the characteristic polynomial, you write lambda I minus this. So, you will get lambda minus a 1, lambda minus a 2, lambda minus a 3 something, something, something here, but below you have got everything 0. So, when you try to expand this fellows determinant you get you expand from the first column, then you get lambda minus a 1 into something plus all zeros then that something again gives you lambda minus a 2 into something plus all zeros and so on.

So, for a triangular matrix you will find that; obviously, the characteristic polynomial we will emerge as a product of these factors; that means, that you have got the characteristic polynomial already in factorized form that immediately gives you a 1, a 2, a 3, etcetera, the diagonal members of the original matrix as the Eigenvalues, but Eigenvectors is a different question for that you have to do a lot of calculations to find the Eigenvectors Eigenvectors are not so; obviously, visible here. So, when you handle triangular matrices we talk directly in terms of the Eigenvalues only, not Eigenvectors Eigenvectors can be found with some further processing they are not so obviously, visible.

Now, when you take a block triangular matrix, that is if these scalars are replaced with matrices and there are big blocks of 0 zeros sitting below that, and big blocks of other entries perhaps non-zero any of them will be non-zero answering here then we have a block triangular matrix which is look like this. These a block triangular matrix with 4 blocks block A square, block B not necessarily square, block 0 which is also not necessarily square it will be just size of the transpose of B and then block C which has to be square.

Then you say that the Eigenvalues of this is the same as the Eigenvalues of A and the Eigenvalues of C. Now for this matrix the statement that Eigenvalues of this large matrix is the collection of Eigenvalues the matrix A and the Eigenvalues of the matrix C can be easily seen in a similar way in which you saw just now the result related to the diagonal matrix; however, here the statement is made only for the Eigenvalues and not about the Eigenvectors. So, if the matrix A has an Eigenvalue lambda with an Eigenvector v that is this, then we can apply the complete matrix hover a coordinate extension of v 0 and then we defined that the product gives us this; that means, v 0 the coordinate extension of v turns out to be an Eigenvector of the complete matrix H with the corresponding Eigenvalue lambda.

Whatever when you try to ascertain verify that the same wholes for c also, then we cannot immediately apply it on a coordinate extension because that will create the punctuation with this B, because in the product the way this 0 at a help in this case it will not help in the other case. When this particular situation what we do is that we take mu as an Eigenvalue of C and then argue then it is also an Eigenvalue of C transpose and then C transpose w turns out to be mu w for that mu for sum vector w and then we apply not H, but H transpose on the appropriate coordinate extension of w in this manner and then we find at the end that we get mu into this vector 0 w; that means, mu turns out to an Eigenvalue of H transpose, then that will mean that mu is an Eigenvalue of H as well.

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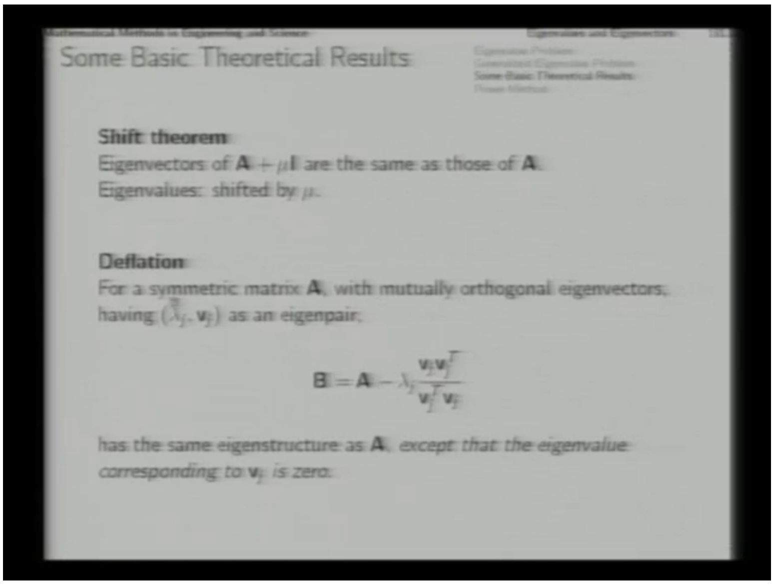

Now, apart from these results there are a few points which we need to keep in mind which will be very useful in many of the methods, one is that if we add a scalar times identity to a matrix then all the Eigenvalues get shifted by that scalar value and this is called the shift theorem. This is very easy to verify and so, I am not going into that I am leaving it for you than the other important we show that we must keep in mind it actually applicable only for a symmetric matrix that is for a symmetric matrix A which mutually orthogonal Eigenvectors a fact that we will verifying the next lecture, for a an Eigenvalue lambda j with corresponding Eigenvectors as v j, we find that if construct another matrix B from in which forma we have subtracted this part then this resulting matrix B has exactly the same Eigenstructure as A, Eigenstructure means same Eigenvalues with the corresponding same Eigenvectors except that the Eigenvalue corresponding to that particular Eigenvector v j is now more lambda j but it is reduced to 0.

That means, the information worth of that Eigenvalue only has been removed from A, the all the rest of the information of the Eigenvector remains as it is. Now this is an important issue to which we will come back after studying the symmetric matrices in detail in the next lecture before that I will try to explosive right now to an important.

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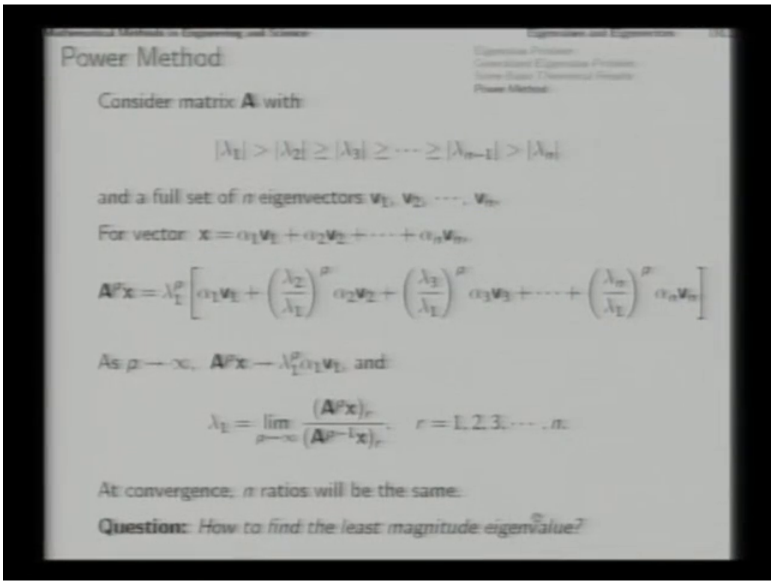

Quick and easy method for solving the Eigenvalue problem and that is for power method. This helps you in finding the Eigenvalues of the matrix when you are not interested in finding all Eigenvalues of a large matrix.

But you are interested in finding only a few largest magnitude Eigenvalues or perhaps the largest magnitude and the lowest magnitude Eigenvalue; Eigenvalues. Now this is very quick and easy method easy to understand is to implement, but note that it will work only for those matrices which have a full set of n Eigenvectors that is which are diagonalizable and for which there is a single Eigenvalue which has the largest magnitude; that means, that the largest magnitude Eigenvalue has a magnitude which is larger than all the rest that is not too are at the top only one Eigenvalue is at the top.

In that case power method gives you the largest magnitude Eigenvalue very easily what we do for that is, first to understand the way it operates you consider that if the matrix a processes a full set of n Eigenvectors then these Eigenvectors will span the entire space are in and; that means, any other vector x that you can think of, can be expressed as a linear combination of these vectors in this manner. Now it is a different matter that given a vector x we can choose any vector x that will have a representation as a linear combination of the Eigenvectors with alpha 1 alpha 2 (Refer Time: 40:44) etcetera representing the corresponding coefficients. Now even though we do not yet know those Eigenvectors and the corresponding coefficients what we know this much that any vector x that we can think of that we can have picked up we will have some representation like this with alpha 1, alpha 2, etcetera and v 1, v 2, etcetera currently unknown to us.

Now, if on both sides we multiply with a the matrix then what happens? On this side x is a non vector with we have picked up. So, we multiply a x we can work out the result on this side we do not known what is happening exactly the numbers we do not know in detail, but we know this much that a v 1 with the lambda 1 v 1, a v 2, v v lambda to v 2 and so on; that means, through a multiplication of a whatever was the representation here now in the coefficients we will get any other additional factor of lambda 1 lambda 2 lambda 3 etcetera. If we go on multiplying the vector the resulting vector with a once more once more once more in after p such multiplications on this side we will have A 2 the power p x which is known which is the result of multiplying a p times over x.

On this side we will have alpha 1 onto lambda 1 to the power p p 1 plus alpha 2 into lambda 1 lambda 2 to the power p b 2 and so on. If we take that lambda 1 to the power p outside then this will remaining side right. Now under the assumption that the lambda 1 Eigenvalue is the largest magnitude Eigenvalue and the next one is a little below that what will happen is that as p goes too high many many many times it has been multiplied then that will mean that in that case lambda 2 by lambda 1; lambda 3 by lambda 1 all being of magnitude less and one after raise to large power all of them will tend towards 0 when p is sufficiently large.

That will mean that after many such multiplications we will have a vector sitting inside this which is in the same direction as v 1 and then after that process has stabilized after that direction has been stabilized one more application of that same multiplication with a will mean at on this side and Eigenvector is being multiplied with a and that will give you lambda 1 into that vector. And that gives you the vector in the direction and the lambda 1 as the scale between 2 successive values. So, as p tends to infinity this fellow tends to this lambda 1 to the power p alpha 1 v 1 then you find that after the process has converged then you will find that the result A p X compare to the result in the previous iteration previous step are 2 vectors which are in the same direction; that means, the ratio between the first components in the ratio between the second components in the ration between the third components will all this m and that ratio is lambda 1 that convergence all n ratios will be same. In fact, that is a test that convergence has taken place.

So, this way you quickly get the largest Eigenvalue largest magnitude Eigenvalue note that it may be negative for that matter it does not matter it. So, you will get the largest magnitude Eigenvalue and the corresponding vector will be the Eigenvector. Now we will make 2 points here, one is that other than the largest if you need the least magnitude Eigenvalue also then how to do that? For this purpose we can use the shift theorem. So, how to find the least magnitude Eigenvalue; what we can do is that after finding this largest magnitude Eigenvalue we see its sign this is a ratio which may have sign.

So, whether it is positive or negative that he has been found here. So, if for example, suppose that lambda 1 transfer to be positive, say the largest magnitude Eigenvalue is 23 then what we can do is from the original matrix we subtract 23 from all the diagonal entries, that is application of the shift theorem that is we subtract 23 I from the original matrix that will mean that all the Eigenvalues have got shifted left word by 23, that is whatever was 23 earlier that become 0 now, whatever was 21 earlier that becomes 19 and so on in that case the smallest magnitude smallest algebraically, that turns out now as the largest magnitude Eigenvalue largest magnitude, then we can apply the same power method once more and then we will find that which is the largest magnitude Eigenvalue and then as we shift the think back 23 steps on the right side, then will get the appropriate correct Eigenvalue for matrix a with the corresponding Eigenvector, right.

So, this is one way to find the largest and least magnitude Eigenvalues which has a lot of practical significance. Now one more possibility of a important of an important question maybe that for example, if you are not interested in finding all Eigenvalues, what we are interested in finding a top few, the largest magnitude once lambda 1 lambda 2 lambda 3 lambda 4 etcetera some say 6 of them, 6 top Eigenvalues, we want to find out and the corresponding Eigenvectors there also for example, the matrix suppose is 100 by 100 we are interested in all the 100 Eigenvalues and there Eigenvectors, but only top 6 or if you top once with some conditional requirements.

Then what we can do after the finding the largest one we can use equation this will work in the case of symmetric matrix, which is quite often encountered in practical situations. By deflation what we can do is that we can subtract the part which is contributed by this particular Eigenvalue lambda 1 and the corresponding Eigenvector, then the resulting matrix will have the largest magnitude Eigenvalue as lambda 2 which can be found to power method and so on. Now this is a very state forward method which can be applied if you are sure that the matrix does satisfy this requirements otherwise the process may not operate as expected or as desired rapper from these things there are 2 important concepts which will go long way in our discussion in the coming lectures on it is the Eigenspace.

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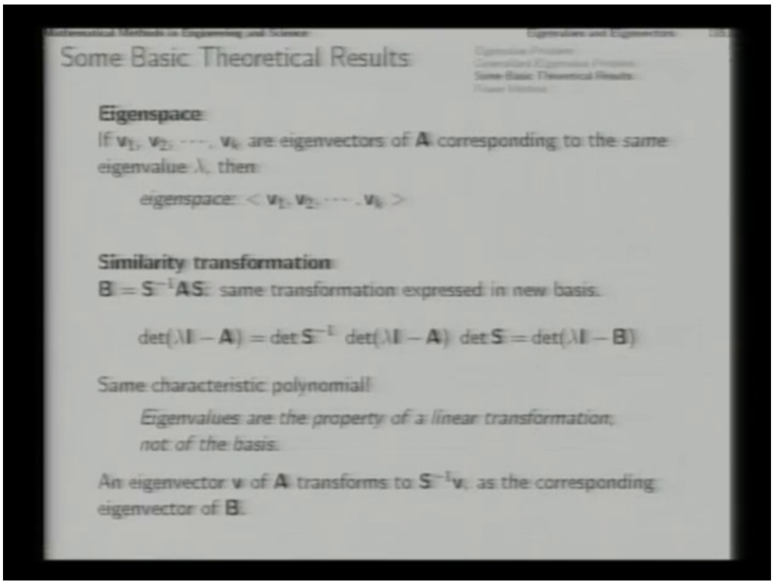

This is a done to in use for representing a subspace of R n which is composed by the Eigenvect Eigenvectors of a matrix corresponding to the same Eigenvalue lambda for example, suppose a has an Eigenvalue lambda corresponding to which there are k Eigenvectors v 1, v 2, v 3 up to v k, then that will mean that any linear combination of these Eigenvectors is also going to be an Eigenvector becomes verify that very easily suppose corresponding to Eigenvalue lambda there are 2 Eigenvectors v 1 and v 2.

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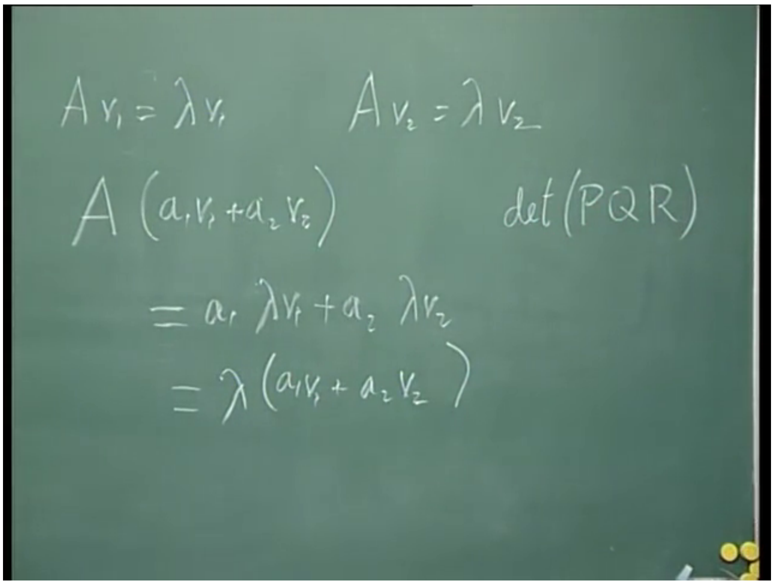

That will mean that A 1 A v 1 is lambda v 1 and A v 2 is lambda v 2, then if we apply a on a linear combination of these 2 Eigenvectors then we will find that this will turn out to be a 1 is scalar. So, we can take it out and then we will have a 1 into A v 1 which is lambda 1 lambda v 1 plus a 2 into A v 2 which is lambda v 2 from here and taking lambda scalar outside this common we find that we have got this.

That means the matrix a multiplied over this vector gives us lambda into this vector; that means, if v 1 and v 2 are 2 Eigenvectors corresponding to the same Eigenvalue lambda not that it is applicable for same Eigenvalue, then any linear combination of them is; obviously, going to be an Eigenvector with respect to further particular matrix a. Now this is not an a linear linearly independent Eigenvector, but this is certainly an Eigenvector it does not come in the counting of Eigenvectors, but whenever required this vector does operate like an Eigenvector and that is means that if these a Eigenvectors are corresponding to the same Eigenvalue lambda, then the complete subspace spend by these vectors gives you a subspace in which every vector every vector is an Eigenvector and therefore, this particular subspace is also called the Eigenspace of A corresponding to that Eigenvalue.

There is important theoretical point that will be quite in our discussion in coming lectures, that is similarity transformation. This is something which we have already earlier seen once and here we look at some important properties of it. If we decide to represent the vectors of a space R n in a different new basis s and therefore, the matrix representation of a linear transformation changes from A it becomes B, for B is S inverse AS this we have seen earlier. Now note that determinant of lambda I minus A which is the characteristic polynomial of the matrix A.

Now, we already know that determinant of a matrix and the determinant of its inverse are reciprocals of each other; that means, that if we multiply this with determinant of S and also with determinant of S inverse we are actually making no change because this will be reciprocal of this. We also know that determinant of the product of 3 matrices of the same size is same as determinant of P into determinant of Q into determinant of R. Now what we have got here is determinant of P into determinant of Q into determinant of R that means, this is same as determinant of PQR that means, a single determinant with s inverse inserted from this side and s inserted with in this side will be the same as this.

Now, when s inverse and s are inserted on from the 2 sides on this, they cancel each other because identities remain inside that is why this is lambda I, on this it will have the effect which is different that is S inverse a S which is B; that means, we have got this whole thing same as determinant of lambda I minus B what is that? That is the characteristic polynomial of the matrix B now that shows us that. So, the similarity transformation the matrix might has been changed, but its characteristic polynomial remains same as earlier the characteristic polynomial of A and the characteristic polynomial of B turn out to be the same is the entire polynomial is same for A and B then all the roots will be same; that means, that Eigenvalues remain unchanged through a similarity transformation because similarity transformation comes out only as a result of a change of basis.

No geometrical entities being changed only its representation is being changes and Eigenvalues are the property of the underlined linear transformation not of the basis and therefore, Eigenvalues remain constant through all this similarity transformations. How do Eigenvectors change? Geometrically even Eigenvectors do not change, but then their representation in the new basis will change as the vectors as any other vector would change its representation in the new basis through the multiplication of S inverse which we have already studied in the same manner and Eigenvector of A will transform to s inverse v in the new basis which is given by S. So, if v is an Eigenvector of A the corresponding Eigenvector of B will be s inverse v because there the new basis S has appeared.

So, the basis change of vectors takes place through this relationship and the same will apply to Eigenvectors as well. Now let us quickly summarize what are the points that we have discussed in this particular lecture.

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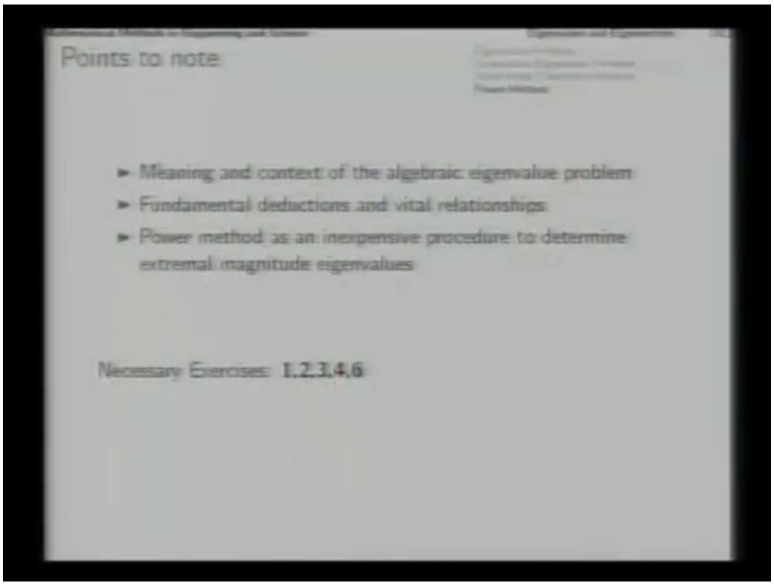

First important point is that meaning and context of the algebraic Eigenvalue problem that we have discussed. Second is that we have studied the fundamental relationships deductions which are vital for the solution of the algebraic Eigenvalue problem and third we have a been exposed to a quick and easy method for power method as an inexpensive procedure to determine the extremal magnitude Eigenvalues, only the largest or largest and lowest or the largest few in all these situations we can use the power method with a little bit of help from the shift theorem or the deflation technique. But then while applying power method you must be careful that the power method does not apply to arbitrary matrices, but on 13 matrices having particular kinds of Eigenstructure if a matrix false in that category then power method will be very handy for you in many situations, but otherwise it may not operate as desired.

So, in the next lecture we will build up on what we have develop till now and see the detailed discussion on the theoretical developments on Eigenvalue problem, which will be then use in different categories of methods for solving the algebraic Eigenvalue problem.

Thank you.