Mathematical Methods in Engineering and Science
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Module – VIII
Overviews: PDE's, Complex Analysis and Variational Calculus
Lecture – 05
Singularities and Residues

Good morning this is our last lesson in the theory of complex analysis; in the preceding two lectures we discussed first the analytic functions and their properties. Cauchy's Riemann conditions and harmonic functions and in the second lecture we discussed integrals in the complex plane, integrals of complex functions and established Cauchy's integral formula, Cauchy's integral theorem and Cauchy's integral formula. Now, till now our study has been mainly focused on analytic functions and today we concentrate on those situations where analyticity is lost that is for those functions which are by enlarge analytic expect at some points so those points called singularities.

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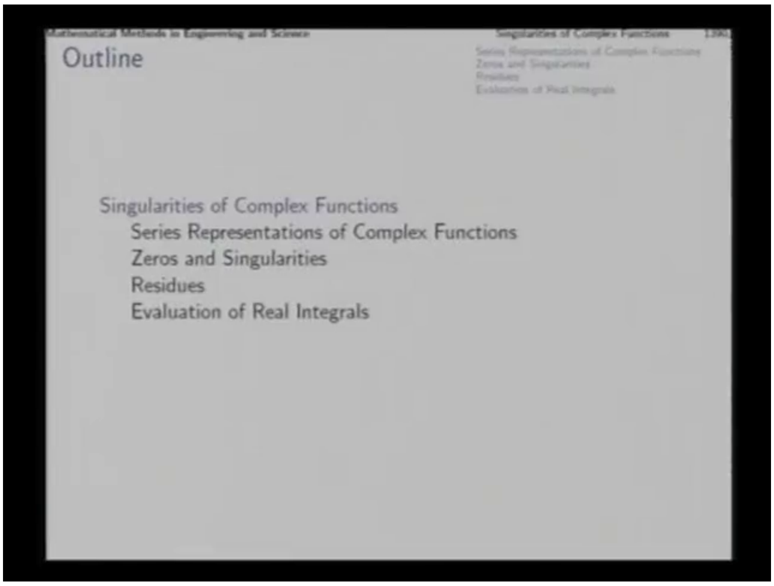

So, today we are going to concentrate on those singularities of complex functions, first we start with a series representation.

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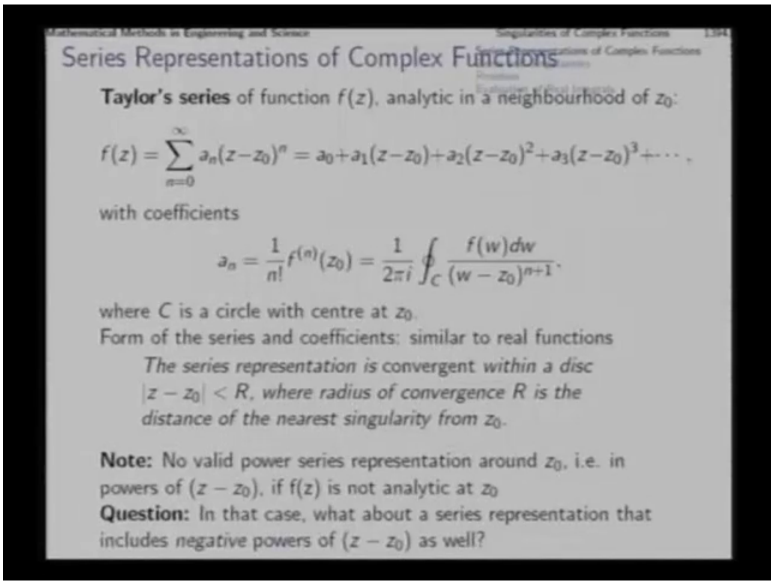

If a function is analytic then for that function we can work out at Taylor's series in the neighborhood. So, for example, if the function f z is analytic in the neighborhood of a point z 0 then around z 0 we can work out at Taylors series for it in this manner, exactly the way in which we use to create we used to develop a Taylors series for real functions. So, here the variable is complex the rest of the things remain almost similar. So, here as we try to work out the Taylor series we start with a 0 plus a 1 into z minus z 0 plus a 2 into z minus z 0 whole square and so on exactly in the fraction of real function, in which the coefficient are also found from similar expressions.

So, the coefficient to z minus z 0 to the power n is a n given by this formula, which is the nth derivative evaluated at z 0 divided by factorial n and from the preceding lesson, from the previous lesson we know that this can also be represented in this manner this we derive from Cauchy's integral formula through n differentiations. Now, here this curve c for the contour integral is a circle with its center at z 0, now form of the series and coefficients we can see are similar to real functions. Now, we must make note of the region of validity of this Taylor series, now we started with the analyticity of the function in the neighborhood of z 0, but how large is a neighborhood. So, the series, this series representation is valid that is it is convergent within a disc of radius r, this is the equation of the disc in inequality there is the representation of the disc.

So, it is valid or convergent within a disc of radius r which is also cause the radius of convergence which is the distance of the point z 0 from the nearest singularity; that means, if within a disc r everywhere the function is analytic then it will have within that distance it will this series representation will be convergent and therefore, valid. Now, if the neighborhood includes a point of singularity that is if analyticity is lost at some point then enclosing that we cannot work out a power series representation like this, in which the constant linear quadratic cubic etcetera terms will not suffice to represent the function which is not analytic. Now, we consider another option that is if then a function is not analytic then as we know that the Taylor series will not be valid that is it will not be enough to represent a function then we can ask what will be enough.

So, we ask this question that in that case what about the series representation that include negative powers as well. So, indeed answer is yes, we can do that in the corresponding series is called Laurent's series.

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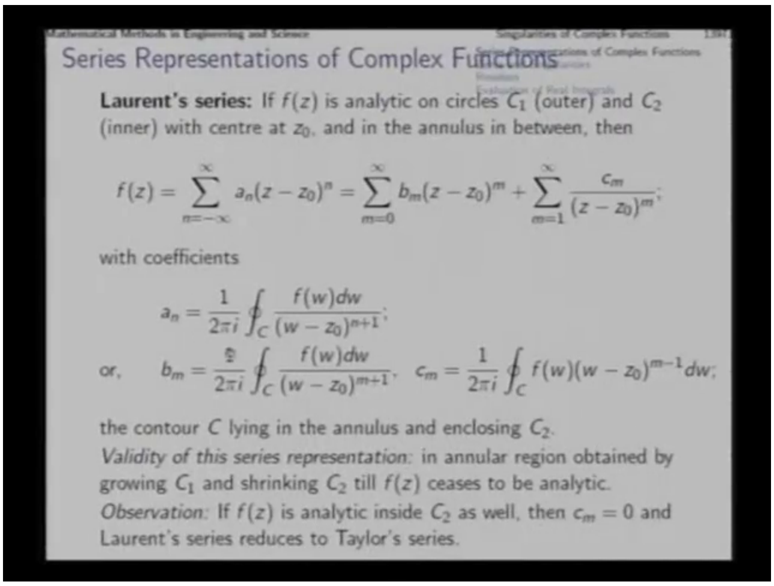

So, if f z is analytic on an outer circle c 1 and an inner circle c 2 and everywhere in the another region within it, say you have 2 concentric circles c 1 and c 2 it is the inner circle c 1 is the outer circle. Then they will enclose an another region then if the function is analytic within that another region everywhere as well as on the two bounding circles the outer and the inner circles then we can work out a series which is resembling at Taylor series expect that negative powers are also included and that kind of a representation can be made such a series called Laurent's series. Laurent's series will be valid even if there are singularities within that is in the integer of that inner circle.

So, in the integer of inner circle we do not need the function to be analytic. So, in that case the series representation will be exactly like that as in the Taylor series, except that the series will start not from n equal to 0, but from n equal to minus infinity. That means, here you will have terms which are a 0 plus a 1 z minus z 0 plus a 2 z minus z 0 whole square etcetera till n equal to infinity on the other side you will have a minus 1 by z minus z 0, a minus 2 that is minus 1 minus 2 are subscripts. So, some constant divided by z minus z 0 some other constant divided by z minus z 0 whole square and so on. So, when we include such negative powers also then the corresponding series is Laurent's series and that is valid if we have the analyticity of the function in the another region inside c 1 and outside c 2 and in that case inside c 2 we do not demand analyticity.

So, if we break this into 2 parts 1 is based on non negative powers that is b 0 plus b 1 into linear term plus b 2 into quadratic term and so on. So, a 0, a 1, a 2 are exactly the same as b 0, b 1, b 2 and a minus 1, b a minus 2, a minus 3 are here included as c 1, c 2, c 3 etcetera. So, these are the negative power term so this is, these are the terms which are extra. Now, if a function is analytic not only in annulus, but inside that inner boundary, inner circle also that is it is if it is analytic within the outer circle everywhere then for such a function all the c 1, c 2, c 3 coefficients will turn out to be 0 and the special case of Laurent's series will appear as simply the Taylor series. The coefficients here are obtain from the expressions which are similar to this the same expression as we worked out here the same expressions will appear here also for a n.

Now, if you spit the non negative and negative coefficient as we have done here in a corresponding b ms will be found like this and c ms which are actually a minus m. So, they will turn out to be like this; obviously, because in that case the minus m will take it about to the numerator. So, in this itself if you put in place of n if you put minus m then you get this and if you put m then you get this. So, m here is everywhere positive in this expression n is positive for these terms and negative for these terms and here the contour c should lie in the annulus and it should enclose c 1 that is it should not go like this and come back like this without enclosing c 2 completely. So, it should be completely within c 1 and it should completely enclosed c 2; that means there will be 3 contours circle c 1, outside inside that contour c under question here and inside c the inner circle c 2 will be there.

So, the contour c for this integrations should enclose the inner circle c 2. So, now, is this series representation valid within the annulus, certainly is there any region outside the annulus where this series representation is valid answer is yes if you go on striking the inner circle c 2 and if you go on expanding outer circle c 1 then you increase the width of the annulus and the series representation turns out to be valid that is convergent as long as this strinking and that expanding does not encounter a singularity. So, you can go on expanding the outer circle and striking the inner circle till you hit a singularity, within that much domain within that much region this series representation will be convergent which means that representation will be valid, this we have observed already that is in place in the case of c m being equal to 0 if it is analytic inside c 2 as well then this series will simply reduce to the Taylor series.

Now, we can try to establish this result that it is indeed a convergent series representation under the premises made.

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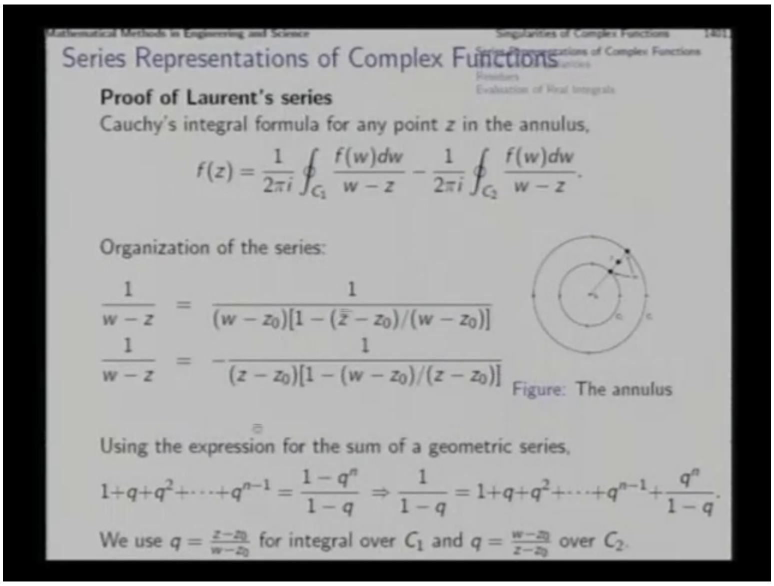

Say we already know that Cauchy's integral formula for any point in the annulus gives us this, that is if the singularities whatever they are, are enclosed within the inner boundary c 2 and outside c 2 inside c 1 if there is no singularity that is everywhere the function is analytic then from Cauchy's integral formula we know that the function f z at any point z in the annulus is given by this outer integral minus this inner integral. This we have already seen in the last lesson. Now, in order to handle this w minus z term in the denominator we observe that it can be written in this manner see here this point in the annulus is z and that is any point where we are taking the function and this point on the outer circle c 1 and this point on the inner circle c 2 will serve as w in this integral and in this integral respectively.

So, w is one of the boundary points either the outer one or the inner one, outer here inner here and z 0 is this point right. Now, when we try to write w minus z as w minus z 0 minus z minus z 0 that is all the positions we are referring to this center of the concentric circles. So, the position of this z is written as z minus, z minus z 0 and this w as w minus z 0 and w minus z 0 then this w minus z is simply w minus z 0 that is this vector and similarly z minus z 0 will be this vector. So, you see for the outer circle z minus z 0 will have radius less than the radius of c 1; that means, radius less than the z minus z 0 size absolute value will be less than w minus z 0 for the outer circle for the inner circle it will be the other way around.

So, as we write w minus z as w minus z 0 minus z minus z 0 then taking w minus z 0 outside we will have 1 minus z minus z 0 by w minus z 0. Now, this ratio of absolute value this magnitude, the magnitude of this ratio of the 2 complex numbers will be less than 1 because z 0 z minus z 0 is smaller than w minus z 0, for w lying on the outer circle. For the inner circle it will be the other way around therefore, in the case of inner circle what we do we first say that 1 by w minus z can be say called as can be considered as minus 1 by z minus w, because here z will be giving the larger part. So, then again we say then that that if this minus this here. So, that z minus w can be written as z minus z 0 minus w minus z 0, taking z minus z 0 outside will have this in this again for c 2 this will have absolute value less than 1 and why we are insisting on less than 1 because the binomial expansion of 1 minus q to the power something will be convergent for absolute value of q less than 1 ok.

Then we try to evaluate this integrals with this w minus z for this 1 by w minus z for this integral and for and with this 1 minus w by z for this integral and then you see one by one minus something we have. So, if we take this geometric series say up to n terms then we know that the some of this geometric series turns out to be this and that will tell us that 1 by 1 minus q will be the sum of all this things plus q n by 1 by minus q that is taking this q 1 by 1 minus q term on the other side. So, the remaining is 1 by 1 minus q that is this which will be this entire series plus this term q n by 1 minus q here. So, for the integral over c 1 we use q as this ratio and we use this expression for w 1 by w minus z and for the integral over c 2 we use this as q that is this ratio and use this expression to put here, as we do that we get this 1 minus z expanding from here 1 by w minus z 0 remains.

And this 1 minus q to the power minus 1 for c 1 we use this, 1 minus q to the power minus 1 with this value of q and as we expand it we get this right.

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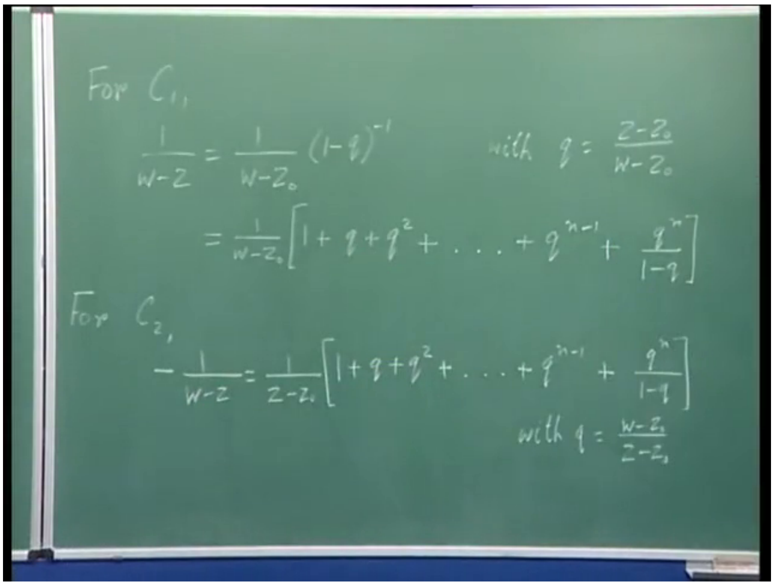

And similarly for c 2 we will use minus 1 by w minus z including this minus here. So, that will mean that this minus goes off and we have the rest every 1 by z minus z 0 1 by z minus z 0 and as similar sum because that also will be 1 minus q to the power minus 1 only with a different expression for q. So, that is this, as we do that there are first handle the first 1 this 1. So, here as we put q as this and then term by term we list out and the first will be 1 by w minus z 0 that is here.

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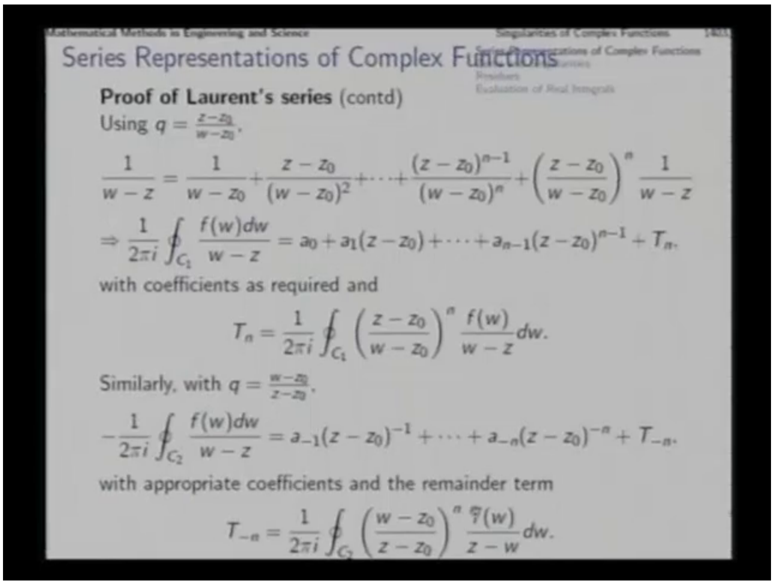

Second 1 will be q by w minus z 0; that means, z minus z 0 divided by w minus z 0 whole square. So, that is this and in the next 1 we will have q square w minus z 0. So, we will have z minus z 0 whole square plus w minus z 0 whole cube and so on.

It will go on continue, continuing till this point there is q to the power n minus 1 by that outside w minus z 0. So, that will be z minus z 0 to the power n minus 1 divided by w minus z 0 to the power n 1 extra power in the denominator because of that term outside. Finally, we have this term q to the power n which is here into what is 1 by 1 minus q, but we have already seen that 1 by w minus z 0 into 1 minus q happens to be the same old w minus z; that means, by including. So, many terms here we have postponed this w minus z which is now coming here anyway with this factor along with it, how does that help? Since the absolute value of q is less than 1. So, this q to the power n will turn out to be small we are going to handle that later.

So, currently we can say that now if 1 by w minus z has this expression then the first term in the integral expression from Cauchy's integral formula is which is this will need that we multiply this series with f w, d w and in integrate around c 1 outer circle and divide by 2 pi i right so; that means, we will multiply each on each of them 1 by 1 and continue to evaluate these integrals. So, let us say this multiplied with f w d w. So, that will give us integral of f w d w divided by w minus z 0, over this circle. So, that will and then divide by 2 pi i that will give us a 0 according to the formula note this formula here a 0 1 by 2 pi i integral around c, f w d w and w minus z 0 it will be n equal to 0. So, that will give us a 0 right in this manner we go on evaluating the next 1 multiply with f w d w integrated divided by 2 pi i will give us a 1 that is only the 1 by this part multiplied with that will give us a 1 and z minus z 0 will remain there.

Next 1 will give us similarly z minus z 0 whole square and there will be a factorial 2. So, as you evaluate this terms I suggest that you evaluate this terms on your own and verify that this will come as describe and this part is easy to evaluate the expressions and the corresponding a 0, a 1 etcetera terms will appear as the coefficient expression goals. Finally, this term will remain, let us call it t n and that t n is this right 1 by 2 pi i here integral integrated this entire stuff into f w, d w. So, this remains as the n th term similarly for the next 1 that is for the inner circle that minus sign has been already taken. So, minus 1 by 2 pi i this whole thing. So, that will similarly turn out to be like this in which the other coefficient will appear as expected and there will be a remainder term t minus n that will be like this ok.

So, that will come from this term. So, now, we say that the entire stuff that we are looking for that is this integral plus, this integral we are saying plus because the minus sign has been included already. So, this integral plus this integral will turn out to be the sum of all this terms plus the sum of all this terms plus t n and t minus n the remainder terms. Now, the sum of all this terms and sum of all this terms turn out to be the summation from minus n to n minus 1 and these are the 2 boundary terms, highest considered here lowest considered here ok.

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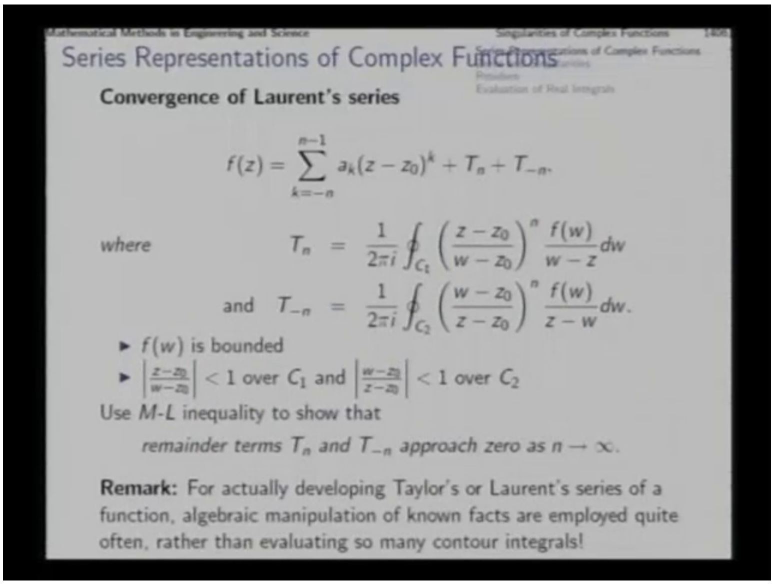

So, that will mean that a k z minus z 0 to the power k has been included from k equal to minus n to n minus 1 and these start a 2 remaining terms. Now, these terms we concentrate on and see as n tends to infinity then what happens to these 2 terms.

Now, you can see that the way we organized that 1 by w minus z we have ensured that this over the outer circle has absolute value less than 1 and this on the inner circle has absolute value less than 1. That means, as n goes very high this tends to 0 and this also tends to 0 and then since f w is analytic and this w minus z on the circles are finite they cannot be extremely small therefore, this whole thing convergent that that is these, these cannot be infinite because the function is analytic and these cannot be extremely small because they are finite there is known already. So, that means, and this term tends to 0 as n tends to infinity; that means, that as n tends to infinity these these terms approach 0 and therefore. So, these are the arguments over which we find that these terms will approach 0 as n tends to infinity that you can show from the m l inequality.

Now, this is the way the proof was; however, for actually developing Taylor series or Laurent's series we do not actually go on evaluating so many integrals to find the coefficients. Quite often we use known algebraic and analytic fact and based on that we manipulate the expressions for known functions to develop the series and the validity or convergent of that series lies on this property this theoretical background.

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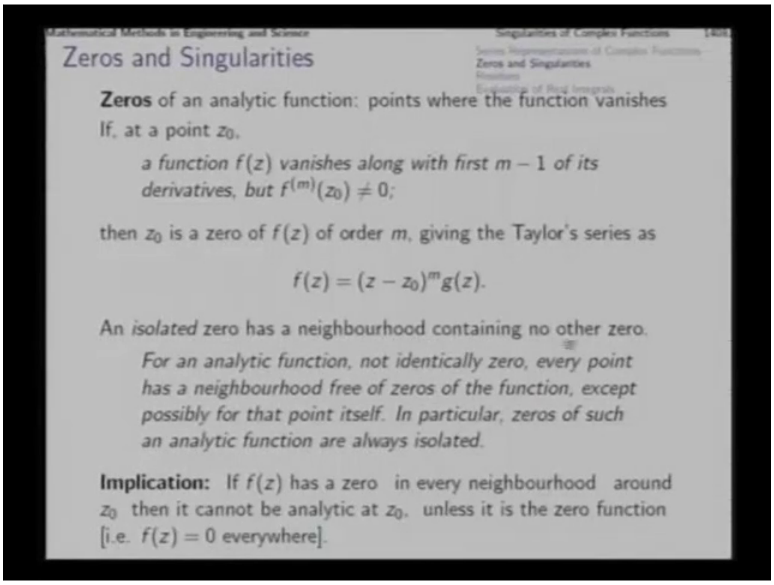

Now, we come to after having a look at the series representations now we come to the analysis of 0's and singularities of complex functions. What are the 0's of an analytic function those points where is the function vanishes, it is a same thing as the 0 or root of a real function. So, roots or real 0's of a real function are those values of x where the function value turns out to be 0 same thing here for the, of this functions also.

So, if at a point z 0 a function f z vanishes then it is a 0 of that function, now if the function itself vanishes and some of its initial derivatives also vanish at that point say first m minus 1 of its derivative vanish, but the next derivative does not vanish then we say that this particular point is a 0 of order m. If m is 1 that means, only the function vanishes no derivative not even the first derivative then there will be a simple 0, if the up to the first derivative vanishes then you call it a double 0. So, if first 5 derivatives vanish along with the function value then you say this is a 0 of order 6 and so on and in that case the Taylor series can be simply worked out as this in which g z does not have a 0 at z 0; that means, g z evaluates to a non 0 complex number at z equal to z 0 and; that means, that the initial few terms in the Taylor series are 0.

Now, there is a concept of a 0 being isolated or not. So, an isolated 0 has a neighborhood containing no other 0; that means, if there is a 0 of an analytical function and then you can include a neighborhood around it ,in which there is no other 0 in that case you call that 0 as an isolated 0. Now, if you include a large neighborhood, if you try to examine a large neighborhood you may find that there is 1 or 2 more 0s inside that neighborhood then it is. So, this large neighborhood is not good you work out a small neighborhood still 1 0 in this side you work out a another small neighborhood. Now, if you can work out a neighborhood whatever small in which there is no other 0 then that particular 0 is called an isolated 0 ok.

Now, it is possible for a function to have a 0 in such a manner that around it whatever small neighborhood you try to develop in that there are other 0's also and whatever small you make that neighborhood you still get more 0's inside that neighborhood. In that case you say that 0 is not isolated, in arbitrary closed neighborhood of it there is another 0 and in that case you can show that the function cannot be analytic; that means, that for an analytic function which is not identically 0 of course, f z equal to 0 is also analytic function for that all points are 0's, but other than that if the function is not entirely 0. If the function is not identically 0 then every point in the domain has a neighborhood free of 0s of the function, except possibly for that point itself; that means, for every point whether it is a 0 or not you can certainly workout a neighborhood at which there is no 0 of the function.

So, for analytical function which is not a which for an non 0 analytic function you can always find such a neighborhood except for that situation in which that point itself around which you working out with neighborhood the itself is a 0. So, in that case there will be no other 0 in that; that means, every 0 of an analytic function is an isolated 0. So, if you can find out f 0 for a of this function which is not an isolated 0; that means, that function is either the 0 function or it is not analytic.

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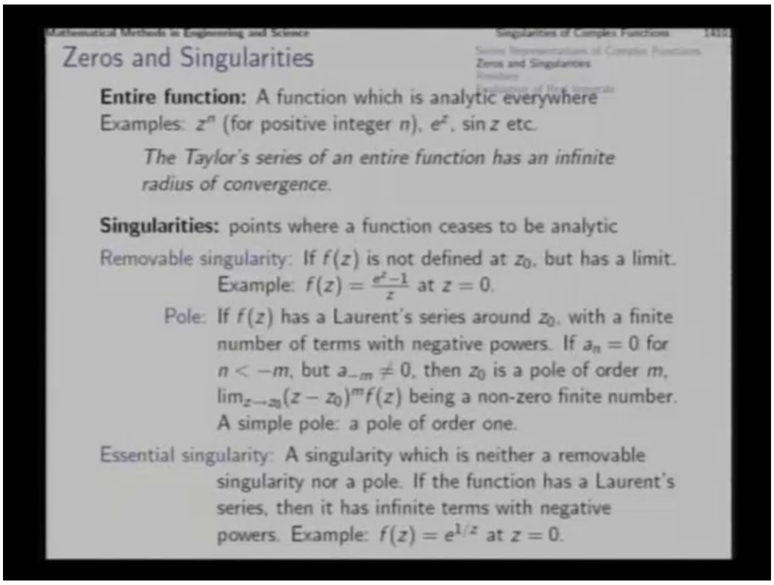

A function which is analytic everywhere is called an entire function, examples there are quite a few examples z to the power n for positive integer, n the exponential function, the sin function etcetera these are entire functions that is they are analytic everywhere and the Taylor series for analytical functions you can work out a Taylor series.

So, the Taylor series of an entire function has an infinite radius of convergence, because it is analytic everywhere, you can go on expanding the domain of expansion a domain of heredity of the Taylor series. Now, the point remains that if you want correct value then you need to include more and more terms, but taking enough number of terms you will be able to always ensure that the series is series representation gives you reliable function values; that means, the function the series convergence. Now, those functions which are not analytic everywhere they will have certain singularities, what are the different types of singularities that you can have? The simplest 1 is removable singularity now removable singularity for many practical purposes are not really singularities, for example if f z is a function is not defined at z 0.

But if it has a limit then you say that this singularity is removable, in the sense that if you consider say this is example at z equal to 0 you have 1 minus 1 which is 0. So, you get this 0 by 0 form. So, this function is not defined at z equal to 0 it is a it becomes undefined, but it has a limit. So, as you try to find out the limit of this function as z tends to 0 you will get a limit and this kind of situations are referred to as removable singularity, the genuine singularity can be either pole or essential singularity. Now, what is a pole in the Laurent's series of f z in principle you can have infinite terms on the positive powers of z minus z 0 and then infinite powers in the negative power, infinite terms in a negative powers of z 0 that is z minus z 0 to the power of minus 1 to the power minus 2 to the power minus 3 you can actually have infinite terms.

Now, if in the case of a particular function at a singularity at there is around a singularity there is Laurent's series is such that only a few, only a finite number of terms is negative powers are non 0 that is say which negative powers only a finite number of terms are there; that means, if a n is 0 for n less than minus m. That means, a minus 1, a minus 2, a minus 3, a minus 4, up to a minus m are there and beyond that a minus m minus 1 a minus m minus 2 all the lower 1s turn out to be 0; that means, that only up to m terms on the negative side are really there. Beyond that on the still lower side all the other coefficients are 0 then you call it a pole and in that case you call it a pole of order m; that means, that in the Laurent's series with negative power suppose only 1 term is there that is 1 by z minus z 0.

If there is a single term then you say it is a simple pole if the highest negative power is say z minus z 0 to the power minus 2, to the power minus 3, minus 4, minus 5 etcetera are all absent then you say it is a double pole so; that means, it is a pole of order 2 and so on. So, if up to a minus m coefficients are non 0 in between 1 or 2 still may be 0 that does not make any difference, but if the lowest coefficient that is non 0 turns out to be a minus m and lower than that all other coefficients are 0 then you say it is a pole of m order m (Refer Time: 33:00) that if we multiply the function with z minus z 0 to the power m and then take the limit then we get a finite limit, finite number. So, that is the case with the pole, if on the negative side on the negative powers you have infinite terms actually infinite terms there is you never stop getting more and more non 0 coefficients then you say that no such multiplication and taking the limiting value will suffice to get the correct representation that is to remove the singularity.

So, even multiplication of such terms will not remove the singularity and therefore, you call it an essential singularity for example, if you try to expand this you get a Laurent's series in which all the negative powers will remain you will not be able to trunk it anywhere without living some of the non 0 terms. So, this function has an essential singularity at z equal to 0, now 0's and poles have a little complimentary in nature to each other. So, as we have seen that for analytic functions the 0s are always isolated, for non 0 analytic functions 0's are always isolated and poles are also necessarily isolated singularities they cannot be continuously distributed. So, between 2 poles there must be some distance, you cannot have infinite poles situated in continuous distribution that you cannot have.

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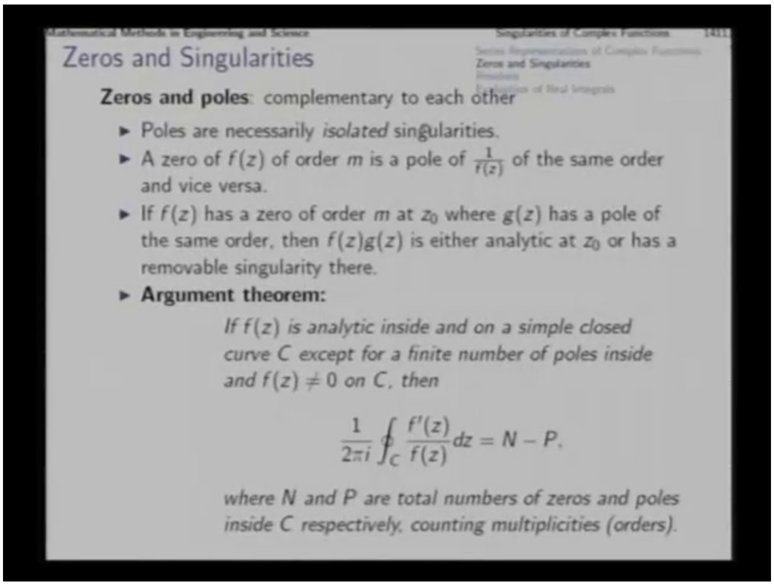

So, poles are also necessarily isolated singularities just like 0's, then a 0 of f z of order m turns out to be a pole of the same order for the function 1 by f z and vice versa right. Then again if f z has a 0 of order m at z 0 and where g z has a pole of the same order if you consider 2 analytic functions f z has a 0 of order m at a point and at the same point g z has a pole of order m then the product of f, f z g z product of these 2 functions in a manner will get rid of the singularity. Will get rid of the 0 as well as the pole and how that will happen that can happen in 2 ways 1 is that the factor which was making f z 0 at this point and the factor in the denominator of g z which was making that point a pole of this of order m. If they turn out to be the same factor then they will cancel each other and there will be the function there will be analytic, on the other hand it may happen that they are not the same factor.

But 2 different factors both of them turning out to be 0 at that particular point in that case they will not directly cancel each other, but in the limit they will cancel each other as we saw in this particular case. Suppose this is 1 factor which is 0 at z equal to 0 and this is another factor which is also 0 at z equal to 0 and if f z turns out to be this numerator and g z turns out to be 1 by z. So, f z has a 0 at z equal to 0 an d g z has a pole (Refer Time: 36:40) equal simple pole at z equal to 0 then their product in this case is not analytic at z equal to 0, but it is a removable singularity. So, these 2 cases are possible, either it is a removable singularity or it is outright analytic there is an interesting theorem in this context and that is called the argument theorem, if f z is analytic inside and on a simple closed curve c except for a finite number of poles inside.

And f z is not equal to 0 on c then this expression gives you the difference of n and p where n is the total number of 0's inside c and p is the total number of poles inside c of course, counting multiplicities or counting orders. So, examine what are the premises here, f z is not equal to 0 on c; that means, on the contour there is no 0 of the function and f z is analytic inside and on the simple closed curve c. So, that means, that f z is analytic on the simple closed curve also that is that f z has no 0 no singularity on c and inside also f z is analytic everywhere except for f finite number of poles and a finite number of 0's. So, then that number of poles number of 0's and number of poles if they are represented as n and p then the difference of total number of 0's minus total number of poles is given by this integral, this is called argument theorem and this has interesting applications in the nyquist stability theory.

So, in the exercises of the textbook in 1 exercise the steps to establish this theorem has been given and I advise you to go through that because this is an interesting theorem and it has a quite important use and right now we proceed forward for another important concept which has lot of practical significance and that is the concept of residues.

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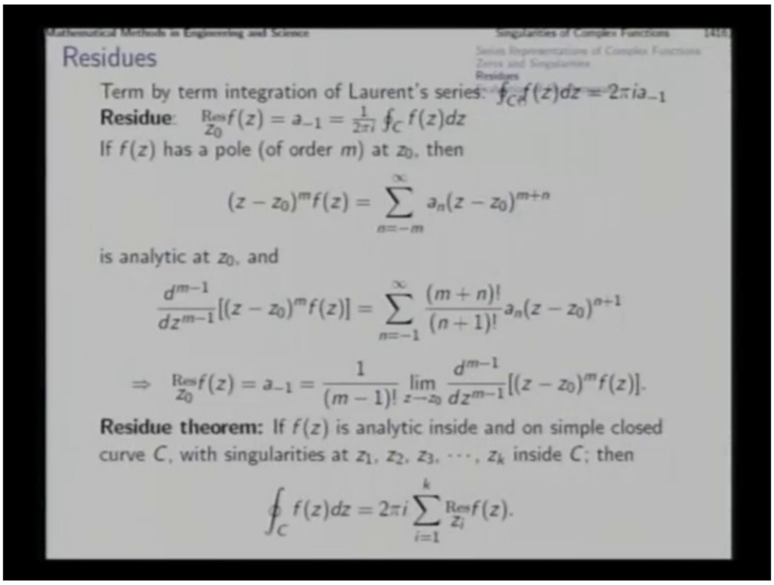

If you consider the Laurent's series anyway there are large number of terms constant term, linear term, quadratic term, cubic term and so on and then 1 by z minus z 0, 1 by z minus z 0 whole square and so on. Then we will find that if we put that entire series representation here and over a contour if we evaluate this contour integral then on the positive powers as well as the constant term we will find that they constitute the analytic part of the function.

So, the contour integral will certainly turn out to be 0, on the other hand the negative powers also all of them will turn out to be 0. In fact, this particular case z to the power n d z integrated over the circular contour we have already seen much earlier in the previous lesson. So, in that case all the all such integrals turn out to be 0 for n not equal to minus 1 for n equal to minus 1. We find that the integral evaluates to a non 0 number and from there we get this 2 pi i a minus 1. So, the coefficient remains and the integral gives you 2 pi i this we have seen earlier if you make note of this here.

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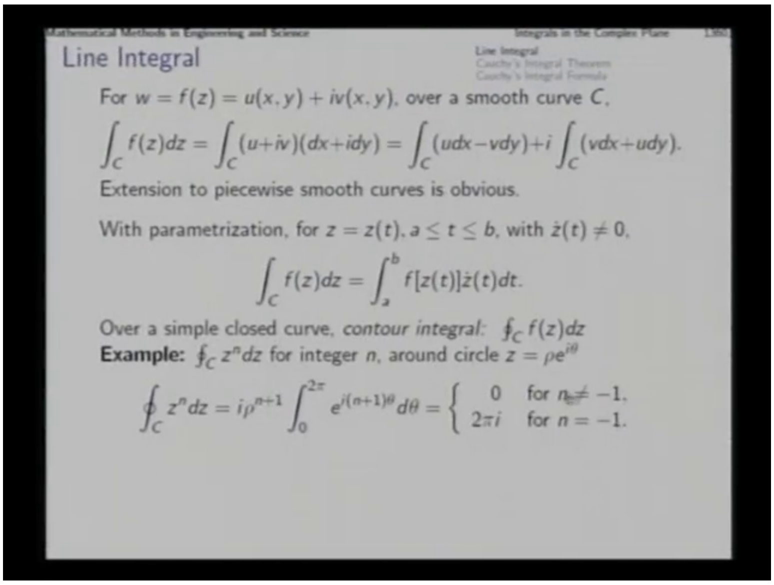

We worked out the integral when we are discussing the line integral c. So, this 2 pi i term we get for n equal to minus 1 and for all other values all other integer values.

So, we get 0. So, using that we will find that this is the only term that will remain in this integral and this is the term which is the only term that remain after everything else evaporates of and that is why we call it the residues. This coefficient a minus 1 that is the coefficient of 1 by z minus z 0 that term is called the residue because of this reason and we can define it like this residue of f z at z 0 is this coefficient which turns out to be 1 by 2 pi i into this integral these all other terms will vanish. Now, if you find that the function f z at z 0 has a pole then to work out the residue say it has a pole of order 1 in that case you multiply it with z minus z 0 and then what will happen is that this series which is originally like this.

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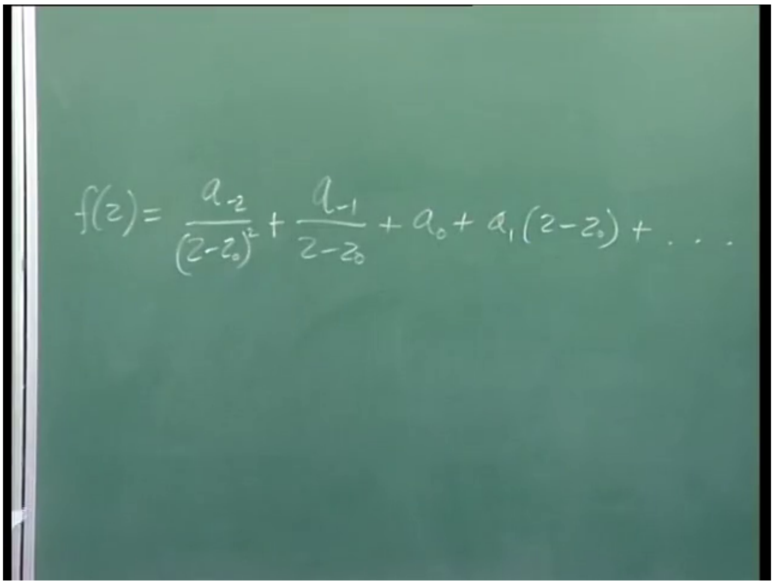

Then if you want to evaluate this residue a minus 1 for that if you multiply this entire series with z minus z 0.

That is z minus z 0 into f z, then what you will get you will get a minus 1 plus a 0 into z minus z 0 plus a 1 into z minus z 0 whole square and so on and in that then simply substitute the value z equal to z 0 all this terms will go off and a minus 1 you will be able to get. So, that means, that if f z has a simple pole at z 0 then the residue you can find out by multiplying f z by z minus z 0 and then taking the limit, if the function has a if the function does not have a pole there is at that point z 0 if the function is analytic then; obviously, a minus 1 will turn out to be 0 as the lower coefficient also will be 0. Now, if it has a pole of order 1 then this will be the situation and in that case multiplication with z minus z 0 will give you this upon substitution of z equal to z 0 if it is a pole of order 2 then you can multiply with z minus z 0 square.

And then a minus 1 will appear in the correct place where then you can evaluate the function at z minus z 0. So, what will happen is that in the case of pole of order 2 you will have, see the expression of a 2 here let me supply the expression this. So, if the function f z has a pole of order 2 then this will be its Laurent series right below z minus z 0 whole square other term that is, below z minus z 0 to the power minus 2 the lower terms are absent this is the situation with pole of order 2.

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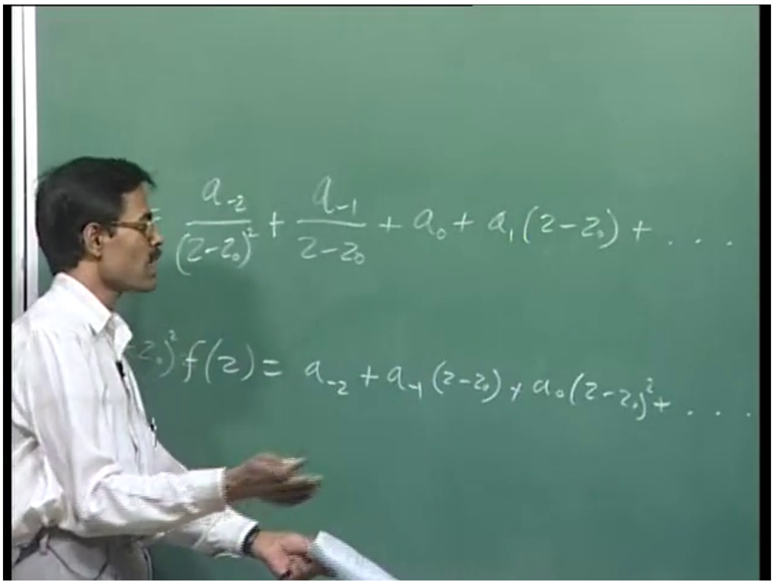

And for this if you want to find out the residue at z minus z 0 then if you multiply it with z minus z 0 whole square then you will get a minus 2, plus a minus 1 z minus z 0 plus a 0 z minus z 0 whole square and so on.

But then when you want to find the residue your intension is to find this. So, immediately you do not substitute z equal to z 0 because then you will get this and not the residue. So, what you do you differentiate it once, as you differentiate once this goes to 0 this goes off a minus 1 lets expose and other terms will remain with a factor z minus z 0. So, then you are substitute z equal to z 0. So, then you will get the value a minus 1 that is the residue. So, for an order, for a pole of order m to evaluate the residue you first multiply the function with z z minus z 0 to the power m and in differentiate m minus 1 times and then what you get turns out to give you the residue directly because here a minus 1 will remain as the leading term with no z minus z 0 factor n equal to minus 1 putting here there will be the leading term with no factor z minus z 0.

So, then you substitute the value or take the limit. So, this is the where to find the residue at z 0 depending upon whether it is analytic at that point in which case the residue is 0 or simple pole or pole of order m. So, this single formula gives you all the cases, but this is so only for poles, not for essential singularities. Now, since this thing divided by 2 pi i gives you a minus 1 there is a residue and we have seen that the integral formula of a over a contour outer contour and inner contours we have already seen that this integral minus the integrals over the inner contours is 0 that we have seen from the Cauchy's integral theorem and we have also seen that the contour integral over an outer contour turns out to be equal to the sum of the contour integral over the inner contours inside which the singularity are enclosed.

So, these small integrals are 2 pi i the corresponding residue and therefore, the residue theorem tells us that the contour integral over large contour can be evaluated as 2 pi i times the sum of residues at all the singularities. So, if we consider these residues which are small contributions of immediate neighborhoods of every isolated singularities then together they will constitute the entire integral, this is given by the residual theorem and this is the reason why this residue is so important. Now, this residue theorem along with this definition of residue is very important in the evaluation of quite a few very important real integrals.

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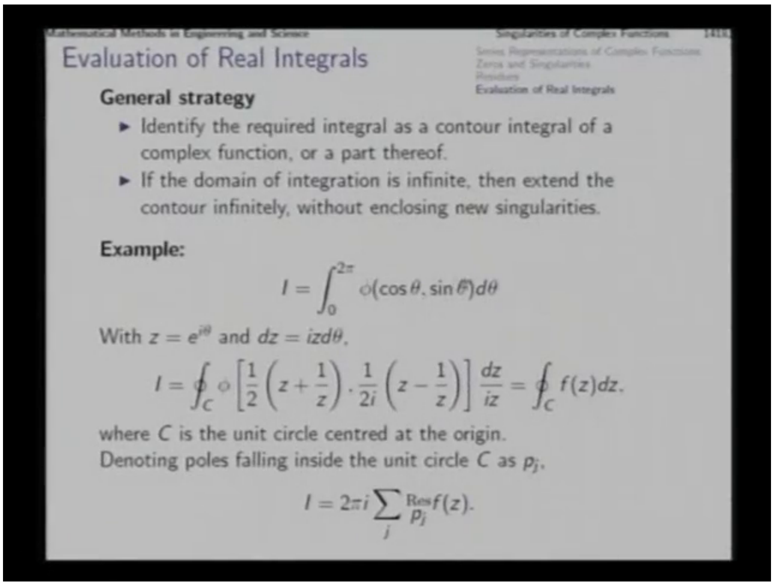

So, general strategy in evaluating such real integrals is to identify the integral in the form of a suitable contour integral of a complex function or a part of that.

Now, if the domain of integration is infinite then we work out a domain which can be easily extended to the infinite contour without enclosing any new singularities and then we as we find that the part of the contour is our actual real domain of integral and the integral over the rest of it vanishes then we can evaluate the real integral by the use of the contour integral. Say 1 generic example is this, we want to evaluate the integral of a function of theta which involves cosines and sins of theta over the entire domain 0 to 2 pi, then if we use z as e to the power i theta then these are turns out to be i z d theta and then cos theta is e to the power i theta plus e to the power minus i theta by 2 that is this and sin theta turns out to be e to the power i theta minus e to the power minus sin theta by 2 i. So, in place of cos theta and sin theta.

We put these expressions in terms of the complex variable z and for d z we use for d theta here we use d z by i z and; that means, this entire stuff turns out to be as we evaluate the function phi this entire stuff turns out to be function of z in this form and the contour c is a unit circle 0 2 pi that is unit circle. So, radius remains 1 and theta varies from 0 to 2 pi and then the poles falling inside the unit circle will denote as p j and then we evaluate this contour integral that is with some of the residues at those isolated singularities and multiply the sum with 2 pi i that turns out to be this integral. Another case is for real rational functions for which we want to evaluate the integral from minus infinity to infinity in proper integrals.

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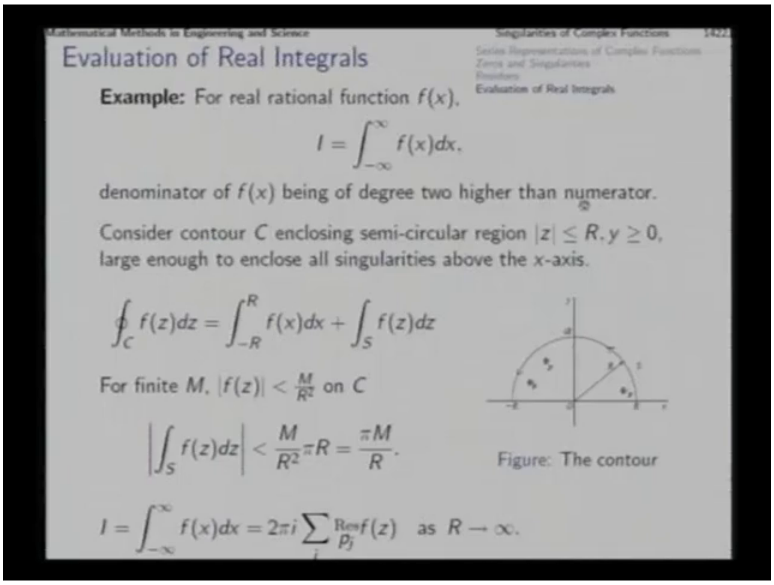

Now, the method that we are going to discuss will work for those rational functions in which the denominator is at least 2 degree higher than the numerator. Let us see how it works, what we do we consider a contour c enclosing this semicircular region minus r to r along the real axis and then along the semi circular path back here, now this will give us this semicircular region as the contour. So, this region is mod z is less than equal to r that is inside the circle and y greater than equal to 0; that means, above the real line. So, this semi circle and we should consider the initial contour large enough to all to enclose all singularities which are above the real line. So, all singularities all poles are here nothing outside here, on the lower side there could be any number of singularities.

We do not need to bother then we consider that this contour integral is actually integral over this line segment an integral over this semi circle, the over this line segment is z is actually x. So, that is this integral, x is z is x and d z is d x over the line segment over this we have this, this integral along x. Now, for finite m we will have the absolute value of f z bounded by m by r square this is the meaning of this the this condition that is denominator of f x is of degree at least 2 higher than the numerator. So, then for a finite m the value of the f z will be bounded by m by r square; that means, as r goes high f z value will go on decreasing. So, then as we consider this integral this part which is not occur interests then this integral will be bounded by m by r square into the size of the path, size of the path is this semi circle pi r.

So, we will get pi m by r; that means, as r is increased indefinitely this will shrink to 0; that means, that in the limit as r tends to infinity this will be 0 this will be the integral that we are talking about and this will be the contour integral which we will find by the sum of all the residue at these poles, it is important to enclose all the poles in the first round itself. So, that new poles are not encountered. So, this is the integral, a similar situation will arise when we try to evaluate the Fourier integral coefficients there also it will work for those functions where which have the denominator at least 2 degree higher in x compare to the numerator.

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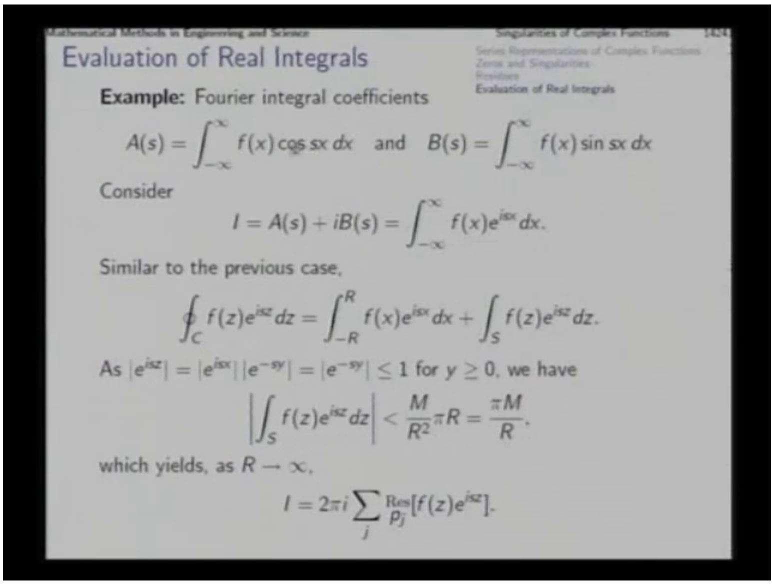

Now, these are the Fourier integral coefficient functions and actually we can determine both these coefficient functions together if we consider it like this a plus i b. So, as we put them together consider single integral then will have cos s x plus i sin s x.

So, it is e to the power i s x, now in the same lines like the previous case we consider semicircular contour and then this contour integral will turn out to be this real integral which is occur interest as r tends to infinity plus the semicircular integral. Now, we concentrate on this as we know that e to the power i s z can be broken up like this e to the power i theta is e to the power i z e to the power i s z. So, here z is s x plus i y. So, the x part will give us this and i y part will give us i square s y which is this i square is minus 1 then we know that this is unit size cos s x plus i sin s x this is unit size. So, this will not change the size. So, this is less than equal to 1, because y is real s is real and this is negative. So, s and y are positive upper half then. So, this is less than equal to 1 for y equal to y greater than equal to 0. So, then; that means, that this stuff we have bounded by m by r square into pi r which is again this.

So, as r tends to infinity we get this part tend to 0 and this part our integral that we want to determine and this is the contour integral which we get by the sum of residues and as r tends to infinity we get the complex coefficient function in 1 short from where you can separate out the real and imaginary parts.

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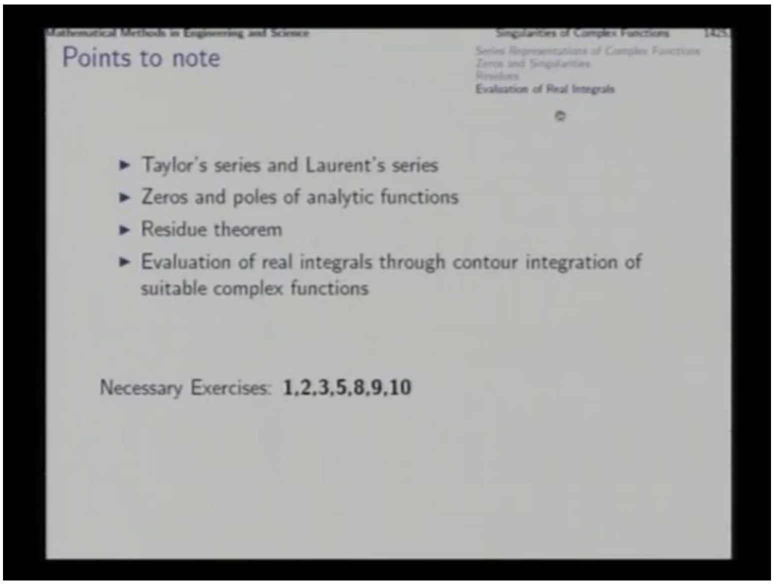

So, these are the points which we have studied in this lesson and there are quite a few interesting exercises in this lesson and some of this exercises actually work out integrals which we encountered in the exercises of previous chapters say previous lessons in Fourier integrals or Fourier transforms or in the solution of partial differential equations. In earlier exercises some of the integrals where left as it is and in some of this examples some of the exercises you will find the step 2 evaluate those integrals by the use of what we have leant in this lesson.

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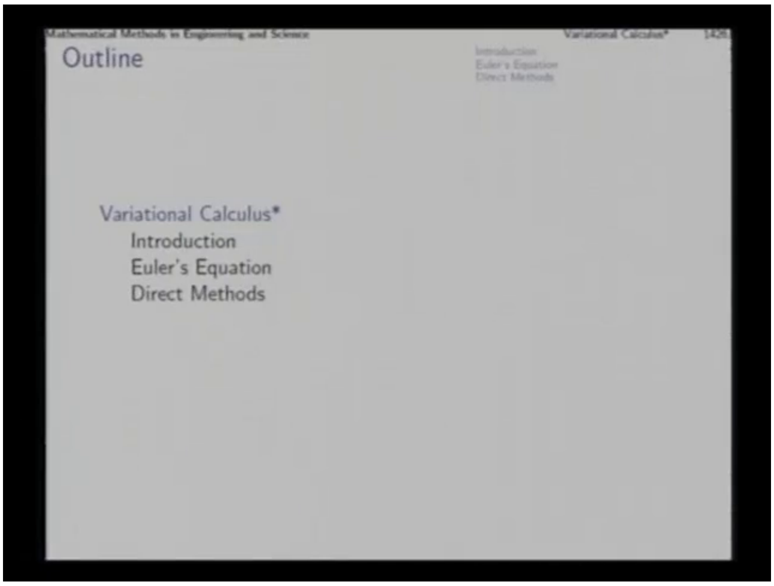

So, this completes our module of complex analysis and in the next lecture which will be the last lecture of the course we will see a quite a few interesting interconnection in different areas of that mathematics and that is our single lesson on variational calculus, that will be the last lesson of this course.

Thank you.