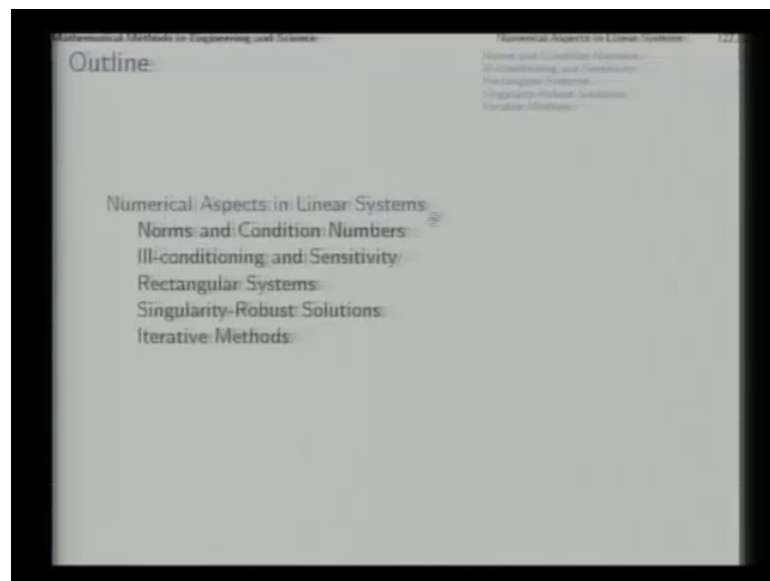


Mathematical Methods in Engineering and Science
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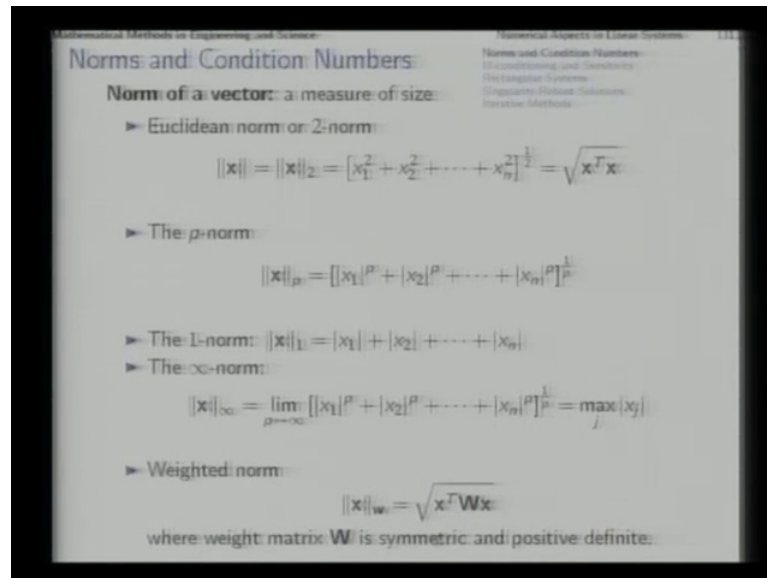
Module - I
Solution of Linear Systems
Lecture - 05
Ill-Conditioned and Ill-Posed Systems

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So, in this lecture we will be discussing numerical aspects in linear systems. In the previous lecture I told you that in linear systems we encounter certain good situations for which we should try to take advantage of it in terms of efficiency and sometimes we encounter bad situations, where there can be numerical errors and overflows and underflows, in this case we need to know how we handle these special situations. So, in this lecture we will be considering those measures which will help us in handling bad systems. So, here first we will define a few terms like norms, condition numbers, ill conditioning and sensitivity of matrices and then we will study rectangular systems and then the most general solutions of linear systems through singularity over solutions and finally, we will have a quick glimpse on iterative methods of solving linear systems.

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First the underlying definitions, norm means size, when you talk of norm of a vector we basically talk of a measure of size of the vector and all of you are familiar with the ordinary Euclidean norm or 2 norm that is represented like this or in general situation we can represent it with a subscript 2. Which means 2 norm and that is simply the length of the vector x in ordinary geometric sense and that is x_1 square plus x_2 square up to x_n square to the power half means square root.

The way you take the distance from origin to xyz point as square root of x square plus y square plus z square, in that same manner this foundation works. This gives you the ordinary Euclidean norm or 2 norm in the same sense you can define the general p norm that is the sum of the p th powers of the absolute values of the coordinates x_1 to x_n and then that sum is taken through a power of one by p means p th root of that sum of the p th powers of x_1 mod x_2 mod and so on. this is the general foundation in which insertion of p equal to 2 will give you the ordinary Euclidean norm with which all of you are familiar.

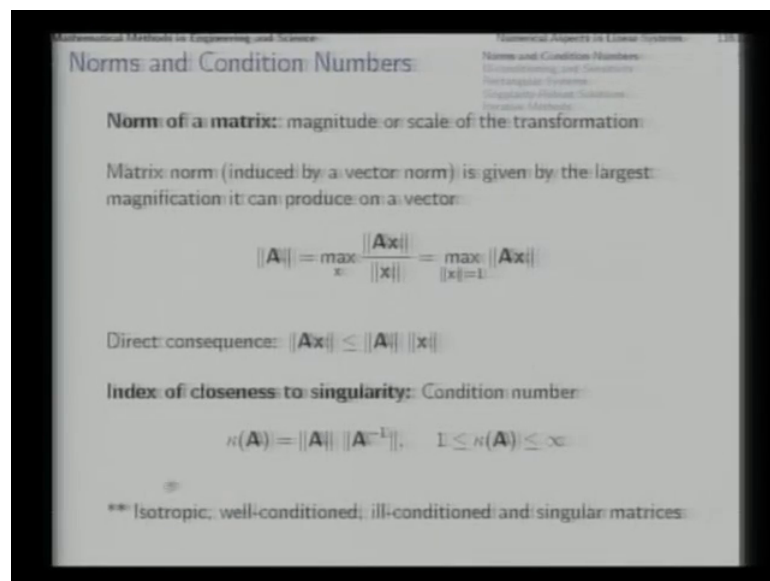
In that sense for other values of p you can define other norms, in particular you can define one norm which is this the sum of the absolute values and you can also define the infinity norm which is sometimes also called max norm. And that is defined in the limit when p turns to infinity and that is defined in this manner and this is also called max norm because finally, what you get out of this is the maximum value of the coordinate

from all the coordinates, because as you raise x_1, x_2 etcetera through a large power then the largest of these magnitudes dominates the sum.

And then when you take the p th root then all others die in proportion and you recover the p th root of the largest which is the largest coordinates and that is why it is also represented in this manner and that is why infinity norm is also known as the max norm. Even in this ordinary Euclidean norm you could define the norm not directly as $x^T x$ under root, but through this relationship where you say that this is a norm defined with respect to a weight matrix W and you can represent it in this manner. The condition is that this weight matrix must be symmetric and positive definite otherwise this function will fail to define a distance measure.

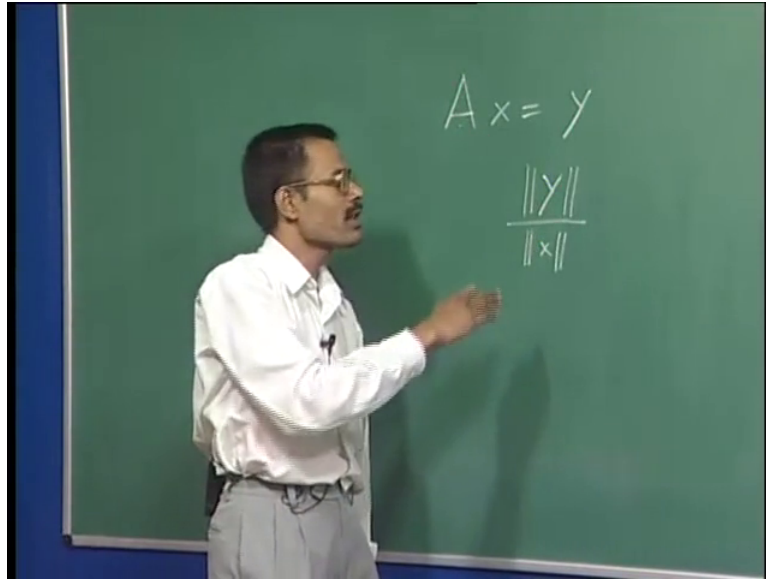
Now, with these kind of definitions of norm of a vector we can then proceed to define the norm of a matrix unless otherwise stated in all our dealings we will be talking about the Euclidean norm, but then in the same context other norms also can be used.

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So, when you talk of norm of a matrix we ask what is the job of a matrix, in what sense we can talk of the size of the matrix? The task of a matrix is to multiply a vector and give another vector.

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This vector has a size and this vector has a size, matrix A will possibly produce a magnification in the size. So, that magnification will be size of the resulting vector divided by size of the original vector and that in a way could be considered in some sense the size of a matrix, but then the problem starts when we find that on different vectors this matrix, the same matrix will produce different magnifications and therefore, to define the size of the matrix we say that we will take the size of a matrix or the norm of matrix as the maximum magnification that it is capable of providing.

That means in this kind of a situation on all vectors x it will produce different kinds of magnifications and the maximum of those magnifications we will take as the measure of the magnification produced by the matrix A . That means, we will define the norm of the matrix or size of the matrix as the maximum over all x of the magnifications size of x (Refer Time: 06:44) x divided by size of x or if you wanted to take all x s which are of the same size.

Then you could also define it in this manner, there is all x s taken of size 1, in that case this will reduce to the value 1 and then you define it in this manner, that is on a units period you take all vectors and out of that the vector which is magnified to the largest extent that would defined the norm of the matrix. As a direct consequence of this we will find that the norm of Ax will always be less than or equal to the product of norm of the matrix defined like this and norm of the vector x because the largest possible of this has

been defined as the norm of the matrix. So, when the largest possible is sitting here on the other side whatever you have that can be at most equal to this most of the time it will be this, that is why as a direct consequence of this definition we have this inequality.

Another important term that we need to define is condition number, we already know that when the columns of a square matrix are linearly dependent then we call that matrix is singular, its determinant is 0 and inverse does not exist. Even when the matrix is not exactly singular we can talk of its closeness to singularity, if the determinant is close to 0 then we say that the matrix is close to singularity and there is a measure of closeness to singularity that is given by the condition numbering or rather it is a measure of being away from singularity. That means, that if this condition number which is the product of the norm of A as defined above and the norm of a inverse then this product is called the condition number. If the condition number is 1; that means, the matrix scales all vectors in equal proportions, in most of the cases that will not be the situation in which case the matrix will scale some vectors less and some vectors more

In that case the condition number will be higher than 1, in the worst case where the matrix is singular some of the matrices will be mapped to 0 in which case the magnification is almost like 0. But, on the other hand there will be some other vectors which are magnified higher and in that case you will have a very desperate spectrum of magnifications and the condition number the product of these two norms will tend to infinity.

For a singular matrix in the limit the kappa will tend towards infinity so; that means, that higher the condition number close is the matrix to singularity, lower the condition number its mapping is well rounded. Now, if all the vectors have mapped to equal magnification then we call such matrices isotropic, iso equal tropic direction the performance of that matrix in all directions is similar; in the other extreme we have similar matrices that gives you the 2 limits of condition number one and infinity.

In between if we find very high condition number for example, 200, 400 something of that sort then you call that matrix ill conditioned, its health is bad, it scales some of the vectors to a very high magnification on the other hand some of the other factors are scaled to very low magnification that is illness, that is ill condition ill conditioning of the matrix. In between if you have small numbers as condition numbers 4, 5, 7 then you say

that this matrix is nice behaving well conditioned you can say. Now, why this situation of large condition number is called ill conditioned, why it is ill in what sense? To see that let us consider a small example here.

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Mathematical Methods in Engineering and Science

Numerical Aspects of Linear Systems

Ill-conditioning and Sensitivity

$0.9999x_1 - 1.0001x_2 = 1$
 $x_1 - x_2 = 1 + \epsilon$

Solution: $x_1 = \frac{10000\epsilon + 1}{2}$, $x_2 = \frac{9999\epsilon - 1}{2}$

- sensitive to small changes in the RHS
- insensitive to error in a guess

For the system $\mathbf{Ax} = \mathbf{b}$, solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ and:

$$\delta\mathbf{x} = \mathbf{A}^{-1}\delta\mathbf{b} - \mathbf{A}^{-1}\delta\mathbf{A}\mathbf{x}$$

If the matrix \mathbf{A} is exactly known, then:

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\| \|\mathbf{A}^{-1}\| \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} = \kappa(\mathbf{A}) \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

If the RHS is known exactly, then:

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\mathbf{A}\| \|\mathbf{A}^{-1}\| \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} = \kappa(\mathbf{A}) \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}$$

We have 2 equations $0.9999 x_1$ minus $1.0001 x_2$ equal to 1 and then x_1 minus x_2 equal to 1 plus epsilon, you can see that for epsilon is equal to 0 that solution will be half minus half, because here you will have 1 and for x_1 equal to 0 half minus half will satisfy half minus half will satisfy this equation exactly, in terms of epsilon you can say that the solution is this.

Now, note that with epsilon equal to 0 you have the solution as x_1 equal to half x_2 equal to minus half. Now we put a epsilon equal to something then you see what will happen, as you put epsilon equal to some small value even a value of 0.0001 will change x_1 and x_2 drastically, why? With a small value of epsilon which is 0.0001 this get magnified here with 10001 and here with 9999 and; that means, there will be a significant change in x_1 and x_2 . That tells us that this solution, solution of this system is very sensitive to small changes in the right hand side, with small changes epsilon there will be significant changes in the solution x_1 and x_2 .

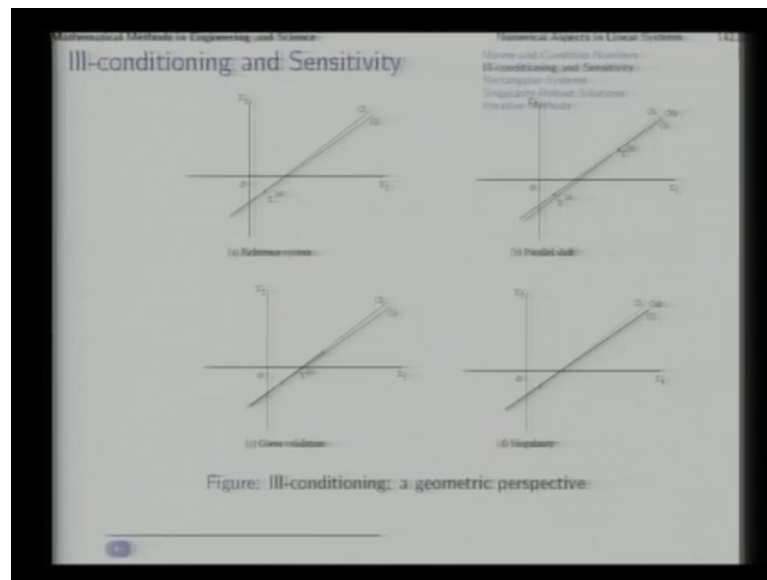
So, this sensitivity to small changes in the right hand side is the ill conditioning and if you try to find out this matrix, this the condition number of this matrix you will find that it is very large. You can see that because the first row is very close to 1 1 and the first

column sorry, the first column is very close to 1 1 and the second column of the coefficient matrix will be very close to minus 1, minus 1 and they are; obviously, close to linear dependence and; that means, the matrix the coefficient matrix will be very close to singularity.

That means ill condition according to our definition and that ill conditioning manifest itself in this high sensitivity to small changes in the right hand side, why this is dangerous? Because all numbers that we get in practical situations are results of measurements or other calculations. Now, any measured data or any result of calculations will be susceptible to some small errors now that will mean that with small errors in the data the final solution will suffer badly and this is the result of that ill conditioning. In this sense it is ill, if there is a person who is not well who is suffering from bad health then small exposures to heat cold or fatigue will put him to other illnesses and in that sense this coefficient matrix will be ill.

Apart from being sensitive to small thing changes in the right hand side there will be another bad result of ill conditioning of this matrix and that is in terms of validation of a guess. Suppose somebody makes a guess of this solution, this solution of this system then with a wide range of guesses the equations will verification through that equation will tell us that it is almost right; that means, small mistakes in the guess of the solution will not be identified, will not be captured in this kind of a system in order to see let us in order to see how it works.

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Let us see the illustration, here the two lines the two equations those two equations in terms of lines in the x_1, x_2 planes are plotted here as 1 and 2.

Their point of intersection is here that is half minus 1 half as I told you that is the correct solution this is the reference system that is the correct solution. Now, with a small change in epsilon the second equation changes; that means, the second line shifts parallel to itself, in this case the line demarcated as 2 b is the shifted position and this 2 is the original dashed line 2 is the original position of that second line.

Now, see where the point of intersection is, point of intersection is here because of a very small angle between the 2 lines to a small parallel ship of the line 2 the point of intersection suddenly jumps from this point to this point and this is the result of small change in epsilon, large change in x_1 and x_2 . That solutions shift by a huge distance, this is one difficulty through a parallel shift the point of intersection changes suddenly, if we had tried to validate a guess which is not right half minus half is the correct solution for epsilon equal to 0, but then if we had decided that we will try a 0.10 which is somewhere here even that 0.10 is extremely close to both the lines and here if you try to put 1 0 as solution in the equations you see what do you what do you get one into this number minus 0.

So, this is 0.9999 almost close to 1 then 1 0, 1 minus 0 is 1 exactly right; that means, if you had tried to use point 1 0 as the possible solution as a guess these equations will tell

you that your guess is almost right, but from the figure you know that half minus have is here, that is a correct solution and 1 0 is here. Not only 1 0 any point in this narrow zone guessed and verified will tell us that it is almost the right answer, though it is far away from the right answer. That is what you show see here in this shaded portion any point in that shaded portion will impersonate as almost right answer, there is the another interesting feature here, rather than shifting the line to a little through epsilon if you had rotated line 2 then that would mean that making changes here and here. Making this 0.9999 other than 1 and making this 1.0001 rather than 1 will mean slight change, slight rotation of the line and if you anchor that line while rotating about this point then you will get the two lines coincident in that case there will be infinite solutions exactly singularity.

On the other hand if you anchored at any other point say at 1 0 while rotating the line then you will get a line which is parallel and in that case you will find that there will be no solution, another manifestation of singularity of the coefficient matrix. So, all these different qualitatively and quantitatively different scenarios can arise through small changes in the right hand side or in the coefficient matrix in case of an ill conditioned system which will not be so if the condition number of the coefficient matrix is small that is if the matrix is nice and healthy. To analogically see these things let us consider this system $Ax = b$ for which the solution is $A^{-1}b$ and let us analyse the first variations for this $Ax = b$.

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The image shows a green chalkboard with handwritten mathematical equations. At the top, the equation $\delta Ax + \delta x = A \delta b - A \delta A x$ is written. Below it, the following inequality is derived:

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\| \|\delta b\|}{\|x\|} \frac{\|b\|}{\|b\|}$$

$$= \|A\| \|A^{-1}\| \frac{\|b\|}{\|A\| \|x\|} \frac{\|\delta b\|}{\|b\|}$$

If we consider first variations then we say first variation of A into x plus A into first variation of x is equal to first variation of b right. So, this is what you get as the first variation of the equation $Ax = b$, from here we will try to find out an expression for Δx and for that we take this first on the other side and then pre multiply over all with A^{-1} .

As we do that we get $A^{-1} \Delta b - A^{-1} \Delta A x$ that is here $A^{-1} \Delta b - A^{-1} \Delta A x$. Now, this we use as our important relationship which tells us how Δx changes with small errors in right hand side and small errors in the coefficient matrix. We consider the 2 issues separately, first suppose the matrix is exactly known, in that case ΔA will be 0 and we have to consider the effect of Δb . Now, when we try to consider the effect of Δb only then Δx norm we take the size, we take the size.

So, in the first case we are considering that the matrix is exactly known; that means, ΔA is 0, Δb is something and we are interested in seeing its effect right. So, Δx norm, now we know that norm of this will be less than equal to norm of this into norm of this right. So, this will be less than equal to the product of the norms right, now both sides we can divide by this now what we do we consider the division and multiplication by mod of Δb , norm of Δb sorry norm of b like this right.

Our intention is to compare the fractional change fractional error in x to the fractional error in the data that is b . So, we will compare mod of Δx by mod of x with mod of Δb by mod of b this is our intention right. Now, this we can write as these two have been taken care of, this we put here this has been taken care of these two remain right, what we do is that we multiply with this and we also divide by that, this multiplication division has taken care of each other and the rest of the 2 terms which apply back this b norm here and this x norm here.

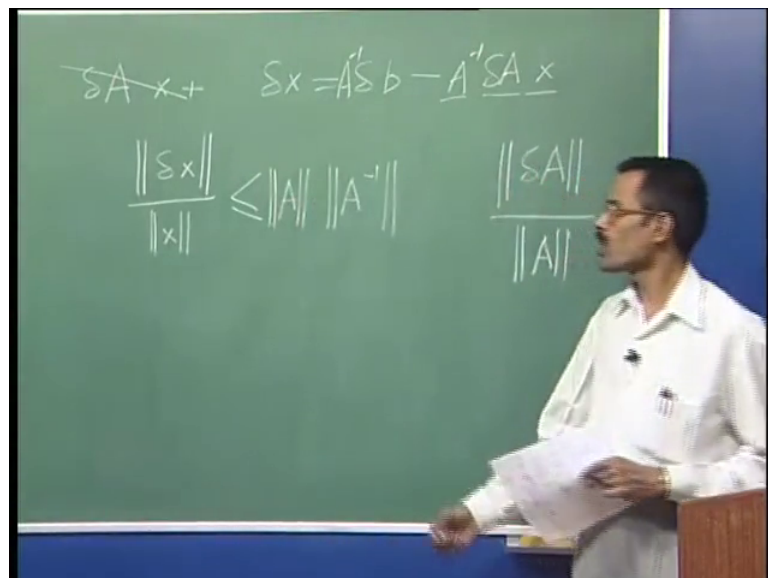
Now, again remember that compared to norm of A into norm of x norm of b will be less that is because of that same old relationship that is norm of x is less than equal to norm of A in the norm of x . That is why this will be less than equal to this so; that means, that that is this whole fraction is less than equal to 1 so; that means, that this side is less than equal to this which is equal to this, which is further less than equal to this into this and that is shown here that is this fraction is less than equal to this.

What is this stuff here defined as condition number of A ; that means, we have this relationship; that means, that the fractional error in the result x is limited, is less than equal to κ times the fractional error in the right hand side. Now, if κ is large if condition is large condition number is large; that means, if the matrix is ill conditioned suppose the matrix is 1000 that will mean that by a small change in b , small error in b the result can be erroneous by 1000 times that small error.

If there is a 1 percent error in b in response to that as a result of that as a consequence of that the error in x can be 1000 percent this is a result of ill conditioning what we saw in the illustration. In the second situation if right hand side b is known exactly and Δb is 0 and ΔA is something, then when we try to do a parallel operation then see we reach the same conclusion in this case we are taking Δb as 0 right hand is exactly known this may be something.

So, the norm of Δx will be less than equal to norm of this into norm this vector right, again the norm of this vector will be less than equal to norm of this matrix into norm of this vector; that means, the twice less than equal to further less than equal to, those twice less than equal to things we get here also right.

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So, then again we find that Δx norm will be less than equal to norm of this into norm of this into norm of this and this norm of x we have already divided here. So, that will mean that we will get here right and as we multiply and divide again with norm of A then

you see again, what we have got here is this. This is kappa of a condition number and this is fractional change in the size of A , norm of A again here the fraction is the fractional change in size of x ; that means, again if condition number is something like 1000 that will mean that a one percent error in the matrix coefficients matrix values matrix entries or matrix norm rather will allow a 1000 percent error in the result in x and this is again the result of insensitivity; that means, there is a sensitivity to the small changes in the matrix.

That means if the matrix is ill conditioned, if this kappa is large that will mean that with small changes with small errors in the right hand side or in the coefficient matrix you can expect large change or large error in the result. So, the solution that you will be finding numerically will not be quiet reliable, now we need to handle such situations and we know that when very large numbers and very small numbers turned up in the same computation together then the numerical errors grow faster and the resulting solution may not be very reliable, how do we handle such situations and try to find out solutions which are skill robust. This is 1 issue that we handle in subsequent discussion; another way other than ill conditioning another way in which a system of linear equation could be bad is by the matrix shape.

If the question matrix is rectangular that is if the number of equations is larger than the number of unknowns or vice or versa then the standard gauss elimination type of processes will not apply directly. We know we can analyse the situation completely through the method which we discussed earlier regarding whether solutions exist, how many solutions are there and whether they are infinity and so on and try to describe them. But quite often you will find situations where irrespective of whether there are infinite solutions or whether there are no solutions you are required computationally to find one solution which is a good solution, which is a representative solution and which is a useful working solution.

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Numerical Aspects in Linear Systems 143

Rectangular Systems

Consider $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\text{Rank}(\mathbf{A}) = n < m$.

$$\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b} \Rightarrow \mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

Square of error norm:

$$\begin{aligned} U(\mathbf{x}) &= \frac{1}{2} \|\mathbf{Ax} - \mathbf{b}\|^2 = \frac{1}{2} (\mathbf{Ax} - \mathbf{b})^T (\mathbf{Ax} - \mathbf{b}) \\ &= \frac{1}{2} \mathbf{x}^T \mathbf{A}^T \mathbf{Ax} - \mathbf{x}^T \mathbf{A}^T \mathbf{b} + \frac{1}{2} \mathbf{b}^T \mathbf{b} \end{aligned}$$

Least square error solution:

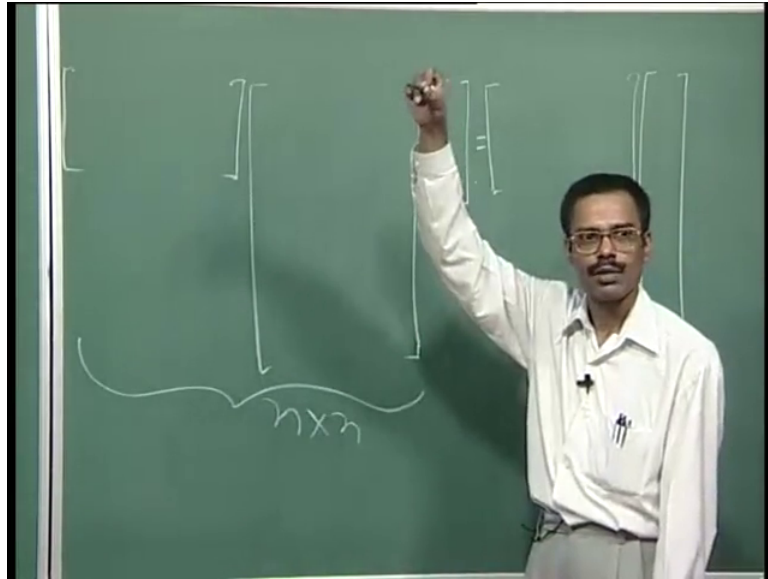
$$\frac{\partial U}{\partial \mathbf{x}} = \mathbf{A}^T \mathbf{Ax} - \mathbf{A}^T \mathbf{b} = \mathbf{0}$$

Pseudoinverse or Moore-Penrose inverse or left-inverse:

$$\mathbf{A}^+ = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$$

How to find that? So, first we consider 2 cases of rectangular systems one in which the number of equations is largest and the other in which case the number of unknowns is larger, in both cases first we consider the situation where the coefficient matrix is at least full rank. Which is relatively easier and afterwards we try to combine all sorts of difficulties together that is the equation system may be rectangular at the same time there may be ill conditioning and we try to see how we work out a solution still in the phase of all these difficulties. First consider the system $\mathbf{Ax} = \mathbf{b}$ in which \mathbf{A} is a matrix of size m by n m is larger; that means, more number of equations; that means, the matrix is actually a tall matrix thin and tall matrix (Refer Time: 29:44) number of rows.

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Than columns, the matrix shape will be something like this number of unknown be less and number of equations will be more, this is the kind of system we are talking about; that means, less number of columns, more number of rows rectangular with this kind of shape this is the first guess that we take up, in which n is less than m .

But then fortunately the rank of A of the matrix a is n ; that means, all these n columns are linearly independent; that means, its rank is n it is a full rank matrix, later we will remove this special situation also and try to handle even worst case. Now, in this situation before applying all these methods we try to see that in only n unknowns there are so many m equations we do not expect them to be consistent we expect conflict as we expect conflict.

That means, that mathematically solution does not exist, but still there are situations where we have to say that whatever little discrepancy, whatever little conflict is there among the equations that is due to experimental errors we do want a representative solution, this kind of situations quite often arise in problems where we need to find the least square error, we need to find a solution parameter set with our moral is right, but the data is erroneous because of experimental measurements. So, in that case one quick remedy we can suggest and that is multiply both sides with a transpose, if we multiply both sides with a transpose then this transpose will be of this kind of a shape, on this side

also we will multiply with a transpose and in that case you will find that the matrix that comes a transpose A will be a square matrix n by n size.

And the product here A transpose b will again be a small vector of size n small size. So, that is what we get here through pre multiplication with a transpose on both sides we will get this system which is a small m by n size and because of the full rank nature of the matrix this matrix coefficient matrix now a transpose a will be a non singular invertible matrix, square matrix, non singular matrix.

So, now, we can apply our cholesky decomposition kind of methods which we studied in the previous lecture and immediately get the solution and that solution is this. Now, this multiplication pre multiplication with a transpose is not something add how, not something arbitrary that we did just in order to solve the system it has a much deeper meaning and in order to discover that meaning let us consider the question completely separately, completely independently.

The question of minimizing the error norm, we say that we want to find out that x which minimizes the square of the error $Ax = b$ was the system of equations, we expected conflict most probably there will be conflicts. So, the error will be $Ax - b$ now we say we want to find out that x with which this error is minimum or the square of that error is minimum as you open this square of norm we find it is $Ax - b$ transport $Ax - b$ which gives rise to this expressions.

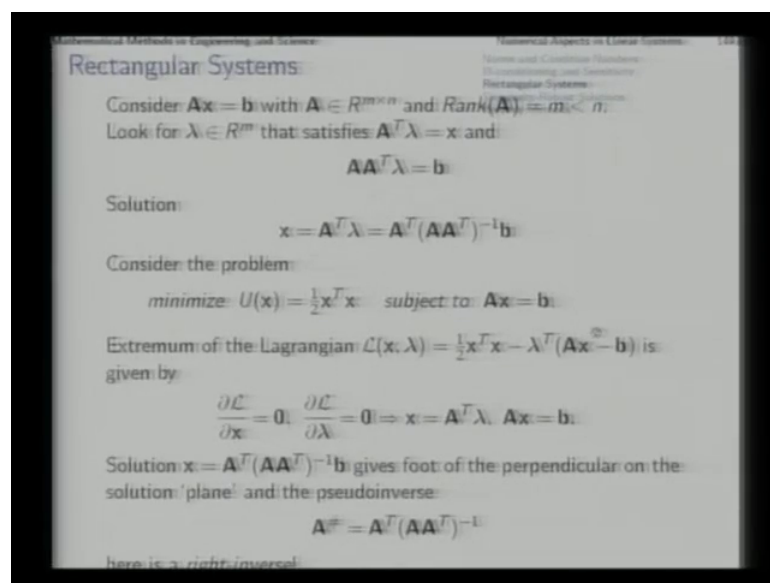
Then you say that for this function to be minimum this errors square to be minimum its first derivatives with respect to x that is its gradient with respect to x must be 0 and when we find the gradient we find it is this and this equal to 0 is nothing, but the equation system which we actually solve. A transpose Ax is equal to a transpose b ; that means, through this small trick of pre multiplying the equation system with a transpose on both sides, what we have done is that we have filtered out a solution which minimises the error square.

This is why the resulting solution x is called the least square solution and the matrix here from here to here before b is known as the pseudo inverse or Moore Penrose inverse because in a way that is acting like an inverse. Because it is giving you something like a solution of $Ax = b$ to the multiplication pre multiplication of b something into b is the solution of $Ax = b$; that means, that something is in a way acting like

the inverse of A that is why it is called pseudo inverse it is not rightly inverse, but it is some sort of an inverse it is called pseudo inverse or Moore Penrose inverse in this case it is also called the left inverse because multiplication of this on the left side of a will give you identity.

Now, this is one case in which the shape of the matrix is like this, now you consider the other case in which the shape of the matrix will be fat short, that is a case where we will have m less than n; that means, you have got a like this.

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So, again here in $\mathbf{Ax} = \mathbf{b}$ this will mean that you have got less number of equations in larger number of unknowns; that means, too many unknowns to determine this problem is typically indeterminate that is infinite of solutions you will expect if the rank is full as is shown in this particular case. The rank is full m; that means, m only rows are there with too many columns n columns, but those m rows which are there they are linearly independent, with that situation you will have infinite solutions for the system of equations, but then there may be many situations in which you are not interested in those infinite solution you want to find only one situation with in a way will be a very nice good useful solution. For that how to find out one solution you want a square matrix, but then like last time you cannot multiply with A inverse on the left side, A transpose with the left side on the left side.

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$$\begin{bmatrix} & & \\ & & \\ & & \\ \hline & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ \hline & & \\ & & \\ & & \\ \hline \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} x = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

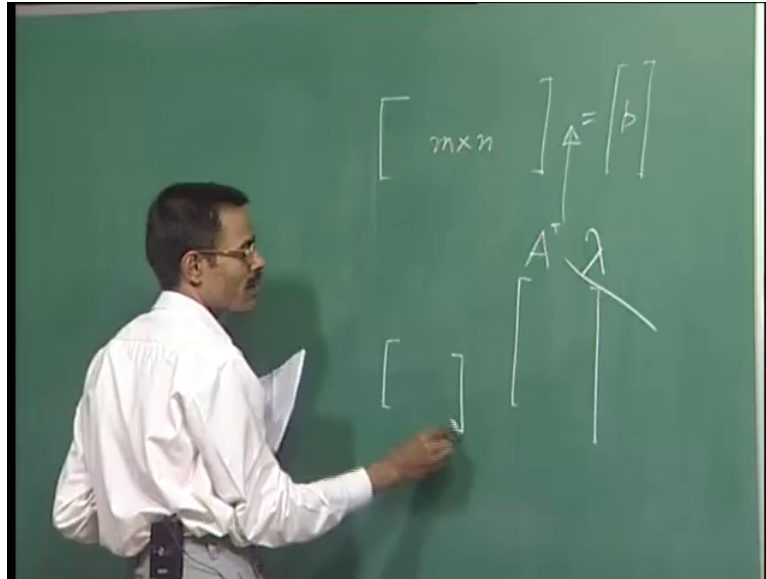
$n \times n$

Because here matrix a is of this shape, x is large if you multiply with a transpose then you will be essentially multiplying with a matrix like this and as a result you will get a matrix which will be of this size, huge matrix you will get.

So, this matrix was m by n with m less n larger this is m by n and the resulting matrix a transpose A that you will get will be n by n which is large and this matrix is sure to be singular because this matrix of rank m and this matrix of rank m in product will never give you a matrix which is of rank higher than m. So, at most it can have rank m and which is less than n; that means, this matrix will be certainly singular and your methods like gauss elimination or lu decomposition will certainly fail. So, this is not the way in which you should go in this kind of a situation.

So, what to do? You apply a different trick in this case, what you must do here is not to pre multiply with a transpose, but you look for a vector lambda m dimensional vector lambda that satisfies a transpose lambda and as you do that then a transpose lambda equal to x; that means, you look for an m dimensional vector lambda which will satisfy a transpose lambda equal to x and then in place of x you try to insert this a transpose lambda. That means, that here in place of x you try to insert a transpose lambda and what is this a transpose a transpose is the transpose of this matrix.

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That means transpose of this short fat matrix will be a long, tall thin matrix and the product of AA^T will be again short of this size square size small matrix which is m by n .

So, then we have this system $AA^T \lambda = b$ and rank of A is m that will mean that this matrix AA^T is symmetric and non singular that is positive definite and then you can solve this and get λ that will be $AA^T^{-1} b$ that is from here to here will be the solution and that λ when multiplied with A^T will give you x . So, you get this solution which is somewhat different from the earlier solution in which you got $A^{-1} b$ after multiplied with A^T here that $s b$ multiplied with A^T , earlier it was $A^{-1} b$ after multiplied now it is $AA^T^{-1} b$ after multiplied with A^T .

So, this is the solution now again last time we found that the particular trick with multiplication with A^T had a meaning in terms of minimization of error, what is the meaning here? To discover that meaning again you consider this small optimisation problem, this is a constrained optimisation problem you try to minimise this size of the solution itself $\|x\|^2$ half of that subject to these constraints these equality constraints which is actually the system of equations that you wanted to solve. So, here because of infinite solutions of this system you are asking for that vector x which

satisfies this equations as a required condition and among all the infinite solutions it tries to find you the smallest size solutions.

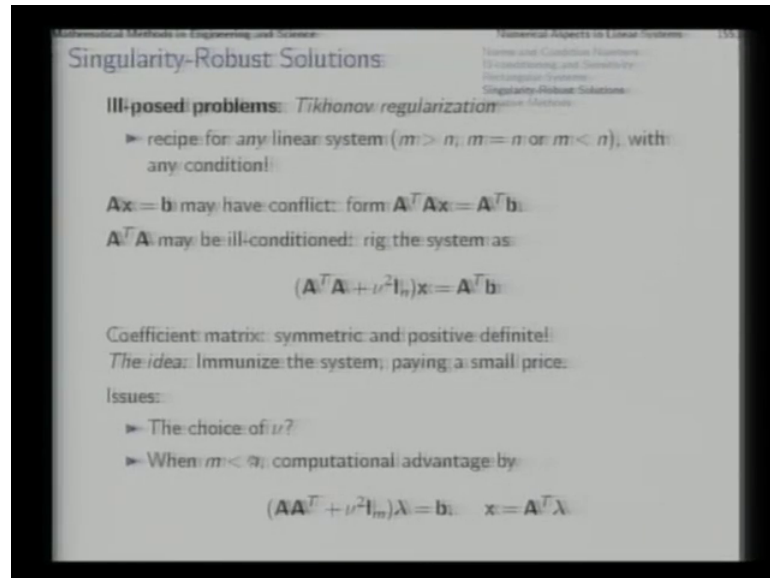
If you try to formulate this constrained optimisation problem one possible formulation is TWO find the extremum of the Lagrangian that is lagrangian is this objective function minus lambda transpose A x minus b and the extremum of this lagrangian is found at that point x and lambda where the derivative of this with respect to x and the derivative of this with respect to lambda the corresponding gradients are both 0. When you apply that condition through differentiation of these and those gradients you find then the gradient with respect to x gives you x minus A transpose lambda equal to 0; that means, this which is what we assumed here and the gradient with respect to lambda equal to 0 tells you A x minus b equal to 0 which is the original system of equations and these are the 2 systems of equations their solution of which has been found here in this manner right. That means, through this small trick you find that x which is the minimum sized x that satisfies these system this systems of equations, apart from many others which you are not bothered with.

Now, this particular solution gives you the foot of the perpendicular on the solution plane; that means, a lot of solutions are possible for this system because this system actually defines a plane like entity in the space of x and out of all the points that you can take on that plane like entity the one which is closest to the origin, the size of which is the least that is actually the one which will minimise this subject to this constraints. The constraints come in the equation of the plane and this minimization actually gives you the foot of the perpendicular to that plane right and this matrix A transpose AA transpose inverse which is the entire thing working like some sort of an inverse that is the pseudo inverse in this case or the Moore Penrose inverse or in this case it is called the right inverse because from the right side if you multiply with this matrix to a then the result is identity. Now, we found 2 situations in of rectangular systems, rectangular systems of equations in one case number of equations is more in other case number of unknowns is more.

Now, in the most general situation where it could be either way or that way compounded with the additional problem of ill conditioning that is rank of a is not even m is less in the other case rank of A is not even n even less. That will mean that you have got the problem of the system being rectangular combined with the additional problem of rank

deficiency or singularity of AA^T in this case and a transpose A in that case compounded with that could be the ill conditioning issue; that means, the most general situation.

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This is the situation quite often encountered in ill posed problem, in which the statement of the problem is not enough not completely clear and in that sense, in that situation we still can find out a solution which is a very good working solution and which is quite robust not so sensitive and that is called the singularity robust solutions. The method to do that is called tikhonov regularization and this particular technique gives you a recipe for any linear system, m greater than n or m equal to n or m less than n with any condition. It could be well conditioned, in it could be ill conditioned or it could be even singular in all cases one recipe can be useful with a little additional computational cost which will and with a little extra error, but how much is error that is in your hand, but a singular recipe will be able to solve all cases of sizes and shapes and with any condition.

The way to handle that is you first make the observation that the system of equations $Ax = b$ may have conflict. So, in order to handle that conflict you pre multiply with a transpose and get this so called normal system of equations which is certainly consistent. Now, then you say that still the coefficient matrix that we have in hand a transpose A that maybe singular or that may be ill conditioned then what you can do is that in order to handle this equation, this matrix which may be ill conditioned you rig the

system of equations. Add a little new square in the diagonal entries that is equivalent to adding μ square identity to this matrix, add you as you add this you enrich the diagonal entries by a little amount. Now, you see that a transpose a even in the case of ill conditioned or singularity would be at least positive semi definite; that means, its value for any x with x transpose this matrix x would never be less than 0 in the worst case it could be equal to 0.

Now, with this addition it will not even be 0. So, now, this matrix this coefficient matrix is symmetric and positive definite you can prove that, you can prove this that this matrix is symmetric and positive (Refer Time: 47:44) in the book some of the, one of one or 2 of the exercises advise you to prove this particular statement that this matrix is always symmetric and positive definite. Symmetric is of course visible, positive definiteness you can prove, now if it symmetric and positive definite then ordinary methods (Refer Time: 48:02) or any other method will (Refer Time: 48:04) to give you a solution of this system and; that means, a value of the vector x .

So, the idea of this method of techno tikhonov regularization is that expecting or being concerned of some level of ill conditioning in the system you first immunize the system with this little dose of errors, that is the little price which will result in slight amount of error, how large you will allow that will depend upon this magnitude of μ . You can choose μ to be as small as you like, you can choose it to be 10 to the power minus 3 , 10 to the power minus 6 that is based on your understanding of up to how much level your computation is going to proceed without any trouble. Based on that your error value will be large or small now with a little immunizing error here you make the system solution process safe, there will be no further difficulties.

So, the issues at the choice of μ which is in your hand and another small issue is that if m is smaller than n that is the number of equation is less then rather than handling an n by n matrix here you could get computational advantage by considering an equivalent system. Rather than solving this you could rather solve this rather than n by n system here you will be then solving an m by m system get λ and then evaluate x separately. So, if m is significantly smaller than n then this alternative way will give you an advantage computationally, less number of computational operations and you can show that this calculation will result in the same final result as this calculation, there is a small exercise which I leave for you. Apart from these methods which are non iterative

methods in which you do not need any initial guess there are a few methods which are by useful in many application problems they are iterative in nature, 2 iterative methods for solving linear systems are quiet well known one is Jacobi's iteration method and the other is Gauss Seidel method.

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Iterative Methods

Jacobi's iteration method:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1, j \neq i}^n a_{ij} x_j^{(k)} \right) \text{ for } i = 1, 2, 3, \dots, n.$$

Gauss-Seidel method:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right) \text{ for } i = 1, 2, 3, \dots, n.$$

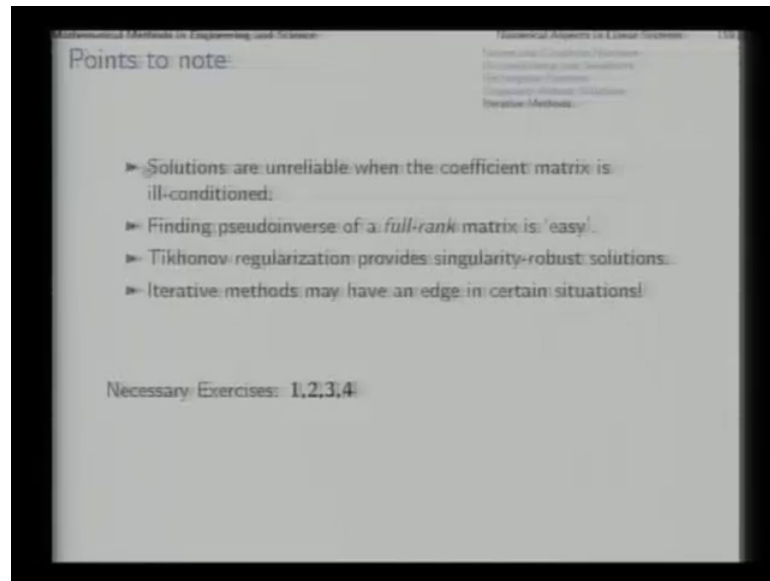
The category of relaxation methods:
diagonal dominance and availability of good initial approximations.

Table of Contents (right side):
 Sparse and Ill-conditioned Systems;
 Preconditioning and Conjugate Gradient;
 Perturbation Systems;
 Singular Value Decomposition;
 Iterative Methods.

Now, in both of these methods you start with a guess and through iterations you would try to improve, at every equation you change one of the unknowns and through iterations after iterations you try to improve the estimates of the unknowns. Now the iterations will result in improvement only when we organised the equations in a particular manner and that particular manner is related to the concept of diagonal dominance, when the diagonal entries of the coefficient matrix are significantly larger than the off diagonal entries then you call that kind of a matrix as a diagonally dominant matrix.

And in that case these kinds of depths will actually lead to improvements in the values of x_s , these methods work very well in the case of the availability of good initial approximations. A situation which quite often happen in solutions involving the linear equations that arise from differential equations and therefore, these Gauss Seidel and Jacobi's methods are quite often used in the methods for solution of differential equation and these methods in general are called relaxation based methods.

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So, in this chapter, in this lesson we have considered these important points that is we have noticed, we have noted that the solutions are unreliable when the coefficient matrix is ill conditioned and we have found that finding pseudo inverse and finding a quick solution for full rank position matrix. Even rectangular ones is easy and when that is not the situation then we have found tikhonov regularization method to find singularity over solutions and finally, we have seen the iterative methods which may have an edge in certain situations where good approximations are available.

Now, I will take you briefly to the original contents of this course in these lectures that we have covered till now; we have covered these chapters, these lessons which completes one module of our course that is the module of systems of linear equations.

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Mathematical Methods in Engineering and Science

Contents II

- Eigenvalues and Eigenvectors
- Diagonalization and Similarity Transformations
- Jacobi and Givens Rotation Methods
- Householder Transformation and Tridiagonal Matrices
- QR Decomposition Method
- Eigenvalue Problem of General Matrices
- Singular Value Decomposition
- Vector Spaces: Fundamental Concepts*

Next we will be taking up, in the next lecture we will be taking up the problem of algebraic Eigenvalue problem that will go through 1, 2, 3, 4, 5, 6. So, these 6 lessons we will cover the algebraic eigenvalue problem and here you will note that we have covered quite a few topics in the system of linear equations and if you have covered these lesson extremely fast then I will remind you that without going through the, without going through the exercises will sometimes make you lose contact with the subject matter that we are discussing.

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Mathematical Methods in Engineering and Science

Logistic Strategy

Tutorial Plan

Complex Submatrix	Exercises	Complex Submatrix	Exercises
1	1.1	26	1.1.1.1
2	1.1.1.1	27	1.1.1.1
3	1.1.1.1	28	1.1.1.1
4	1.1.1.1	29	1.1.1.1
5	1.1.1.1	30	1.1.1.1
6	1.1.1.1	31	1.1.1.1
7	1.1.1.1	32	1.1.1.1
8	1.1.1.1	33	1.1.1.1
9	1.1.1.1	34	1.1.1.1
10	1.1.1.1	35	1.1.1.1
11	1.1.1.1	36	1.1.1.1
12	1.1.1.1	37	1.1.1.1
13	1.1.1.1	38	1.1.1.1
14	1.1.1.1	39	1.1.1.1
15	1.1.1.1	40	1.1.1.1
16	1.1.1.1	41	1.1.1.1
17	1.1.1.1	42	1.1.1.1
18	1.1.1.1	43	1.1.1.1
19	1.1.1.1	44	1.1.1.1
20	1.1.1.1	45	1.1.1.1
21	1.1.1.1	46	1.1.1.1
22	1.1.1.1	47	1.1.1.1
23	1.1.1.1	48	1.1.1.1
24	1.1.1.1	49	1.1.1.1
25	1.1.1.1	50	1.1.1.1

And I will draw your attention back to this tutorial plan; that means, in the chapter 2, 3, 4, 5, 6, 7 in the book that we have covered till now in these few lectures these are some of the problems in the exercises which you must attempt in order to keep the pace of understanding at a good position and these are the problems which are particularly important which are plugged here as possible tutorials problems.

So try to cover as many problems in the exercises in the book as possible and certainly including these with special attention to these problems which are backed here as problems in the next lecture we will make an introduction to the eigenvalue problem and continue forward.

Thank you.