Mathematical Methods in Engineering and Science
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Module - VIII
Overviews: PDE's, Complex Analysis and Variational Calculus
Lecture – 04
Integration of Complex Functions

Good morning, in the previous lecture, we discussed the differential calculus part of complex analysis.

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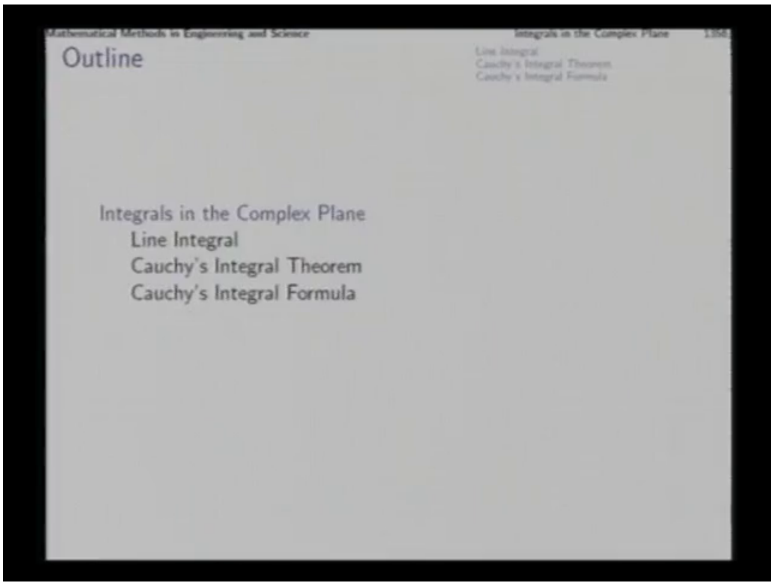

Today in this lecture we go into the integral calculus part of it and we study integrals in the complex plane. Our 2 main topics of discussion will be Cauchy's integral theorem and Cauchy's integral formula before going into those 2 topics we spend a little time to build the ground for it first we discuss the concept of line integral in the complex plane.

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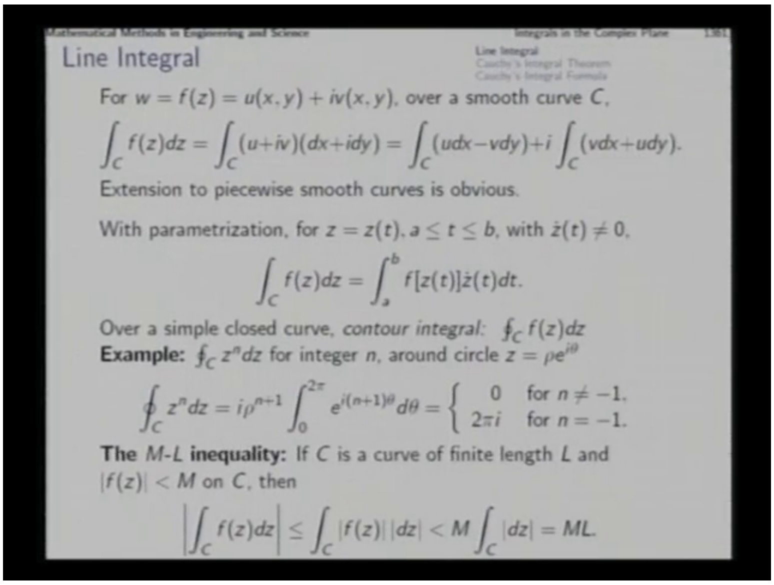

Now if fz is a function of z that is w is equal to fx which is expressed in this manner u and v are the real and imaginary components respectively, then we can define its line integrals over a smooth curve C in this manner that is we consider fz as u plus iv and dz is as usual dx plus idy and then the product of this, these 2 will give us the real part as udx and minus vdy which is here and the imaginary part will be vdx plus udy which is here now this real integral plus i into this real integral will give us the line integral of fz. So, this will be a complex function; obviously, because this is the real part and this is the imaginary part.

Now, this we have defined for the time being over a smooth curve C, now extension of this same definition over piecewise smooth curve is obvious because if the smoothness is interrupted at say a finite number of points, then considering those finite number of points as the endpoint of one segment of the curve at starting point of the next segment of the curve, we can build the same line integral over several segments. So, now, if particularly if we have a parameterization of the curve z in the form of z of t with respect to a real parameter t which varies from a to b that is as this parameter t varies from real number a to real number b over a an ordinary interval, then if the corresponding z of t traces that curve with this derivative never vanishing, then this line integral can be reduced to a simple definite integral because in that case we can put fz as F of z of t which means this turns out to be in terms of t and then dz can be put in terms of z dot into dt and in that case this entire thing F and z dot the product of F and z dot becomes the integral and the integral is with respect to this real parameter t and then it is a definite integral.

So, with the parameterization of the curve C available the line integral can be reduced to a definite integral in real variable quite easily now if the closed curve if the simple if the smooth curve C is taken as a closed curve simple closed curve without self intersections, then you can talk of contour integrals which is integral over a closed curve and then the symbol is used like this, let us take an example which will serve 2 purposes one to demonstrate the evaluation of integrals using this kind of formulae and second to have a little result in hand which we will be using later for a very important step in later derivations, we try to find out this particular integral for integer n know that is z to the power n simple function z to the power n we find try to find the integral of this around a circle which is centred at 0 at origin and with radius rho, then here you see that the variable theta is the parameter the it takes the place of t in this formulation in that case this for evaluating this contour integral over the circular contour we consider z as this and then this z will be i into this whole thing right because the derivative of this will be rho into a to the power i theta into i.

So, from this step to this step n powers of rho come from this and the last another additional power comes from here from the dz. So, rho to the power n plus 1 and since rho is constant it is a circular contour. So, that we can take outside the integral sign and then similarly e to the power i theta n powers of that come from here and one power comes from here. So, e to the power i theta also is raised to n plus 1 power and in the derivative of this factor i comes which is here. So, this entire thing has been now reduced to this and now since here the variable of integration is theta.

So, the integral is evaluated over theta with the limits 0 to 2 pi one circle will have 0 to 2 pi interval of the parameter angle theta now you see that the contour integral has been reduced to a ordinary definite integral in this manner right now if we try to evaluate this integral, we notice that if n is equal to minus 1, then this enter power vanishes and this integral integrant becomes one and in that case it is a simple integral of a constant function one and that will give us 2 pi and therefore, for n is equal to minus 1 you get 2 pi from here and you get 2 pi i the as the integral.

On the other hand, if you have n as anything other than minus 1 say 0, 1, 2, 3 and so on, then this turns out to be an integer which is going to give us give us e to the power i phi where phi is n plus 1 theta. Now we know that e to the power i phi is cos phi plus i sin phi and as we try to integrate that we will find that both cos phi and sin phi integrated over 0 to 2 pi will produce 0. So, real part as well as the imaginary part will give us 0 and therefore, for n not equal to minus 1, we get this integral as 0. So, this small result we will keep in hand which will be useful later; now another small result which will be quite useful in later discussion is conveniently named as the ml inequality.

Now here what do we say is that if c is a curve of finite length l and fz is a function which is bounded and the bound is m bounded means its absolute value is bounded. So, now, here is the value of fz of absolute value of greater than m. So, if the value of fz its absolute value is bounded by m over that curve, then we say that the actual value of this integral will be always less than equal to what we would get if we replace this fz and dz by their absolute values, right. So, then this is one less than equal to that is this is one inequality and then rather than fz absolute value if we put its supremum m, then this side will be further increased.

So, therefore, this will be less than equal to actually less then because of this strict inequality this thing will be less than the integral that we would get if in place of this absolute value of fz if we put the supremum value m and that value being constant can be taken out of the integral and then the absolute value of dz integrated over c actually gives us the length of the curve and that is ml. So, what we get as a result the result is that if the value of the function absolute value of the function is bounded by this m and the curve of finite length l, then this absolute value of the line integral is bounded by the product of ml and l. So, this is a convenient result which will be of use later also.

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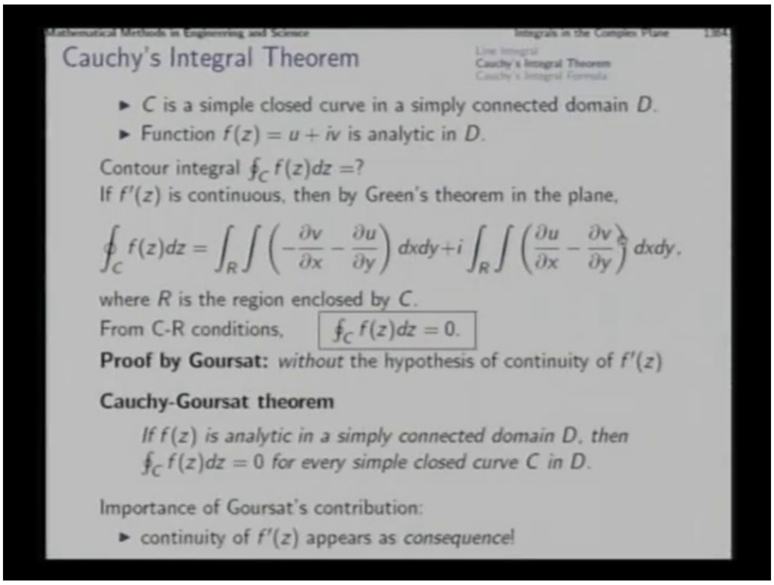

Now, we come to one of the important topics of our discussion and that is Cauchy's integral theorem that C is a simple closed curve closed curve and non intersecting in a simple connected domain D, we have already discussed what is a simple connected domain in the context of vector calculus here in the complex plane, it is the similar domain in which there is no interior boundaries no interiors whole a single boundary which is also non self intersecting that is simply closed curve simple closed curve, then if we take this function which we consider as analytic in this domain now what do we have we have a simply connected domain D in that we have an analytic function which is every analytical everywhere in this domain and we have a simple closed curve C in that domain and then we want to evaluate what is the contour integral of that function over that particular closed curve.

So, if the derivative now one point is already clear that since the function is analytic. So, it will possess derivatives. So, if that derivative is continuous then we can apply greens theorem to develop this integral this is the integral, right. So, here we talk of this integral and we remember; what is greens theorem in the plane.

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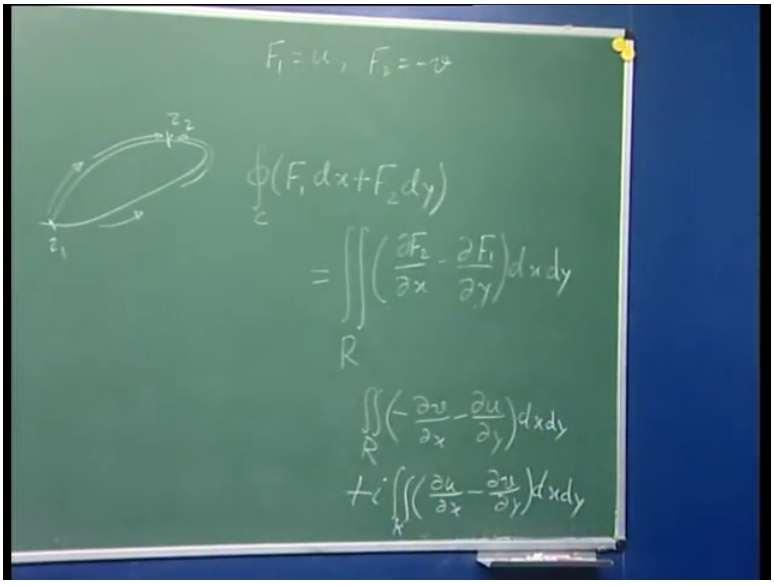

And that is the line integral of a function with components F 1 F 2 which is given like this F 1 dx plus F 2 dy over a simple closed curve C is given by the double integral over the region enclosed by c of del x 2 by del x minus del x 1 by del y right simply stated line integral of function F over the closed curve C turns out to be equal to the surface integral of its curve over the surface bounded by c now that is the Stokes's theorem and the special case is greens theorem in the plane. Now in this particular case with this stuff udx minus vdy and vdx plus udy when we try to apply this greens theorem in the plane on this say on this we try to apply first udx minus vdy. So, for that u takes the place of this F 1 and minus v takes the place of this F 2, right.

So, F 1 is u and F 2 is minus v right for this line integral right; so, then if we insert that. So, here we will get udx minus vdy the real part and then here we will have the right side we will have as del F 2. So, minus del v by del x right minus del F 1 by del y del F 1. So, F 1 is u. So, we will have minus del u by del y this will be the first term and then the second term the imaginary term for that v will take the place of F 1; F 1 and u will take the place of F 2. So, we will have here v is taking the place of a F 1 right. So, and v is and u is taking the place of F 2. So, u here del u by del x minus del v by del y this will be the imaginary part right and then you will find that since the function F is analytics; that means, u and v its real and imaginary components will satisfy the Cauchy Riemann conditions which means del u by del x is equal to del v by del y and del u by del y will be equal to minus del v by del x; that means, both of these integrants will vanish and therefore, the entire integral will vanish

So, by applying greens theorem in the plane in this case we get this as we have developed here and then Cauchy Riemann conditions will imply that both of these integrants will vanish and will get the integral as 0. Now one important premise that we have used is that the derivative is continues now this makes the proof simple; however, there is a more rigorous proof by goursat which does not use the hypothesis of continuity of F from z that gives a little formal advantage to the proof by goursat in the sense that in that case if the continuity of the derivative is not used even without using it if it is proved then the continuity of F prime appears as a consequence.

So, therefore, many authors many authors consider name this Cauchy’s theorem as Cauchy’s Goursat theorem and what it means it means that if fz is analytic in a simply connected domain D then a contour integral over a simple closed curve is 0 for every simple closed curve in C in D; that means, that if in the domain the function is analytic everywhere then whatever the closed curve simply closed simply a simple closed curve we can frame in that. So, around every contour that analytic that integral contour integral will vanish; that means, there is no information content of the function in a domain in which it is analytic everywhere this integral will always vanish over every contour.

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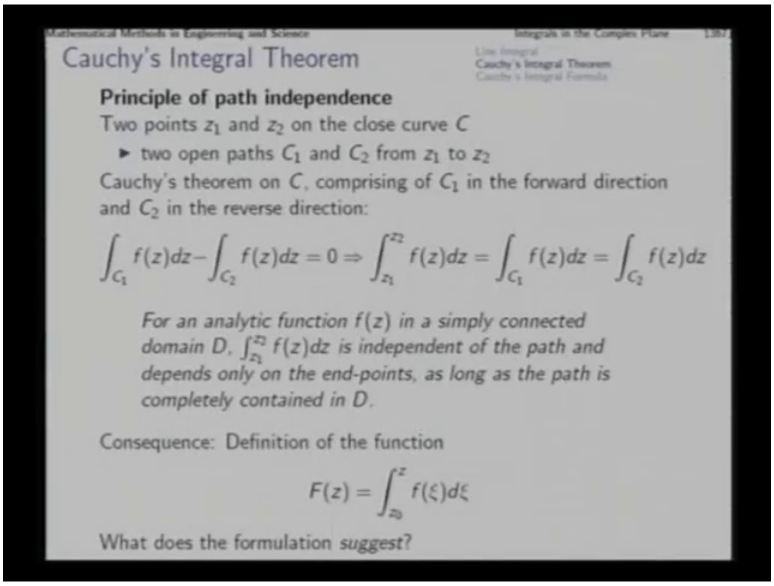

Now, as you continue in that path. So, we find that on a simple closed curve we if we take 2 points say this is simple closed curve. So, we take this as one point and this as another point now around this contour if the contour integral turns out to be 0. So, then this is one line integral from z 1 to z 2 through this path and z 1 to 2 3 though this part is another contour integral. So, if we take a contour another line integral. So, now, this contour integral from z 1 to z 2 and back to z 1 is actually the sum of 2 line integrals z 1 to z 2 along this curve and z 2 to z 1 along this curve which is the second one is equivalent to the negative of the line integral from z 1 to z 2 along this curve. So, this minus this; this line integral minus this line integral will mean this line integral plus this line integral which is a contour integral which should vanish which is 0.

So; that means, if you take 2 points z 1 and z 2 on the closed curve C then these 2 points will open the contour open the closed curve into 2 open curves open paths C 1 and C 2; C 1 as this and C 2 as this both from z 1 to z 2 and then as we apply Cauchy's theorem on C with the 2 parts C 1 in the forward direction and C 2 in the reverse direction then we will find that the contour integral turns out to be integral over line integral over C 1 minus line integral over C 2 which should vanish by Cauchy's theorem, then that will mean that this line integral and this line integral are equal and then we can say that this line integral depends only on the endpoints and not on the curve along which we are going because as long as we keep these 2 endpoints in hand for any other curve, we can argue in the same manner.

So, this is the important result that we get that is for an analytic function in a simply connected domain this line integral from one point to another is independent of the path and depends only on the endpoints as long as the path is completely contained in d because outside d there is no guarantee of analytical analyticity in d we have taken the premise of analyticity. So, analyticity is the fundamental requirement here now as a consequence we can talk of this that is keeping one fixed point z 0 as say z 1 in place of z 1 we keep one point fixed z 0 and the other point if we keep a variable point z, then we can define a new function of z and call it a capital F. Now what does this formulation suggest this formulation is looking similar to the definition of a definition of an indefinite integral because we are keeping this z as the complex variable itself and as a result we not getting a value of the integral, but we are getting a function of z right now this will suggest.

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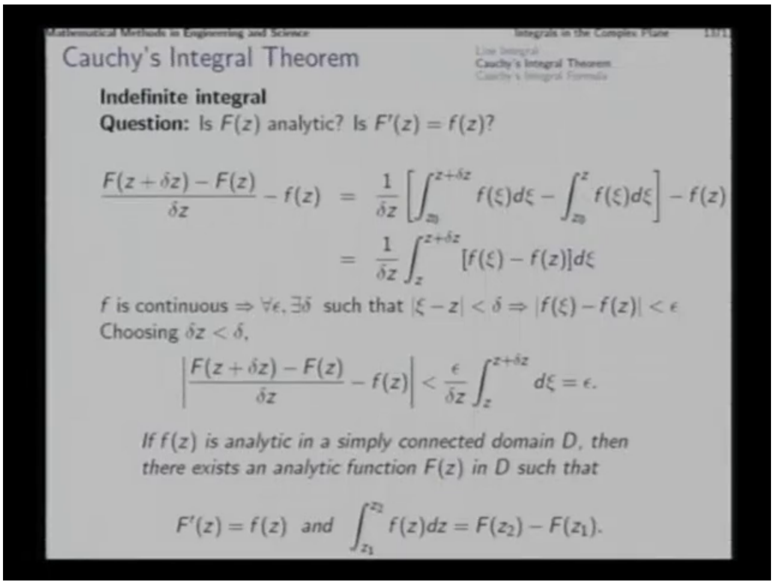

The idea of indefinite integral, but then the implication of a definite integral the direct meaning of a definite integral is the sum of infinite number of small infinitesimal contributions that we have already considered; however, that idea of indefinite integral is that of a function capital F whose derivative turns out to be to be this integrant then before formally accepting this function defined in this manner as the indefinite integral of small F we should ask if we define a function like this and call it capital F, then is capital F differentiable then because if capital F is differentiable and if its derivative happens to be this integrand, then we will be able to accept this formulation as the indefinite integral of small F because the idea of indefinite integral is comes through anti differentiation that is the reverse process of differentiation. So, we ask this question is fz differentiable or if fz analytic and is the derivative equal to small F to get an answer of that we consider this from z we make a little variation delta z and then F of z plus delta z minus fz by delta z we try to evaluate the difference of this quantity with the value of small f.

Now, if Fz is actually differentiable and if its derivative happens to be F, then the limit of this as delta z tends to 0 should be same as f; small f and in that case this difference should tend to 0 as delta z tends to 0. So, we try to first evaluate this difference that is simplify it and then see whether it has a limit and what is that limit that limit should be 0 for our purposes. Now for this one by delta z is common. So, we keep it common capital F at this changed point and the original point unevaluated based on this formula. So, in the one in one place in place of z we use z as it is in other place we use z plus delta z. So, capital F of z plus delta z is this integral evaluated from z 0 to this point and capital F of z is here that is the argument of the function appears only in the limit of the upper limit of the definite integral and everything else is same now here since F is analytic. So, the integral from z 0 to this minus z 0 to this will turn out to be the integral of F from z to z plus delta z from the difference of the 2.

And if we say that this fz for the purposes of this integral business is constant because it is outside the integral sign, but we can always say that this is fz into d psi integrated from this to this now that is basically verified from the evaluation of this second integral. So, consider this integral of fz this part integral of fz from here to here fz is independent of psi. So, fz into psi and the difference of psi will come out to be delta z. So, integral of this will turn out to be fz into delta z and delta z cancelled it if fz only so; that means, this fz can be inserted into integral sing in this manner now as we get this then you say that this difference is equal to this and then if the function is continuous which it must be because the function is analytic. So, analytic function must be continuous. So, since F is continuous then for whatever small epsilon neighbourhood you demand between the function values that small epsilon neighbourhood you can achieve by taking a suitable small delta neighbourhood or the variable itself that is the direct meaning of continuity.

So by taking a suitable delta neighbourhood you can achieve an epsilon neighbourhood of the function value whatever small epsilon is demanded and in that case we choose delta z to be smaller than this delta which will ensure that this is always less than epsilon right. So, in that case the epsilon constant is taken out. So, you get epsilon by delta z and in the integrand nothing remains just one so; that means, that the integral will then be simply delta z and that delta z will cancel this delta z and will give you epsilon so; that means, this whole thing is less than epsilon and what we have discussed already for whatever small epsilon neighbourhood is demanded; that means, epsilon can be indefinitely reduced in size and; that means, this will tend to 0 as delta z tends to 0 and; that means, that indeed capital F is analytic and the limit of this as delta z tends to 0 turns out to be equal to fz because the difference tends to 0.

So, therefore, we say that if fz is analytic then there exists another analytic function capital F whose derivative happens to be f; that means, if the function f; small f is analytic then we knew earlier that its derivative exists now we come to know that its integral also exists which is capital F whose derivative happens to be this and the earlier definite integral that we were talking about earlier definite integral which we define based on path independence that turns out to be available from this formulation.

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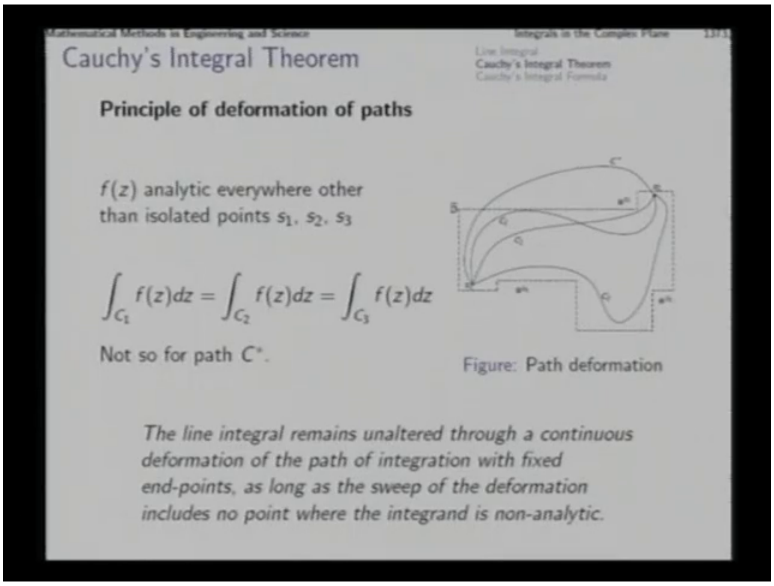

Now, this gives us the same path independence gives us another important tool and that is for principle of deformation of paths say we take an analytic function fz which is analytic everywhere other than for isolated points like this point s 1, this point s 2, this point s 3 at which the function is not analytic everywhere else the function is analytic I perform those few finite number of isolated points then what we can do if we construct a domain as shown by this dashed line which does not enclose these isolated points where the function is not analytic then we can say that any path connecting 2 points z 1 and z 2 in that can be continuously deformed to any other path without crossing as long as the path remains within the domain D; D bar, then the path say C 1 can be continuously deformed to C 2 or C 3 without sweeping over any of these non analytic points right.

And in that case the integral of the function from z 1 to z 2 along all these path will turn out to be same as long as you do not sweep over these non analytic points if you sweep over the analytic points then this guarantee will not be there that is the path the line integral along C 1 along C 2 along C 3 will be all same because they fall within the domain that does not enclose the non analytic points on the other hand the same will not be true for a path which is c star because from C 1 C 2 if you want to change to c star through continuous deformation then the deformation will sweep through this point. So, this we can say.

So; that means, the line integral remains unaltered through a continuous deformation of the path of integration with fixed end points because we have already seen the path independence of definite integrals as long as the function remains analytic and as you sweep across sweep through s 1 as the curve sweeps through s 1 or s 2 or s 3 that analyticity is not guaranteed. So, you cannot claim the equality of the line integral over c star. So, the line integral will remain unaltered through a continuous deformation of the path of integration with fixed end points as long as the sweep of this deformation does not enclose any of the non analytic points now we try to extend the theorem which is integral theorem to multiply connected domain.

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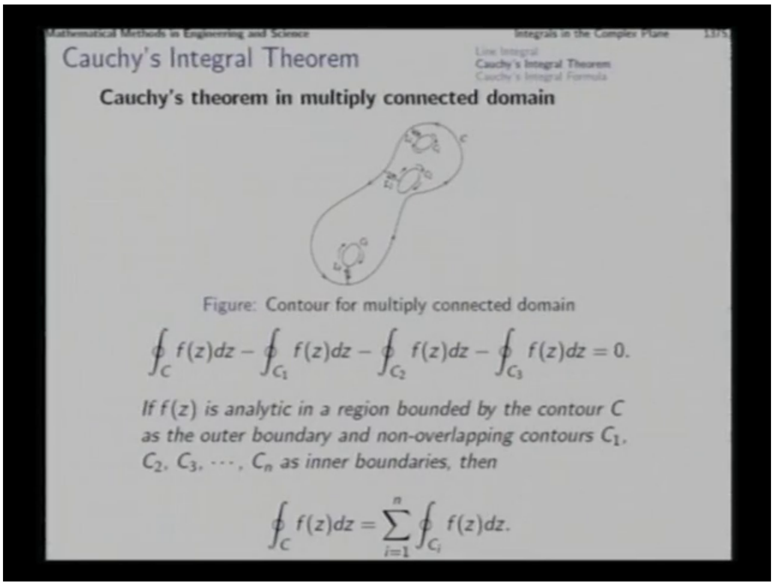

Now here consider this multiply connected domain the outer boundary is this and there are inner boundaries; that means, the domain consists of this region and not the interior of this not the interior of this and not the interior of this.

Now, to evaluate the integral over this we say that it is granted that the function F z is analytic over this domain, but about the interior of these internal boundaries nothing has been seen; that means, the function could have a singularity could be non analytic somewhere inside these interior boundaries. So, what we do is that we consider a little changed contour that is we take the contour as this curve series we start from here we will come back to here at the end start from here from this arrow we go like this and here we may a cut and come along this arrow

And then a clockwise turn along C 1 clockwise means actually negative and then along this same cut line you go back here and then come here many another cut come here another clockwise turn go back and then come here along the cut come along the arrow another third clockwise turn and then here and come back. So, this is a closed contour and this does not enclose any of the singularities this encloses the domain correctly as long as we keep these forward movement and backward movement along these cut lines very very close as close as we want.

Now, the domain which this contour has enclosed is a domain in which the function is analytic everywhere and therefore, we say that over this contour the line integral vanishes the contour integral vanishes and this contour has enclosed has traversed through curve from here to here and then here to here, here to here which is the curve C that is this and then it is enclosed this line segment l one forward and backward cancel this line segment l 2 forward and backward cancelled this line segment l 3 forward and backward cancelled and the 3 interior curves C 1, C 2, C 3 in the negative manner clockwise.

So, that is that is why minus the contour integrals over C 1, C 2 and C 3 this whole thing is 0 because fz is analytic everywhere in the contour that has been actually traversed now this gives us the result that if fz is analytic in a region bounded by the contour c as the outer boundary and non overlapping contour C 1, C 2, C 3 etcetera as inner boundaries in this manner, then over that outer boundary turns out to be equal to the sum of this plus this plus this that is this; that means, whatever are the inner boundaries around those inner boundaries if we develop the contour integrals then the sum of these contour integrals is going to be the contour integral over the outer boundary; that means, that whatever contribution to the contour integral over this curve C is to be made that must be made by the contours enclosing the non analytic regions inside the interior boundary only.

Those places where the function is analytic will make no contribution to this contour integral so; that means, that if we find that in the domain there are a few spots of non analyticity then around that non analyticity non analyticity non analytic small integer parts if we evaluate the small contour integrals and then sum of these then that will account for the entire contour integral over the outer boundary and in the region in which the function is analytic from that region no fresh contribution will be made next we proceed to develop the Cauchy's integral formula.

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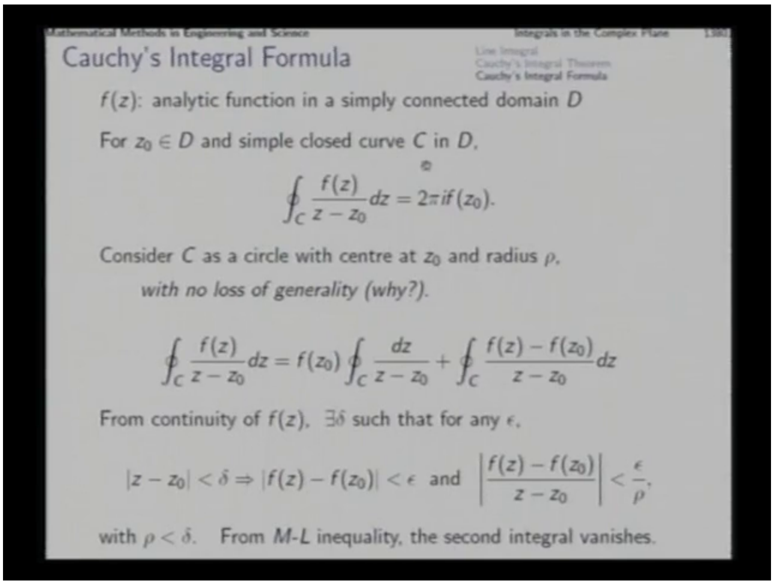

Suppose fz is a function which is analytic in a simple connected domain D and then at a point z 0 in d and a simple close curve C enclosing that point that 0, we want to evaluate this integral a contour integral of fz by z minus z 0 and Cauchy's integral formula tells us that this happens to be the expression for this integral that is integral around the curve C closed curve C is given by 2 pi i times the function value at the integer point.

Now we see that this function fz is analytic, but fz bu z minus z 0 is certainly not analytic at z 0 because at z 0 it is singular. So, which closed curve we consider as c we can consider any closed curve as long as it can be obtained through continuous deformation or any other curve right. So, without loss of generating we consider c as a circle with centre z 0 and radius rho because any other curve could any other such simple closed curve enclosing z 0 could be reduced to this circle through continuous deformation over the analytic domain right. So, if we consider this particular curve then this integral we can evaluate as splitting it first in 2 parts that is we consider this fz as F of z 0 plus fz minus F of F 0 that F of z 0 first part we take here and that is that being independent of z comes out and we get this and the rest of it is here now the this integral we have already evaluated earlier whoever when we are considering the integral of z to the power n. So, now, through a coordinate shift this will give you the same thing and this is the; for the power n equal to minus 1 in which case this was giving us 2 pi i.

So, this part we know and from continuity of fz we can find out a suitable delta in order to make this less than epsilon for any small epsilon and then z minus z 0 we can take less that delta that is a small delta neighbourhood which we give this as within an epsilon neighbourhood and then if we take the radius of the circle that rho radius of the circle as even smaller than the corresponding delta then that will ensure that this whole thing this is less than epsilon and whatever delta is necessary for this if we take even smaller rho as the z minus z 0 as the radius of the circle then this thing will be less than epsilon by rho the lower the denominator is taken as even smaller. So, this side becomes larger. So, with rho less than delta.

So, then this fellow will be less than epsilon by rho and then as we consider the upper bound of this then this will give us less than epsilon by rho into the size of the in into the length of the path and the length of the path is 2 pi into radius 2 pi rho. So, the integrand is bounded epsilon by rho and then the pass length is equal to 2 pi rho. So, this integral will be limited the integral absolute value will be limited by the product of these 2 from ml inequality; that means, epsilon by rho into 2 pi rho; that means, 2 pi epsilon. So, this fellow will be bounded by 2 pi epsilon and that will happen for any epsilon whatever small.

That means epsilon can be demanded smaller and smaller and smaller and accordingly we will get delta and therefore, rho which is even smaller smaller smaller. So, as we make it small enough we will be able to make this in the limit vanish and then integral will be evaluated at this and the integral of this we already know. So, the integral of this will give us 2 pi i. So, 2 pi i F of z 0 that is the result. So, this establishes the Cauchy's integral formula and what are the applications of this formula there are 2 significant applications?

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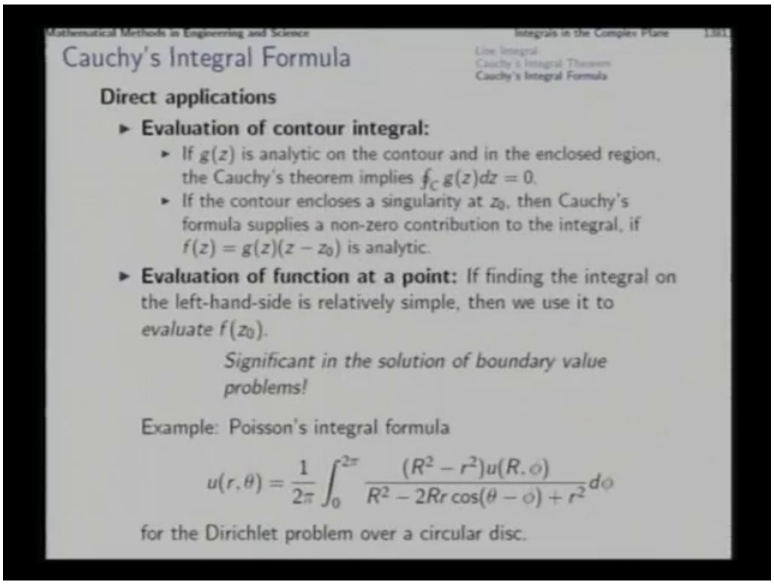

One is evaluation of contour integral as it is proposed that is if we want to evaluate a contour integral like this then by evaluating the function at the integer point at z 0 around witch the in contour integral is there. So, at that point z 0 if we evaluate the function F then we get the integral of this function which happens to be singular note that F is analytic, but fz by z minus z 0 is the integrand which is not analytic. So, when we want to find out the contour integral of this around a contour inside which there is a point of singularity then singularity of this integrand fz by z minus z 0 fz is still analytic.

So, at that point if we evaluate fz that is F of z 0 multiply that with 2 pi i then we get the integral of this singular function singular at z 0 analytic everywhere else because one by z minus z 0 has a single singularity at z 0 and fz itself has no singularity anywhere. So, this rest of the function fz by z minus z 0 has a single singularity at z 0 around which we have the contour. So, evaluating the function at z 0 function F at z 0 will give us the integral of this that is a direct application that is if a function gz is analytic on the contour and in the enclosed region then Cauchy' theorem implies its integral to be 0 on the hand if the contour encloses its singularity then Cauchy's theorem does not give us a result, but then if gz is has a singularity at z 0, but it is removed by multiplying with z minus z 0 that is if fz like this is analytic then this formula Cauchy's formula supplies a a non 0 contribution to that integral around that particular point.

So, this is the purpose of evaluating the contour integral now if it happens that we do not know the function value at the interior points, but we know the function value the boundary points if that is the situation, then the same formula can be used for the reverse purpose and that is the solution of boundary value problems that is evaluation of function at an interior point if finding the interior if finding the integral on the left hand side is relatively simple then we can use the same formula to evaluate fz; fz 0 that is if the function is easy to integrate over a contour and because its value over the boundary points is known and in the integer point we want to evaluate the function then this same integral formula can be used to develop this integral divided by 2 pi i and that gives the function value at the integer point z 0 this is the application of this integral formula for solving boundary value problems. So, this is for example, in the previous lecture we discussed this integral formula Poisson's integral formula which is this now we can develop this formula; now say we take the boundary point as capital R phi in this manner z and the integer point z 0 as this small r theta.

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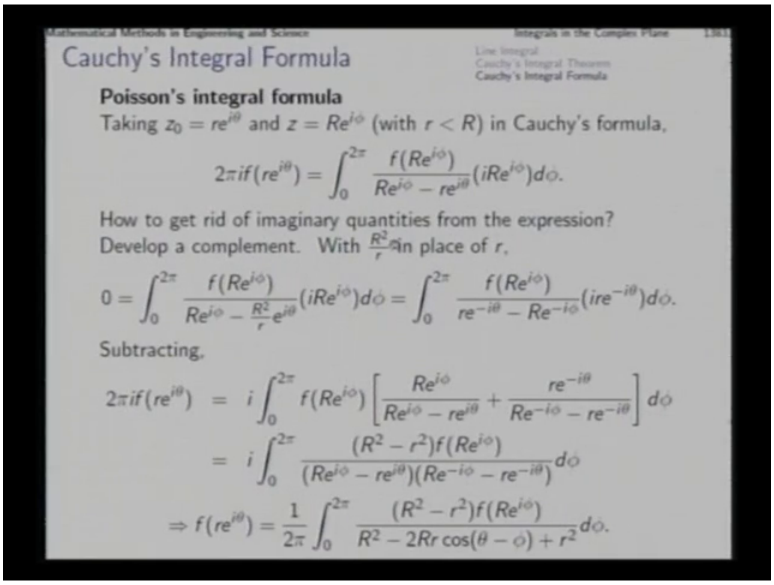

We want the values at z 0 integer point and we can easily find the value of the function on the boundary points. So, capital R is larger than small r small r is integer point. So, if we put it in the Cauchy's formula then we get 2 pi if z 0 is equal to integral over the circle and; that means, 0 to 2 pi off pi that is the boundary; boundary integral and this if fz by z minus z 0 and this whole thing is actually dz, right. So, if z is this then we will get dz as this and through this substitution through this change of variable this phi i will vary from 0 to 2 pi and now this if we try to substitute everything directly and simplify a lot of imaginary terms will remain.

So, to get rid of that what we can do is that we can consider a complimentary term; so, in place of small r if we put capital R square by r then you see here and here we put that small r in case of small r we put capital R square by r and in that case if small r is inside smaller than capital R then this will be larger than capital R right and that will mean that the corresponding point this will be outside this contour where the function is anyway analytic and therefore, the line integral will be the contour integral will be 0.

So, for that 0 is equal to 0 to 2 pi the same thing in place of small r if we write capital R square by r, then we will be talking about the contour integral around the point which is actually outside. So, that integral is 0 and now throughout the simplification what has been done in this simplification small r has been multiplied. So, this small r has gone it has appeared here and this small r will be appearing here this capital R has been cancelled with this r here. So, 1 r will vanish from here the other one r will go from here the capital R will remain e to the power i phi and e to the power i theta with these 2 factors we are dividing it. So, e to the power i phi will go up e to the power minus i theta will remain. So, in the denominator in the numerator that will happen in the denominator e to the power i phi will vanish here e to the power minus i theta will come and the reverse will happen here right. So, we get this.

Now, as we subtract this from here in a left side 0 is subtracted. So, the same thing remains on this side; however, a lot of things happen first of all this is common. So, this remains and this i is common here as well as here. So, this i goes outside that will also remains and other than that what will remain this r e to the power i phi by this and this small r e to the power minus i theta divided by this. So, this is positive powers this is negative powers everywhere now as we simplify this then we will find that those terms which would give imaginary parts they will cancel out and we get this result through simplification and in the denominator as we multiply out we get this and this is exactly what we studied in the what we saw in the Poisson's integral formula earlier.

So, this is F of z that is u plus iv is equal to this; this same u plus iv over the boundary points are here. So, at the integer point the function gets its value through the integral over boundary points integrated from 0 to 2 pi this gives us the poissions integral formula for a disc and if we take the real and imaginary parts separately say writing only for real part then we will get the corresponding u is equal to 1 by 2 pi this integral everywhere the same thing here that function u will come over the boundary points. So, this establishes the Poission's integral formula this is just an example for the unit disc for the disc as the domain, but for other such domains also this kind of formulas for solution of boundary problems can be established.

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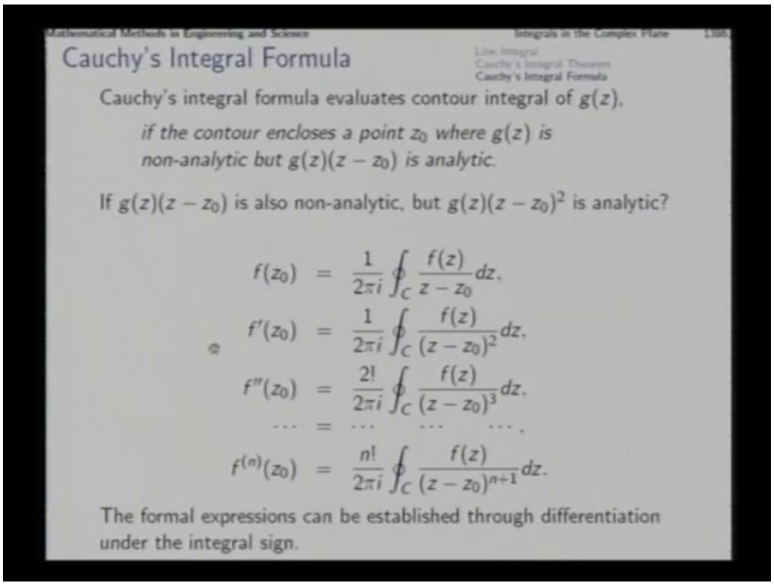

Now, another important aspect in the evaluation of integrals is this, you know we have already discussed that Cauchy's integral formula evaluates contour integral of gz if the contour encloses a point z 0 where the function gz is non analytic, but gz into z minus z 0 is analytic that is one multiplication with z minus z 0 makes it analytic in that case we get the integral evaluation through Cauchy's integral formula that is if gz into z minus z 0 is analytic then gz.

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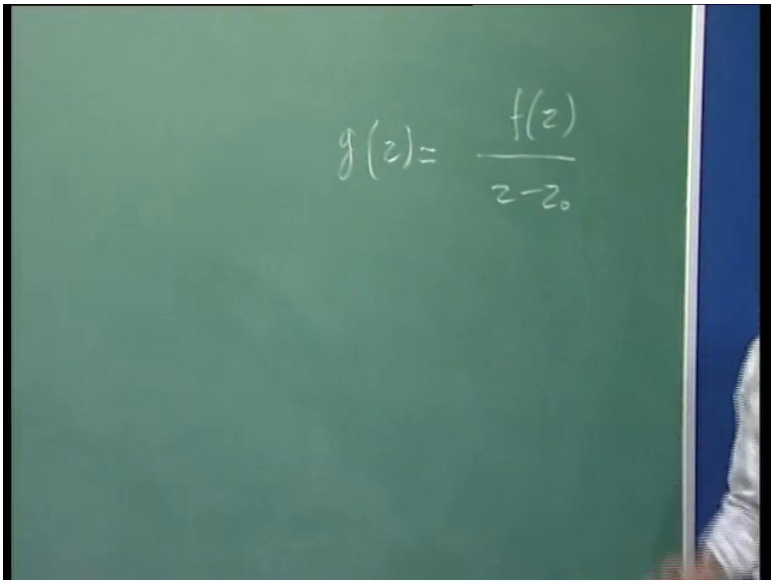

Turns out to be an analytic function divided by z minus z 0 an integral of such a function is given by the Cauchy's integral formula, but then if it happens that even this is analytic, but to make it analytic another multiplication of z minus z 0 is required then what to do through further processing over Cauchy's integral formula we can find a way for that also. So, this is the Cauchy's integral formula that we have discussed earlier now if we try to differentiate this expression with respect to z 0, then we get F prime z 0 and the these for these formal expression can be found by differentiating under the integral sign noting that z 0 and z are 2 different variables z is a boundary variable and z 0 is an integer variable.

So, if we differentiate with respect to z 0 and that is independent of z then differentiation under the integral sign will proceed directly and that differentiation and integration order can be change. So, the derivative of this will be the integral of the derivative of the integrand with respect to z 0 and with respect to z 0 differentiating this is easy because we get fz into 1 by z minus z 0 square negative and another negative sign will come because of this and. So, 2 negatives will cancel and we will get this the further derivative also appear as minus 2 fz by z minus z 0 to the power 3 into minus 1. So, minus minus again will cancel that 2 will appear and this power will get raised to 3 and so on; we can go on differentiating this the expressions of these derivatives can be established simply by differentiation under the integral sign now the actual rigorous establishment that this really happens will require a an exercise similar what we earlier conducted to establish the analyticity of the integral of fz.

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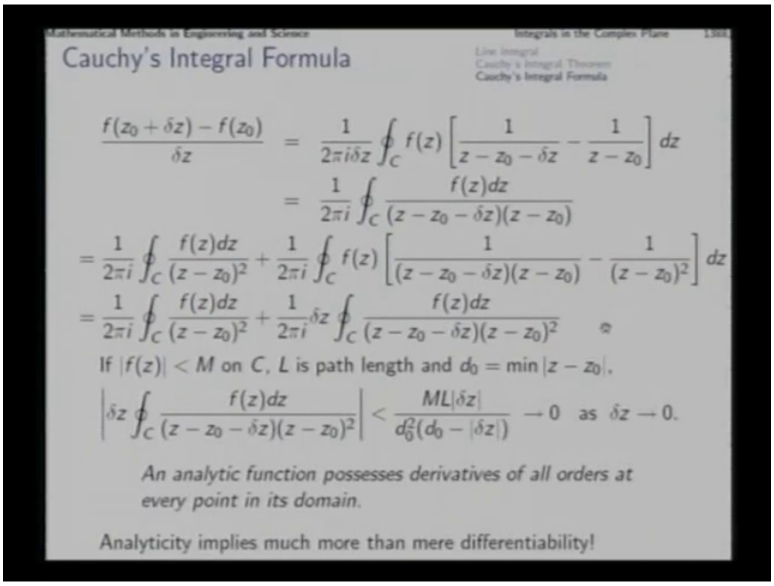

In a similar manner we could consider this to establish the this derivative to establish this derivative we need to consider an expression like this and through several simplifying steps in which we will try to separate out this part and whatever is the difference we will try to expand that and then use appropriate considerations to show that this tends to 0 as we do that finally, through the application of ml inequality we will be able to show that this part vanishes in the limit and that will show that this really tends to this and this is this is this tends to this value as delta z tends to 0 and that will mean that this turns out to be the derivative of fz 0 that is F prime z 0 and if this we can do the detailed derivation you can consult from the textbook.

Then if this part is established then by a similar exercise over this function we will be always able to evaluate this, this, this and so and this shows that and analytic function processes derivatives of all orders at every point it is domain and this is the great property of analyticity that we discussed in the previous example the analyticity is implies much more than mere differentiability it implies differentiability up to any order desired now with this established if gz is non analytic at a point z z 0, but then gz z minus z 0 to the power anything 3, 4, 5, 6, n turns out to be analytic then for that n what we do is now say for n plus 1 say for n plus 1 it is analytic then what we can do we can we use this and then the gz is expressed as that analytic function gz into z minus z 0 to the power n plus 1 which is fz and then this integrand becomes gz. So, integral of gz gz will turn out to be 2 pi i the n th derivative evaluated at z 0 divided by factor of n.

So, that way if as long as z minus z 0 multiplied a finite number of times makes the functional analytic then through the use of some suitable derivative formula in this series we can evaluate the contour integral of gz and this is the great property which will be used in our next lecture also which will be a complete which will completely stress on the singularities of complex functions.

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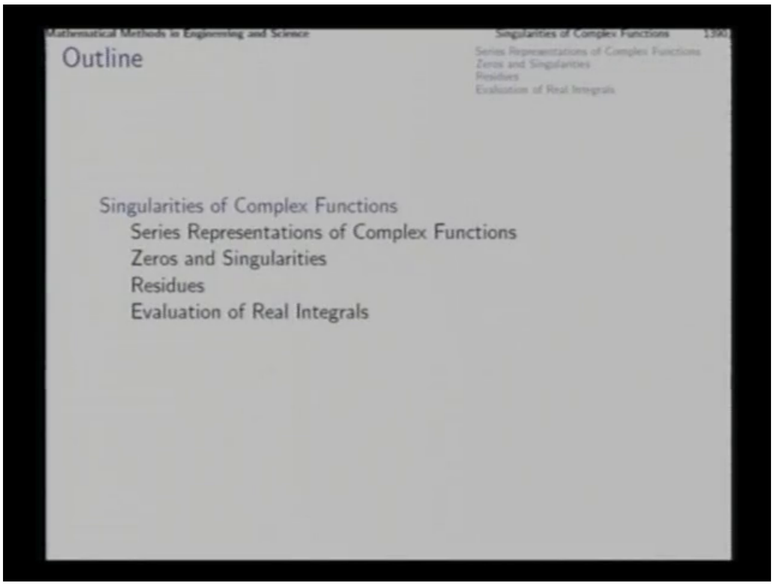

So, we use the results that we have found in this lecture into the study of singularities of complex functions in the next lecture.

Thank you.