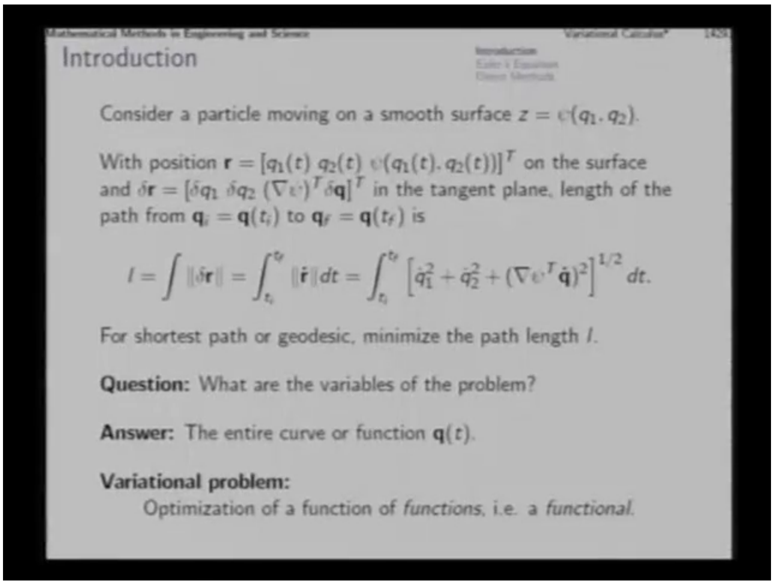
Mathematical Methods in Engineering and Science
Prof. Bhaskar Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Module – 08
Overviews: PDE's, Complex Analysis and Variational Calculus
Lecture – 06
Calculus of Variations

Good morning, we are about to conclude this series of lecture in this course and the way this course is, the way it includes at some detail almost every aspect of applied mathematics, its fitting concluding lecture which discusses variational calculus. The central theme the fundamental idea of variational calculus is in optimization and its formulations are often made in terms of differential equations and in order to solve problems of variational calculus, quite often you will use linear algebra numerical methods and approximation theory as operational tools.

All these aspects; all these building blocks we have already studied in the previous modules of this course, and as we have been seeing interconnections among different areas of applied mathematics this particular topic of variational calculus seems to embody a very large number of these interconnections.

(Refer Slide Time: 01:47)

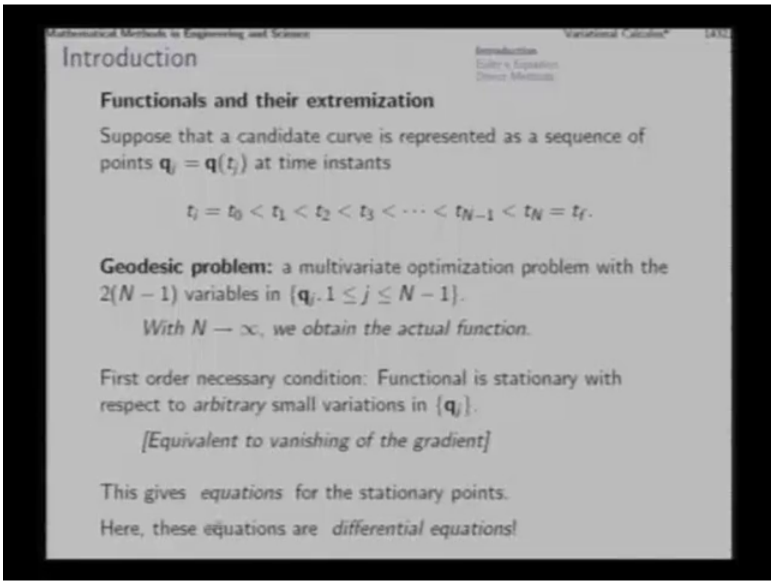


So, first we try to appreciate the nature of the problem of variational calculus. Let us consider a particle moving on a smooth surface, z is equal to psi of q 1 q 2. Q 1 q 2 are the x and y coordinates and this is an explicit equation of a surface. Now with the position of a point on the surface like this q 1 q 2 and the corresponding z on the surface and a small vector delta r given like this delta q 1, delta q 2 and delta z as here; so, such a small step will be in the tangent plane and as we take delta r sufficiently small, the departure from the surface will be accordingly small; that means, as we stay as we consider delta inter decimally small tending to 0 the size of it, then with such steps we will be actually moving on the surface. So, then length of the curve length of the path from the initial point q i to the final point q f on the surface in this manner is given like this, which if you consider this dramatization.

With respect to the parameter t, let us say time then this delta r is replaced with this and this integral is replaced as a replaced by a definite integral from t i to t f and that will give us this and the result is the path length i. Now if we ask for the shortest path from q i to q f on the surface, then we will be basically asking for the solution of the geodesic problem this shortest path is called the geodesic which will minimize the path length i. Now you see this is an optimization problem, we are interested in minimizing the path length i, but with respect to which variables what are the variables of this problem it is which are the quantities, which are the values which we can alter in order to explore the lowest possible value of i the path length and you will notice that it will not be individual points that we are looking for a point q x y z and so on we are actually talking about the entire curve and then when we ask what are the variables of the problem, what are the quantities which we would like to change in order to achieve a lower path length.

In the process of this optimization then we will be talking about the entire curve and therefore, the entire curve of the path is the variable of this problem; that means, this function is the variable of the problem which we want to explore in order to minimize the path length. So, the objective here is not a function of ordinary variables, but it is a function of functions; that means, you take one function q get one path length i, you take another function q get another path length i; that means, i here the objective function is actually a function of functions a function of the this function q, and such a function of functions is called a functional. These are the types of objective functions which we seek to minimize in a typical variational problem.

(Refer Slide Time: 05:12)

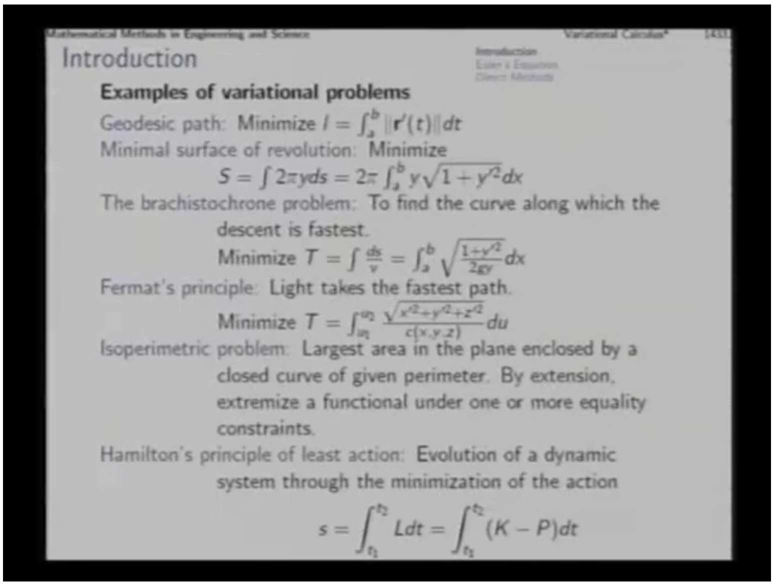


Continuing on this theme suppose that a candidate curve is represented as a sequence of points like this q 0, q 1, q 2 q 3 etcetera at these time instants.

Now, if we make such a discrete representation of the curve then the problem the geodesic problem which was the originally the problem of choosing a function in order to minimize the path length a function of this function that reduces to choosing n minus 1 points or twice n minus 1 variables. the corresponding variable values of q 1 and q 2. So, choosing the variables values of these twice n minus 1 values. So, the q 1 q 2 values at these n minus 1 time instants. So, in that case in that discrete representation only this problem reduces to a simple multivariate optimization problem, which we have studied earlier as n tends to infinity we actually obtain the true function and you know that the function space is an infinite dimensional vector space and this particular representation of it this particular discrete representation of it is just a finite dimensional representation.

Now, if we try to write down the first order necessary condition for a particular curve to be the optimal curve, to be the shortest path the geodesic, then that first order necessary condition will mean that the this functional is at the first order level stationary with respect to arbitrary small variations in these points or in the curve in general. So, this is equivalent to vanishing of the gradient of the ordinary multivariate objective optimization problem. So, this in the case of multivariate optimization problem, would give us equations for the stationery points. In the case of the true problem in this case that will give a give an equation in the function and therefore, this equation will turn out to be differential equation rather than algebraic equations.

(Refer Slide Time: 07:41)

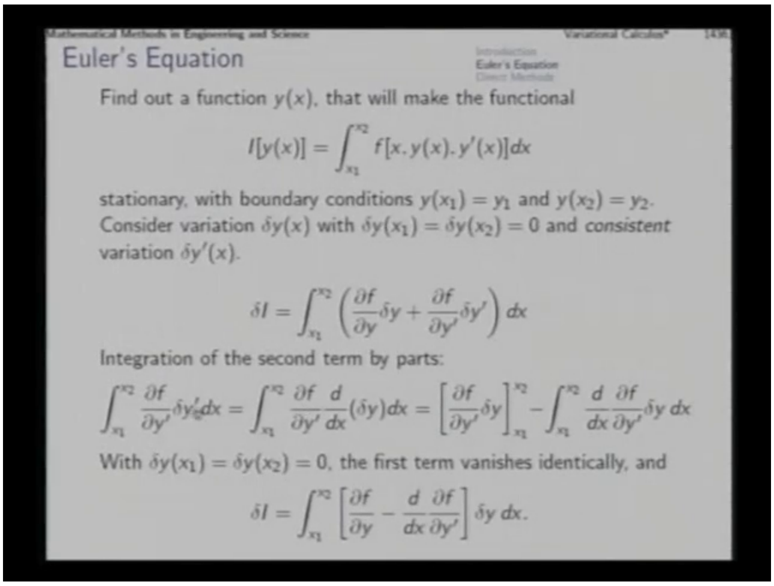


Now, this and similar other problems let do this development of the area of variational calculus or calculus of variations, some of the classic problems which were initially discussed among mathematicians in order to in the process of recognizing this as a particular branch of mathematics where this problems. One is this geodesic problem which we have just now discussed, similar other problems where minimal surface of revolution as you have a curve y of x and then you rotate it about the x axis to get a surface of variation. Now the question is which surface of revolution like this will have minimal surface area and; that means, this integral should be minimized. So, here S w t function is a function of the function y. The brachistochrone problem 2 points on a vertical plane which curve should join them.

So, that a particle starting from this point reaches this lower point at the minimum amount of time that is a brachistochrone problem; brachistochrone shortest time. So, that turns out to give you this integral as the objective function and the variable in this case is the curve y. Formats principle that is light takes the fastest path will require you to find the minimum time that the light will take, and in the process you will end up finding the path from this you can actually derive the laws of reflection and ref refraction. Similarly isoperimetric problem that is how to enclose largest area in the plane by a closed curve of give perimeter or the corresponding 3 d version. By extension these kind of problems give you the problems of extremizing a functional under certain equality conditions or equality constraints. The classic Hamilton's principle of least action also has a variational underpinning

And that is that the dynamic system in nature evolves through that path along with this action is minimized and what is this action action is the integral of the Lagrangian define as K minus P, where K is the potential kinetic energy and P is the potential energy of the system. So, these are some of those classic problems with led to the development of the area of variational calculus and you will notice that in all of these the objective function is actually defined in terms of a suitable integral which involves the function that is in the design space that is supposed to be chosen in order to optimize the objective function which is the integral and here all of these problems can be put in the form of certain differential equations as we try to as we try to derive the first order necessary conditions for those extremums extrema and that set of differential equations or that differential equation is called the Euler's equation.

(Refer Slide Time: 10:55)



Let us try to derive the Euler's equation for the simplest case, in which we are looking for the function y of x which will minimize the functional i defined in this manner. In which f is a function involving x y and y prime note that without involving y prime this kind of a problem will not arise. So, this kind of problem will arise only when up to the first derivative gets involved in this formulation.

So, if this is the integral which we want to minimize and for that we are interested in choosing the suitable y then for that of course, at x 1 and x 2 to begin with we put certain boundary conditions that is from y 1 at x equal to x 1 to y 2 at x equal to x 2, we want the description of the path description of the function y of x which will minimize this; that means, all those functions which have value y 1 and at x 1 and value y 2 at x 2 will be in the design space that is all of them will be permissible functions.

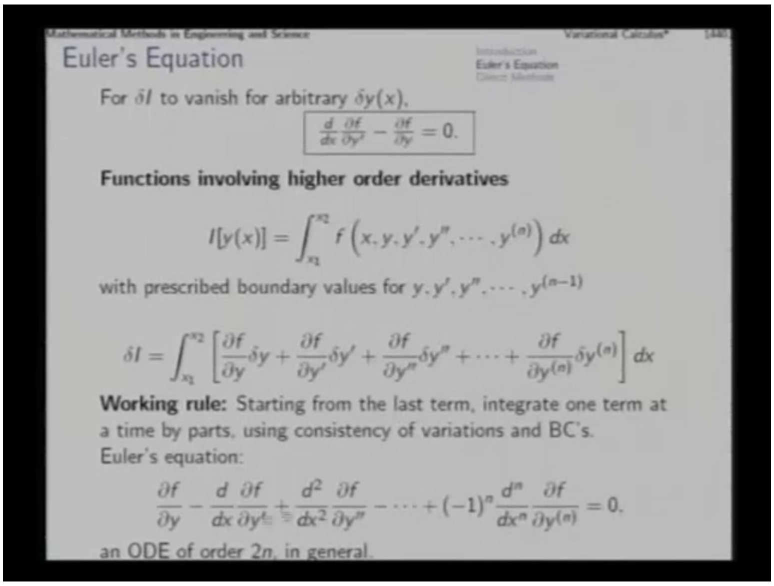
Which satisfy the boundary conditions among these permissible conditions which is the one that extremizes; this in particular we can ask which is the one which is the function around which first order variation will turn out to be 0, that is at which the functional is stationery. So, for that we consider a variation delta y with delta y at x 1 and delta y at a 2 0. So, that the boundary conditions are satisfied all the time and then the consistent variation delta y prime what is meant by consistent variations? That is we cannot chose variation in y and variation in y prime independently, that is the variation of y prime delta y prime should be such that which will be really the derivative of the variation in y that is the prime of delta y must be the delta of y prime that is the variation in the derivative should be truly the derivative of this variation; that means, we were we are looking for consistent variation delta y prime.

So, if we consider variation delta y and consistent variation delta y prime, then the first variation of this delta I will be the first variation of this integral and now since the variation is being considered with respect to y and y prime, while the integral is with respect to x therefore, the integral and diff derivative are taken with respect to different variables. So, we can take the variation inside integral sign and then say that variation of this integral will be the same as integral of the corresponding variation that is integral comes out and then the variations are taken with respect to y and y prime. So, we will have del f by del y into delta y plus del f del y prime into delta y prime, and the integral of this we have to consider. Now integrating this symbolically is not directly possible. So, the trick employed here is that we integrate this by parts and keep in mind we are looking for those ys.

Which satisfy the first order variation equal to 0 condition that is we are looking for that function y which will give us delta i equal to 0 for all variations all delta ys which are permissible that is which are which satisfy this. So, for that we try to integrate this by parts as we do this, this one integrated by parts gives us integral of this first function into this which is second function here. Now we have used here the consistency of the variations, that is delta y prime is d by d x of delta y now the integration of this second part is easy because that is appearing in the form of a derivative. So, integration by parts gives us this first function into integral of the second which is delta y, minus derivative of the first into this integral of that. So, derivative of this is this into this delta y integral of that. So, this is from the formula of integration by parts. Now this one is a term which is going to help us a lot because we know that at the limits of integration at x 1 and x 2 delta y is 0, that makes the delta y permissible variation ok.

So, this one when we put the limit x 1 and x 2 vanishes and we are left with this, and this is the second integral only second term here in this big integral. So, this value of the second integral when we substitute here, then we get delta y common in this and this. So, then del f by del y comes from here and minus d by d x of del f by del y prime comes from here delta y goes outside the bracket because it is appearing in both of the terms. Now we say that we are looking for those ys for which this integral this delta i first variation, is going to vanish for all delta y. Now when we list integral vanish for all delta y only when this integrand itself vanishes that is this vanishes.

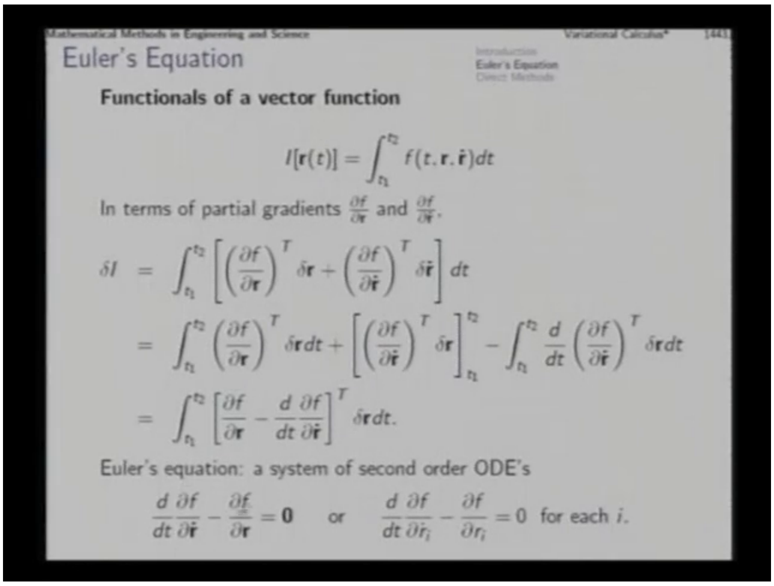
(Refer Slide Time: 16:58).



And this gives us the Euler's equations that it for delta vanish for arbitrary delta. Delta i to vanish for arbitrary delta y this bracketed term must vanish and this is the celebrated Euler's equation.

Now, in this particular case we considered the definition of the functional to involve only up to first derivatives in the integrand, it could involve high derivatives also like this and then the process is exactly the same, but operated quite a few times quite a few integration by parts we have to use in that and in that case finally, we get a differential equation like this. In this case the differential equation that we would get finally, by affecting these partial and ordinary derivatives, will be second order differential equation in this case with up to n th derivative involved here itself we would have an ODE of order 2 n.

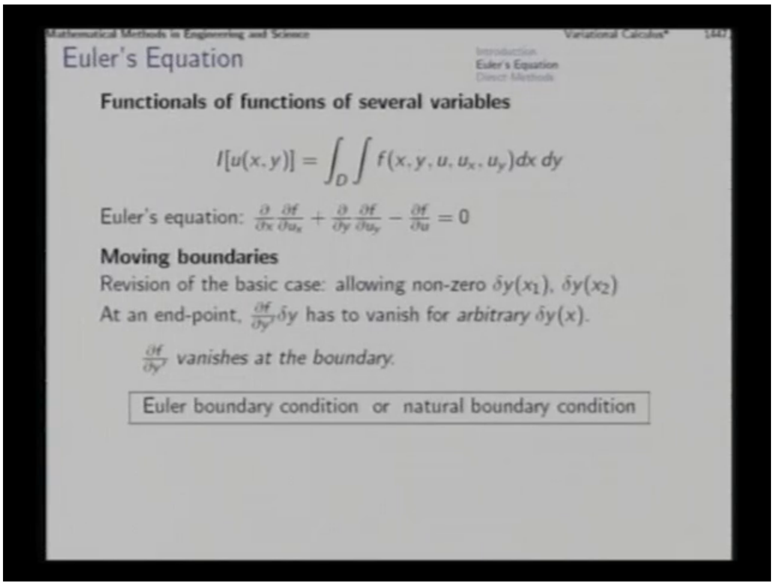
(Refer Slide Time: 17:55)



Now, similarly if we have a vector function of a scalar variable not simply y of x, but this vector function r of t and if the integrand turns out to be this in the definition of the functional, then the process which we carried out just now has to be carried out exactly like that except for vector functions and in that case in terms of partial radiant we will write the first variation in this manner gradient with respect to r transpose delta r plus gradient with respect to r dot transpose delta r dot. And then again we would integrate this part by parts and get these 2 parts this one will vanish.

Because delta r is 0 at t one and t 2 this will vanish and these 2 terms together will give us this and then we will argue in a similar manner and this formulation will be exactly the one which we would need in the case of the problem in which we started this lecture that is the geodesic problem. So, now, here if we argue that this first order variation should vanish for all delta r, then this itself must vanish and that becomes the Euler's equations for this problem and those functions r of t which satisfy this there is a solution of this system of differential equations, will turn out to be the geodesic path. So, you can tell that you can write this Euler equation in vector notation like this in which r and r dot are vectors and that means, these are gradient vectors or you can write them in terms of its individual components for each i. So, this is the way you derive differential equations governing the dynamics of multi degree of freedom systems, in multivariate dynamics and many other phenomena.

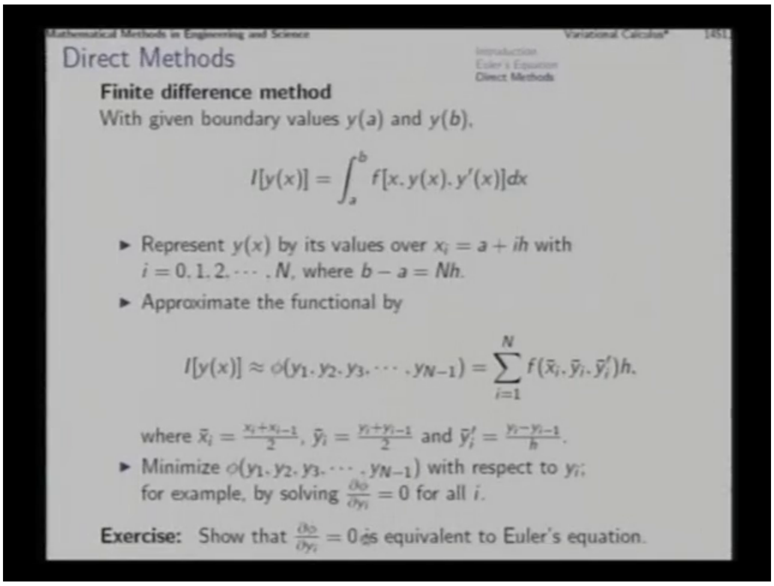
(Refer Slide Time: 20:01)



Similarly, if number of independent variables is not one, in the earlier problems we found independent variable is only one x or t. If the number of independent variables is also large that is x and y in this case 2 independent variables and a scalar function u that we are looking for. We are looking for as a solution u of x y which extremizes this functional, in that case we will get not an ordinary differential equations by a par, but a partial differential equation which will appear in this manner. Similarly if you have u and v as functions of x and y that you are looking for then you would also get a system of partial differential equations; all these kinds of differential equations as the first order necessary conditions for extremization of appropriate functional that is sometimes these conditions delta y at x 1 and delta y at x 2 equal to 0 are also relaxed that is when the boundary condition is not applied.

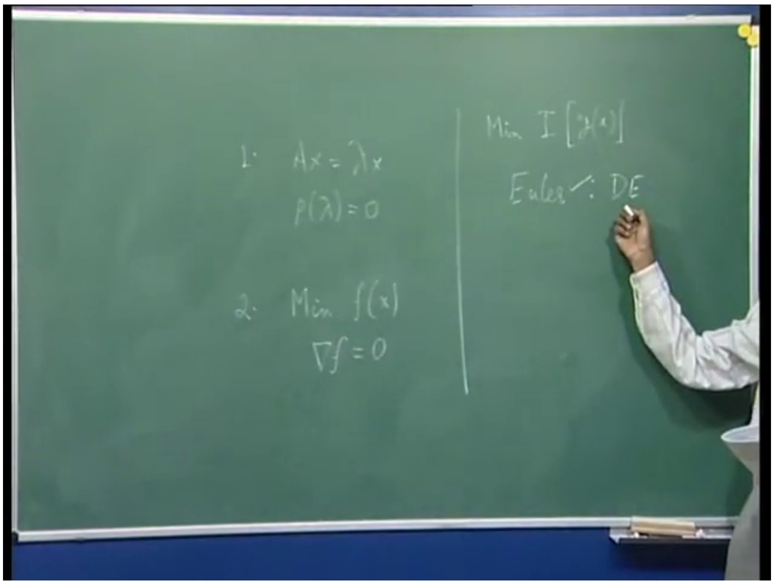
That is the boundary can move the y value at the boundary can be variable, in that case certain natural boundary conditions arise or Euler's boundary conditions arise which in a way are complimentary to the boundary conditions which we have discussed till.

(Refer Slide Time: 21:24)



Now we can also talk about a situation in which we can get the Euler's equations as the differential equations, that arise as the first order necessary conditions of these functionals to be extremum, but then solving differential equations also is not a very simple problem and therefore, people have worked through methods by which you can solve evolutional problems without having to necessarily solve a system of differential equations or an a differential equation. And this process is analogues to what we have done in the case of ordinary equations and the algebraic Eigen value problems and so on. For example, in the earlier modules of this course you would have noticed that we talked about in a equation solving and ordinary optimization, then equations solving and the algebraic Eigen value problems.

(Refer Slide Time: 22:33)



So, algebraic Eigen value problem is like this, the solution of which would require a polynomial equation to be solved first then we said that if the Eigen value problem turns out to be very small. Then it may be a good idea to solve this polynomial equation get lambdas and then get corresponding Eigen vectors from here. On the other hand we said that if this matrix is large, which means the degree of this polynomial is going to be very large, then this is not a very good idea and in that case we said that it would be a good idea to look for the solution of this Eigen value problem directly. Another situation in the optimization ordinary optimization we talked about minimize f of x, for that we worked out the first order necessary condition and said that first order necessary condition turns out to be grad f equal to c. Then we said that for easy problems of minimization of functions, it may be a good idea to work out the first order condition and that is develop the gradient said that equal to 0 and solve this system equations.

And that gives us one possible solution of this minimization problem that is other possible solution is maximum for that matter. Now that was one worthwhile way of solving minimizing problems in case the problem turns out to be easy and in case this equation solving process turns out to be easy; on the other hand quite often equation solving itself turns out to be quite complicated and it is not always easy to solve this system of equations, and then we said that rather than trying to solve this subsidiary resultant equation solving problem, we try to minimize the function through independent means. Similarly in this variational problem, we talk of minimizing I of y of x or some such thing that is the functional minimize the functional and for that we talked of Euler's equations which is differential equation and then solution of this differential equation will give us hopefully the solution of this variational problem.

Now solution of differential equation is also not all the time simple quite often it is very complicated this will give us boundary value problems in differential equations, then if the solution of this boundary value problem turns out to be difficult then we might think of time to solve this minimization problem directly rather than working out this resulting problem by equating the first order derivative to 0, and then solving these equations. So, rather than equation solving we can directly talk of minimization, such an attempt gives rise to the direct methods of variational calculus problem. One of them the simplest that we can think of is the finite difference method. Say with given boundary values y a and y b we have this problem to solve, and the way we earlier discussed that we can represent the function through its values at a large number of selected points.

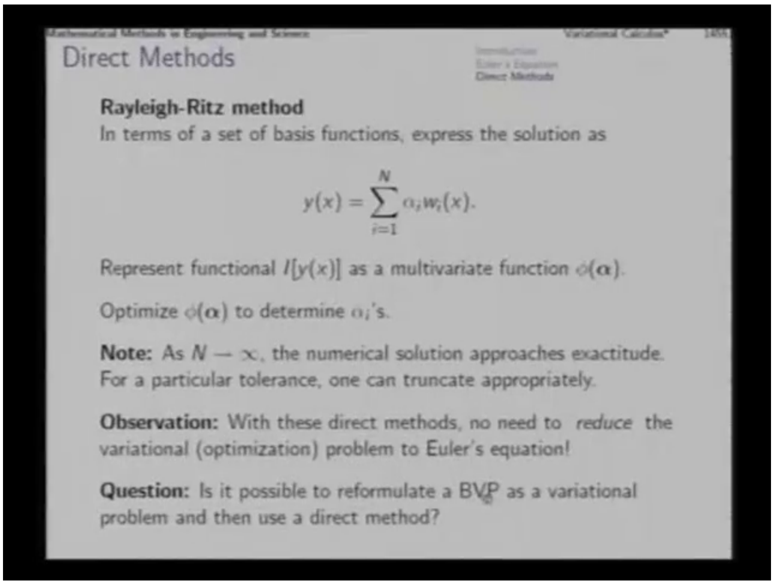
So, we can do that. So, from i equal to 0 to capital N we have n plus 1 values of x that is x 0 x 1 x 2 x 3 etcetera at these values of x, what are the corresponding values of y. If we can get those values, then we have air of representation of the functionals. So, we represent y by its values at these points x 0, x 1, x 2, x 3 at 2 points we have the value already at x 0 we know that is the y of a boundary point boundary value given, similarly at x n which is b we have the value given, at the interior point n minus 1 points we want to know the value that is y 1, y 2, y 3, these are the variables we want to determine. If we determine them then we can say that through the values at n plus 1 points n plus 1 values of n dependent variable x we have represented the function y; then with in terms of these variables which you would like to determine we can approximate the function and therefore, we can make a formulation for this integrand also which is this.

Now we can say that for every interval we use one such value in order to effect this integral through a numerical integration process that is f of x y y prime into h that is delta x, and such contributions we go on adding up to find the sum which is the integral and for that for every interval one such contribution we take in the neces usual step of numerical integration and for that we take the midpoint of the interval that is one possibility. So, midpoint of the interval x i bar, and the corresponding midpoint value of y that is the average of the 2 boundary values of for that particular interval; y i plus y i minus 1 by 2 and the corresponding y prime that is this numerical derivative y i minus y i minus 1 by h.

When we insert these expressions in this integrand here, then we get this that multiplied with h gives us one contribution from x i minus 1 to x i like that all this contributions if we add up through this term then we have got a representation of this functional i and note that this representation is in terms of these values which are till now unknown. And then we say that we want to minimize this phi of y with respect to y; that means, this is a this is the reduction of this variational problem not to a differential equation, but to an ordinary multivariate optimization problem, in which the objective function is this function phi of ys and these ys are the variables of the problem, and we can use any optimization method for that purpose and in particular if we try to work out the first derivative equal to 0 condition, first order necessary condition for this ordinary multivariate optimization problem.

Then we can find these variant components and then you can show that this condition that radiant equal to 0 is actually equivalent to what we have studied earlier as Euler's equation. And that you can show by affecting the derivatives on this and trying to find out the corresponding representation in terms of the function y and you will find that this turns out to be equivalent to what the Euler equation would give us. Now this is one possible method, in which we have try to represent the function with its values at a large number of points giving rise to the finite difference method. Another possibility is to represent the function not by its values over large number of points, but with the help of a set of basis function in the form of a linear combination of a set of chosen basis function, and that gives us the Rayleigh Ritz method. In terms of set of chosen basis functions suppose we express our required function y in this manner.

(Refer Slide Time: 30:34)



In which alpha 1 alpha 2 alpha 3 etcetera the coefficients are the variables that we would like to determine, that is what linear combination of these basis functions w 1 w 2 etcetera will give us the function y that extremizes the functional which we want to extremize. So, what we do? We represent the function i of y x as a multivariate function of these coefficients, that is we our question is what linear combination of these chosen basis functions that is what alpha 1 alpha 2 alpha 3 values should be plugged in here in order to get a y which minimizes the functional i. So, we want to minimize i of y of x which means the corresponding multivariate function phi of alpha with respect to these alphas alpha one alpha 2 is the coefficient, to optimize phi of alpha to determine the corresponding alphas.

Now in this case also as the number of such basis functions ws that is capital n tends to infinity, then we find that our numerical solution approaches exactitude whereas, as more and more numbers more and more such basis functions are selected and we get more and more complete set of alphas to complete the linear combination then we will find that the solution approaches the perfect function that we want as the solution.

For a particular solution for a particular tolerance value we trunk at appropriately that is if you want a representation up to this many places of decimal, then some number of such basis functions will suffice to give us that accurate representation. So, with these direct methods available, there is no need to reduce a variational problem and originally optimization problem to Euler's equation that is we can avoid getting the differential equations and then trying to solve the differential equation, because quite often that differential equation also we will need to solve numerically we may need to solve that boundary value problem numerically.

So, rather than finding numerical solutions of the boundary value problem we can as well try to find a numerical solution to the original problem which is an optimization problem and for that we can use the optimization methods directly or equation solving processes which are equivalent. Now another question arises and that question again has certain basis in these discussions which we had earlier, note again that we first said that in the first step the typical definition based idea of solving an algebraic Eigen value problem is to get the characteristics polynomial and find its roots, then we said that as step 2 we found that as the matrix is large the polynomial here has very large degree and solving that is computationally cumbersome. So, we try to directly track the algebraic Eigen value problem. Now if we can directly track the algebraic Eigen value problem then at the third step those algorithms which track the algebraic Eigen value problem come to help when our original problem is an equation solving problem.

If our original problem at step 3 if our original problem is a polynomial question solving problem and we find it 2 cumbersome, because the degree turns out to be very large then what was our third step at the third step we connected these 2 problems both ways, we said that for every polynomial like this we can work out a companion matrix for which its roots will turn out to be the Eigen values and then we said that for a polynomial equation solving problem actually the genuine Eigen value problem methods can help. So, if the finally, our enlightened conclusion was that if the matrix turns out to be small then get the characteristics polynomial find its roots to get the Eigen value of this. For this equation solving problem is the polynomial is of degree small, then solve it directly. On the other hand if the degree of the polynomial equation is large or the size of this matrix is large then do not bother about the polynomial.

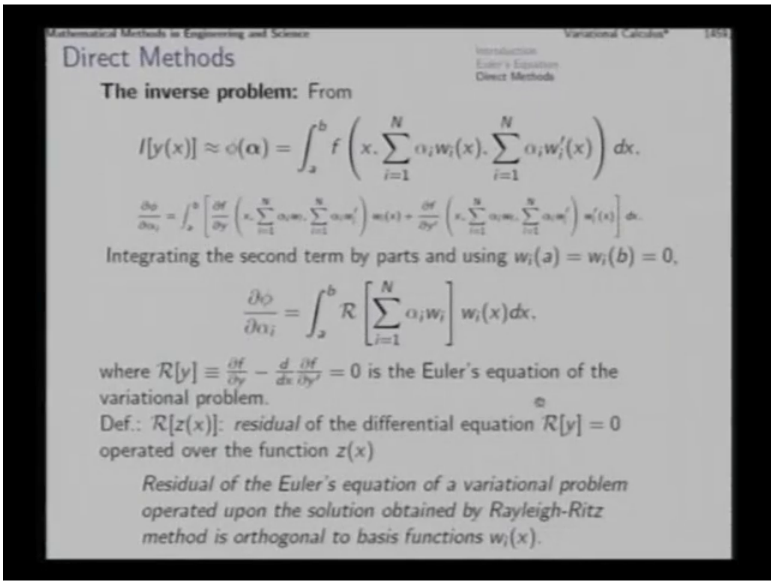
In this case solve the Eigen value problem directly without going into the characteristics polynomial and if your problem happens to be a polynomial root finding problem then find out the companion matrix and find its Eigen values they will give you the solutions of this. That means, between these 2 problems algebraic Eigen value problem and polynomial equation solving problem we worked out a correspondence, and then whenever this is easy we convert this problem there and whenever this problem is easy when we converted the other problem to this side same thing we did in the case of ordinary optimization definition.

Based approach necessary condition based approach says that for minimizing a function, we can try to solve its gradient equal to 0 which is a system of equations, as if a system of equations can be solved directly without any trouble then we said that sometimes that gives a lot of trouble that is all the time the solution of a system of equations is also not easy and therefore, we said that we can try to minimize the function without directly trying to solve it gradient equal to 0 first order conditions.

Then at the third step here also we said that then sometimes it happens that solving a system of equations is. So, complicated that we would rather make an optimization based formulation of it and in solve that optimization problem. In the case of variational problem also we have these 3 steps of argument in the first step we say that the minimization of this functional will be helped if we console the corresponding Euler's equations at the second step we found that solution of the boundary value problem of these differential equations may not be an easy task. So, we talked of direct methods of solving the variational problem. Now if we have the direct method of solving the variational problems, then even if our original problem is the solution of a boundary value problem, can we not try to find a suitable variational problem for which this boundary value problem gives us Euler's equation.

If we can do that then for complicated boundary value problems we can convert it into a variational problem a calculus of variational problems and then use one of the direct methods to solve this, and the answer is yes quite often it is possible to reformulate a boundary value problem as a variation problem and then you use a direct method, then we can avoid trying to solve the boundary value problem of differential equations through the original differential equation methods and indeed this happens to be one of the professionally accepted ways to solve boundary value problems and for that purpose we need to work out the inverse problem, that is if we have got the boundary value problem in terms of the differential equation or a system of differential equations then we first work out the corresponding inverse problem that is we work out the functional for minimization of which the first order necessary condition happens to be that differential equation or that set of differential equations.

(Refer Slide Time: 38:47)



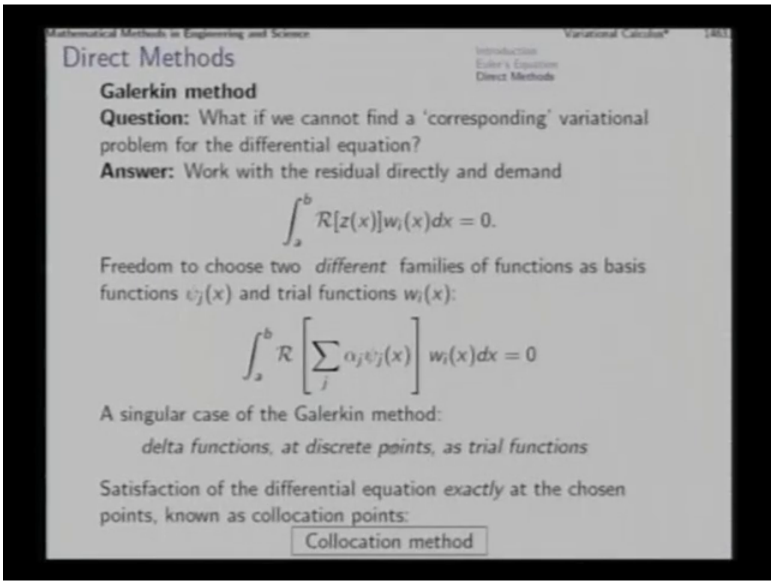
So, that gives us the corresponding formulation of the functional, and then accordingly the multivariate function phi of alpha in this manner f of x y and y prime say this is the basis function based representation of y and this is the corresponding representation for y prime. And now you can actually verify that the derivative with respect to alpha of this turns out to give you this, in that is del f by del y evaluated at this into del of this with respect to alpha i that will give you w y that is this plus del f by del y prime evaluated at this point into derivative of that with respect to corresponding alpha that is w y prime that is this and this is the first order derivative of this with respect to one of this alphas that is alpha i. Now from here you find that you get del f by del alpha i as again for that purpose you will need to integrate this second part second component by parts.

And then you will find that the derivative first order partial derivative with respect to alpha i turns out to be this integral where R of this y is actually this differential equation which is the differential equation for which you are seeking the solution and the corresponding variation problem you have worked out. So, then here you find that the first order derivative condition, first order derivative del f del phi by del alpha i turns out to be that differential equation left hand side evaluated at the current solution is here into w i d x integrated from a to b. So, this is actually the inner product between this large function this R and w i. So, the vanishing of the gradient will give you del phi by del alpha i equal to 0 and that will give this equal to 0 and; that means, the orthogonality of this R with w i right and here we define this R as the residual of the differential equation this ok.

If the differential equation is R of y, r is the differential operator if the differential equation is R of y equal to 0 then this same operator same differential operator R operator over any function z and evaluated is the residual. So, residuals function is that function z operated over with the differential operator which operated over y gives you equal to 0 gives you the differential equation so; that means, if the differential equation happens to be y double prime plus 3 y prime plus 4 y then for any other function z, z double prime plus 4 z prime plus 3 z will give you the residual of this z for that same differential operator right. So, that residual. So, residual of the Euler’s of equation of a variational problem, operator of on the solution obtain by the this method Rayleigh Ritz method is orthogonal to basis function so; that means, the solution of the differential equation solution of the Euler’s equation that you get through the Rayleigh Ritz method turns out to be orthogonal to the basis functions.

That you have selected to represent the y and this gives us a way to determine those alphas; that means, as many ws basis functions are selected that many alphas are there and that many partial derivatives you will set equal to 0 and that many equations you will get that many orthogonality conditions you will get from here. So, that is that is going to give you a system of equations in the alphas that many alphas and from those equations you can find out the coefficients alphas and that gives you the solution of the corresponding boundary value problem of the differential equation as well as the solution of the variational problem of minimizing this I. Now the question is that this will work for those differential equations for you which you can directly work out a variational principles, that is for which you can work out a variational problem of which this differential equation happens to be the Euler’s equation.

(Refer Slide Time: 43:43)

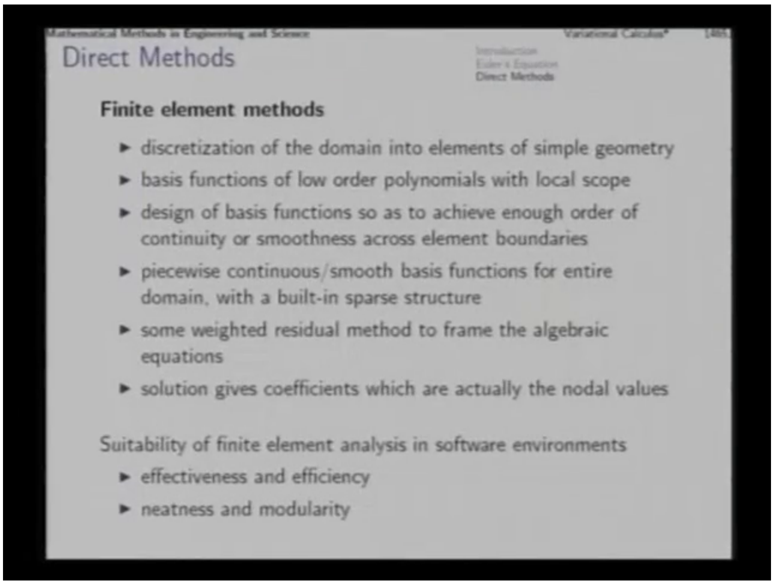


Now, if you do not have that that is what if we cannot find a corresponding variational problem, for the differential equation; the answer to that gives you what is popularly known as Galerkin method, and in that case you do not bother to find out a suitable variational problem, you can work directly with the residuals and you say that I will represent the function y solution of the boundary value problem, as a linear combination of a set of basis functions and then those coefficients we will determine by subjecting it to the same condition over the residual which we just now saw, and here it is not even necessary to have the basis functions and these functions w i be the same. So, it is not necessary for the same functions to represent z as basis and the same functions against which the orthogonality condition is applied it is not necessary. We can actually freely choose 2 different families of functions as basis functions to represent this and as trial functions to set these orthogonality conditions.

That is we can say that as a linear combination of Legendre polynomials, we would try to represent the function and that representation when we subject to the differential operator and get the residual, we will demand that these residual this residual be orthogonal to the family of suffix nomial we can say that, that is in that particular instance the Legendre polynomials serving as the basis we will come here, this will construct a candidate solution and its residual when we evaluate like this, we are demanding that these residuals be orthogonal to the suffix nomial suffix nomials will fit this place as the trial functions. So, the basis functions psi j trial functions w i need not be same they can be different. So, that way Galerkin method turns out to be a further generalization of this process, even the collocation method turns out to be a special case of Galerkins method because in a singular case of Galerkin method.

We can take delta functions as this trail functions; that means, at this discrete points we want exact adherence to the value given values that is equivalent to taking delta functions trial functions here and that will require the satisfaction of the differential equation exactly at those chosen points and this will give you the collocation method. Now a lot of these ideas are now quite popular because of the enormous success of the family of method known as finite element methods, what they so, essentially.

(Refer Slide Time: 46:54)

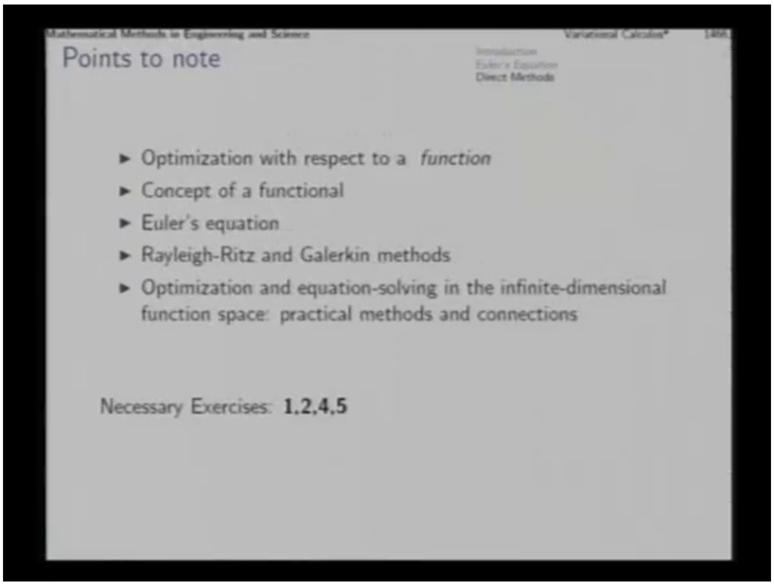


First of all they combine positive features of several areas several ideas, first of all they discretise the domain into elements of simple geometry, they select basis functions of low order polynomial with local scope. So, one polynomial defines a function in one locality, another polynomial defines an at another locality and so on and low order polynomial are easy to handle gives less undesirable perturbations and then we design the basis functions in such a manner that appropriate order of continuity exist across the boundaries.

And then piecewise continuous or smooth basis functions will arise as the com complete representation of the function over the domain and then we use some of these weighted residual methods which we were discussing earlier to frame, the corresponding algebraic equations and the way these equation are framed the solution of these algebraic equations gives the corresponding coefficients, which turn out to be actually the nodal values of the function that we are trying to determine. And combining all these deeper mathematical ideas mathematical and computational ideas, finite element methods have achieve great success in terms of effectiveness as well as efficiency and particularly their implementations in software environments lead to great application, great applicability, great utility because of their neatness of the solution process and modularity of different parts of the solution process differ; utilization of different tools in the process.

So, of course, finite element methods are quite enormous in their scope and application and many of you might be taking or might have already taken a an independent course on finite element methods. So, rather than this more than this conceptual idea, we do not need to discretise finite element methods more in detail as part of this course.

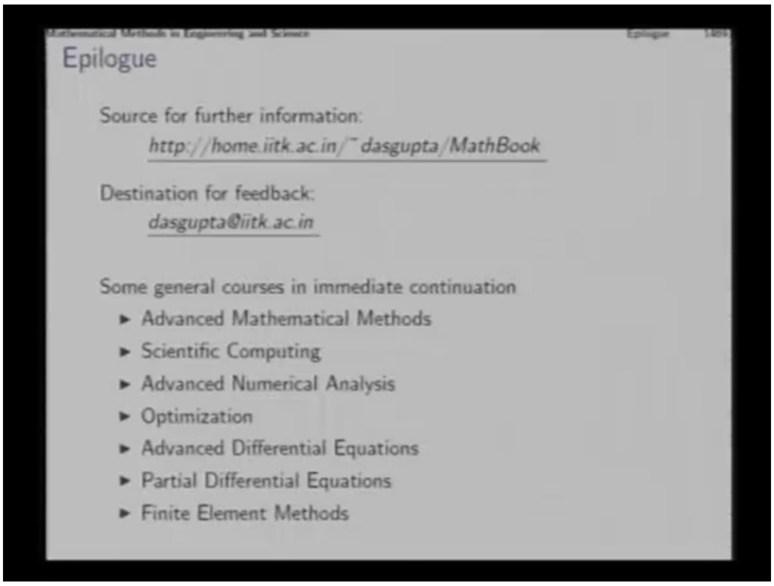
(Refer Slide Time: 48:58)



Now, briefly let me summarize what are the important points that we have studied in this particular lesson. One is the concept of optimization with respect to a function rather than a bunch of variables which lead to the concept of functional and then we found the Euler's equation as the first order necessary condition for this optimization problem and of course, we did not go into the solution of that, because solution of differential equations we have studied earlier separately.

And then when the solution of the resulting Euler's equation is difficult, we need to use direct methods of function functional problem the variational problem and for that we discuss the finite difference Rayleigh Ritz and Galerkin methods and the important issues of optimization and equation solving process in infinite dimensional function base is the conceptual notion that we discussed in extension of our similar earlier observations in the case of finite dimensional optimization problems. Now this completes the topics that we wanted to study in this course and if you closing remarks I would like to make at this point.

(Refer Slide Time: 50:25)

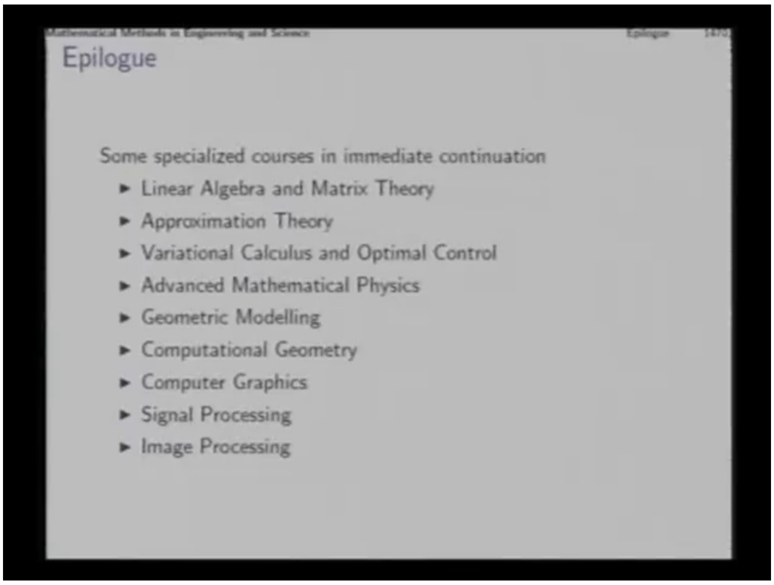


One is that this kind of a course which enables at different ideas of applied mathematics, different areas of applied mathematics in cursory way showing the interconnection between them in a way works as the datum level of mathematical expertise needed to embark on serious analytical and computational research of today's time in continuation to this.

This course many of you may like to specialize in certain areas or improve your general skills through certain general courses as immediate continuations of the course that we are just now concluding. So, these are some of the courses which would work as immediate continuation of the present course that we have finished we are finishing, these are advance mathematical methods which is one level higher than this course scientific computing, advanced numerical analysis, optimization advance differential equations partial differential equations and finite element methods.

These are some of the general courses which operate on the background of our course and go deeper into one direction or the other all these direction are quite general directions. Apart from these general higher level or deeper courses, with the background that we have covered in this course we can you can also think of going into one of these specialized courses depending upon your specific area of study or your specific interest.

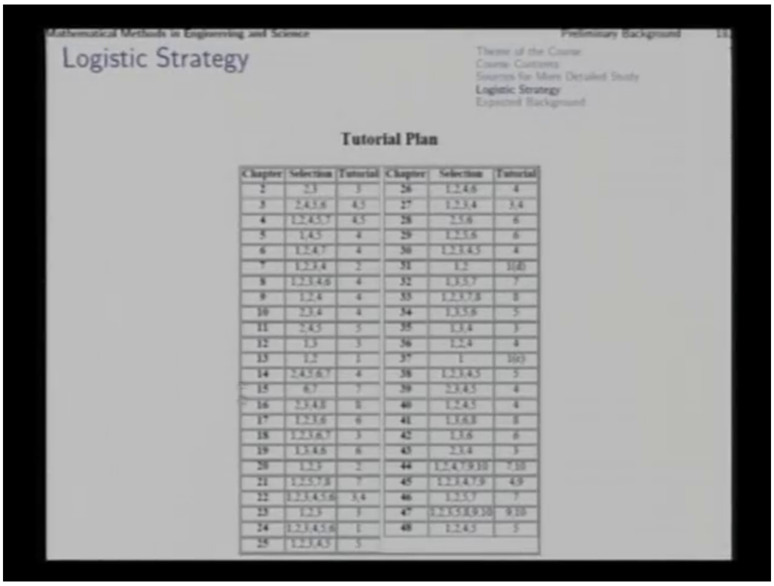
(Refer Slide Time: 52:01)



These specialized courses in immediate con continuation are linear algebra and matrix theory approximation theory, variational calculus and optimal control advance mathematical physics geometric modeling for design, cad, cad cam computational geometry computer graphics, signal processing image processing and so on. So, these are some of the specific specialized courses which will work over the background that we have developed in this course and more details of this course and the associated text book that we have been using you will find here, and let me remind you that in this course we have covered the subject matter rather at a very fast phase and therefore, it is important for you to go through the solutions, go through the exercises a little carefully as you proceed through this lectures.

And if necessary you can go back to the exercises and the lectures and operate back and forth to see how you proceed and in that context I will attract your attention to this plan of exercises that.

(Refer Slide Time: 53:24)



We discussed in the first lecture of this course; in each chapter of this text book the book that we are using for this course. Some of the exercises have been listed here as the selection problems, you should ensure that these problems are quite clear in your understanding for complete comprehension of many of the course material that we have studied here, and keep in mind that the solutions of these problems all the problems all the exercises are also available in the book itself, in the appendix and here in this text book.

Which are we have been using for the subject matter of the course you will find that something like 55 pages of this book are actually devoted to the appendix, in which you have got complete answers solution outlines and further comments which explain the solutions explain the solution methodology and also and make interpretations. So, this 55 pages of appendix is actually something like 13 percent of the text of the matter.

So, many of the concepts are also developed among these exercises. So, some of the exercises we have discussed here in the part of the as part of the lecture itself, but the rest of the exercises also at least the selection part of it, you must complete for developing a thorough understanding of the subject matter covered in this course.

All the best, thank you.