Mathematical Methods in Engineering and Science
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Module - VIII
Overviews: PDE's, Complex Analysis and Variational Calculus
Lecture - 03
Analytic Functions

Good morning. Today we start our module on complex analysis. We start with Analytic Functions.

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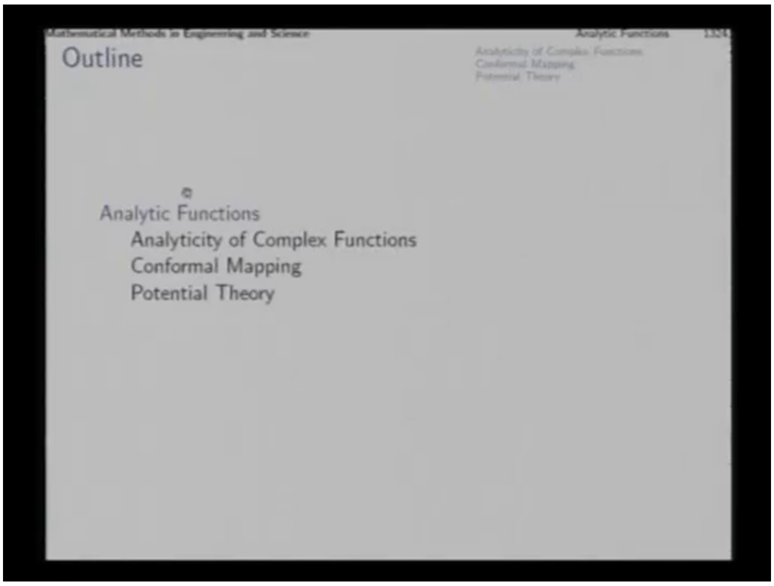

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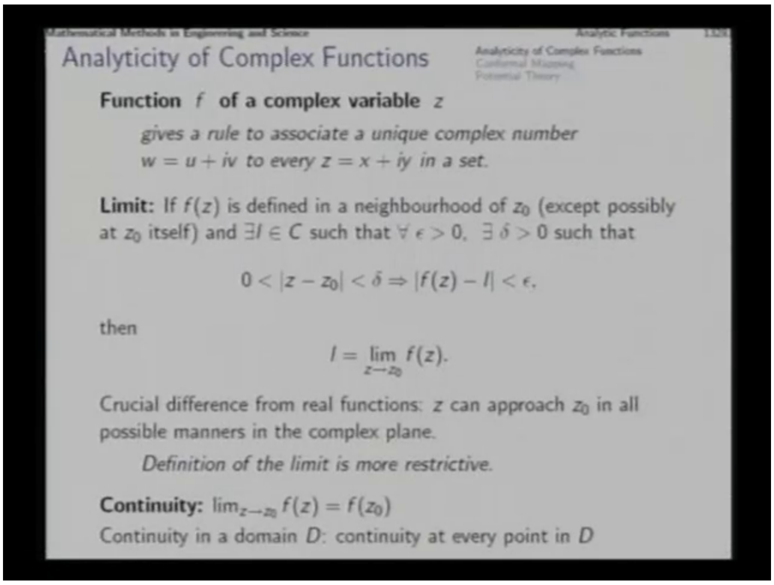

We define the function of a complex variable z as a rule which will associate a unique com unique complex number like this for every z in a domain in the z plane and as for any calculus that is for example, calculus of real functions we start with the definition of limit continuity and derivative. The first limit if the function f of z is defined in a neighbourhood of a point z 0, except possibly at z 0 itself. What it means is that for the definition of the limit at point z 0, it is not necessary for the function to be defined at that point, you can define a limit without the function being defined at that point.

So, if the function is defined in a neighbourhood everywhere in a neighbourhood of z 0 except possibly at z 0 itself, and if there is a complex number i such that for every epsilon every positive epsilon whatever small, you can find a delta such that taking the point z in a delta neighbourhood will keep the difference of the function value from the value i within a an epsilon neighbourhood, then we say that limit exists. That is the around z 0 you can find a small neighbourhood such that you can approach the function value as i with any tolerance required, whatever tolerance you want accordingly you ask for the epsilon delta neighbourhood to be closer and closer. If you can achieve this then we say that the function has a limit at z 0 and that value i is the corresponding limit.

Now, this limit can be defined even if the function f is not defined at z 0, that is at z 0 even if the function does not have a value associated with it and even if the function has a value associated with it at z 0, but the functions value is not i, but along every path if this i value is achieved is approached rather than achieved even then we say that i is the limit; that means, the function value at that point and the limit at that point may be different as in the function of real variable as well.

Now, here you will note that there is a very important difference with real functions, in the case of real functions of a single variable x the point x 0 could be approached only along 1 direction from the left or from the right of course,. So, x 0, point x 0 could be reached either from x 0 minus delta or x 0 plus delta left side and right side, but in the case of complex variable the description of the variable is not along a line, but on a plane. And therefore, there are infinite directions along which the point can be approached and this is the crucial difference which makes the definition of the limit in the case of a complex function much more restrictive. That is it is not on not enough for the limit to be same along two directions mutually opposite, but along all paths in the complex plane.

So, all paths leading to the point z 0 should give us the same limit and in that case we will say same limit i in that case we will say that limit exists and that limit is I. Now this makes it extremely restrictive definition and apart from that if the value of the limit and the value of the function also at that point is same. Then we say that the function is continuous at that point that is the function should be defined at that point and limit should exist in this sense and the function value and the limit must be same. In that case we say that at z equal to z 0 the function is continuous and continuity in a domain means continuity at every point of the domain.

Now, after defining limit and continuity at the next step, we define differentiability and the derivative.

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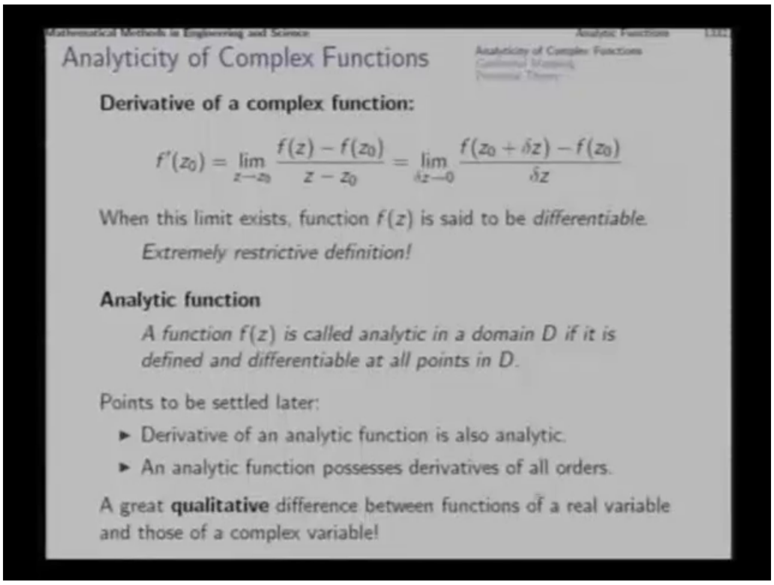

So, just like real calculus we make the first step in the definition of a com of the derivative of the complex function and that is from z 0 if we take a small difference and reach the point z, then this is the difference of the values of the function at the 2 points and this is the difference of the 2 values of z. Now this limit as z tends to z 0 will give us the derivative if this limit exists right.

So, as z tends to z 0, if we show z as z 0 plus delta z then it will look like this. Now when this limit exists; that means, along all paths in the z plane as z 0 is approached a is made closer and closer to 0. That means, as the point z approaches z 0 along all paths then if all the limits turn out to be same then we say that this limit exists and correspondingly we say that f z is differentiable and that value of that limit is called the derivative at z equal to z 0, this again is extremely restrictive definition. That means that for a complex function of a complex variable to be differentiable, it must be extremely nice that is it has to satisfy such a restrictive requirement and that within itself brings in a lot of desirable properties to the function.

So, in that case we find that the function that we are talking about just by in differentiable by being differentiable, it brings in a lot of additional properties all of them together is called is the concept of analyticity. So, we call a function analytic in a domain D, if it is defined and differentiable at all points in D. If there is a domain in which at every point a function is defined and is differentiable in this sense, then we say that the function is analytic and it can be shown that the function being analytic at that point means that it can be expanded in an infinite series of powers of z that can be shown and.

Now, there are a few points which can be said in continuation of this set of nice properties that are brought in and these are points which will be established or settled later as we study the integration also. So, one is that if we can establish that a function is analytic at a point; that means, it is differentiable at a point, then that will also imply that the derivative itself also will be analytic. That means, it will possess a derivative of its own and then by continuing on this argument we can so, show that an analytic function will possess derivatives of all order. And that means that in the case of the real valued functions differentiability did not mean too much, a function could be differentiable once, but that would not mean immediately that the derivative itself will be differentiable.

Now, in the case of complex functions the existence of the first derivative itself requires so much that once the first derivative is established, then it implies a lot of other things which will finally, mean that once the function is differentiable, then it will be differentiable as many times as you need and that is the entire implication of analyticity. So, this is the great quality difference between functions of real variable and those of a complex variable. When we were studying these solution at that stage when we are talking about the coefficient functions being analytic at a point, the sense was this analyticity that is it is not only differentiable, but it has derivatives of all order and that means, that it can be expanded in a power series around that point and that will be convergent.

Now, there are pair of conditions called Cauchy Riemann conditions.

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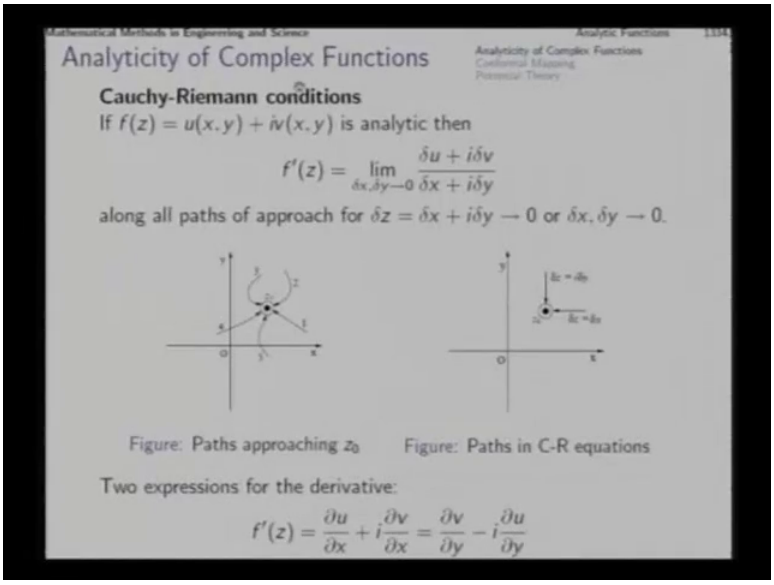

Which are satisfied by a function if it at a point if that point it is analytic and to appreciate that, we a consider this situation. Suppose a function f u plus i v is analytic then we have f prime which is limit of delta f by delta z as delta z tends to 0 now since z is x plus i y. So, delta z will be delta x plus i delta y. So, delta z tending to 0 will mean both delta x and delta y tending to 0 and this is delta f from here and this is the corresponding delta z. Now if the function is analytic then it will mean that the limit of this along all paths should be same; that means, for example, suppose this is point z 0 then 1 2 3 4 5 along all paths the point is approached the limit is same in particular let us consider these 2 paths horizontal and vertical.

In the horizontal case delta z is equal to delta x because y does not change, and in the case of the particular path delta z will mean i delta y because delta because x does not change. Now if all these limits are same then in particular these 2 limits also will be same, now as we consider delta z along this path then delta z is delta x.

So, this limit immediately will be del u by del x plus i del v by del x that is this if we consider this path then delta z will be i delta y. So, keep the i delta y here. So, this will be 1 i del u by del y that is this 1 by i which is minus i because minus i square is 1 plus i by i will go out with gets cancelled rest is del v by del y that is here. So, along this path we have this derivative expression along this path we will have this derivative expression.

Now from analyticity we know that along all these paths the limit is same in particular along these two path limit is same.

So, we get two expressions for the derivative and as we equate these two expressions and separate out real and imaginary parts, then we get del u by del x is equal to del v by del y and del v by del x is equal to minus del u del y these two conditions are called Cauchy Riemann conditions or CR conditions or CR equations.

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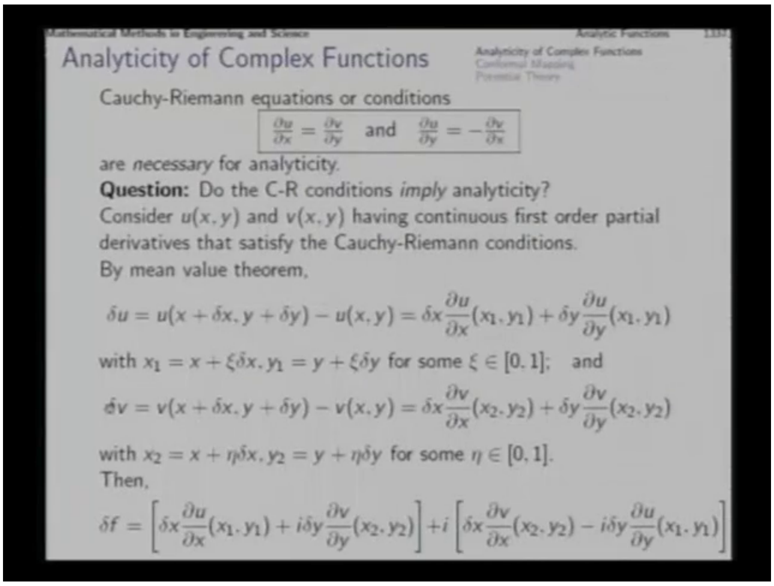

So, Cauchy Riemann conditions are simply this, and in this derivation we first assumed that the function is analytic at z 0 and then we found that these 2 should hold at z 0; that means, that these are Cauchy Riemann conditions are necessity necessary for analyticity or the function at that point. Immediately the second question that will arise that are they sufficient also, that is do the Cauchy Riemann conditions imply analyticity answer turns out to be yes and to establish that we consider 2 function u and v which such have first order continuous partial difference equations coefficients and these conditions among those partial derivatives hold, that is Cauchy Riemann condition hold. Then we want to show that the function is analytic for that we construct delta u delta p.

So, what is delta u delta u will be u at the change point minus u at the current point and up to first order we will get this as delta x into del u by del x at the point x 1 y 1 plus delta y into del u by del y at the point x 1 y 1 right. What is x 1 y 1 here? X 1 y 1 is a point in the line segment joining the original point to the changed point right this is by mean value theorem so that means, that by mean value theorem and that is why we could use this equality without any plus dot dot dot, because we have we are not including a second order com terms.

So, that is why we have to use the mean value theorem and the remainder form of the Taylor series that is remainder form of the Taylors theorem. So, we are keeping only the first order change. So, we use the mean value theorem up to the first order now which is Lagrange theorem for that matter. So, this x 1 y 1 is a point which is in the line segment joining x y to x plus delta x y plus delta y; that means, for a xi its 0 to 1, x 1 is this and y 1 is now this gives us the expression for delta u. Similarly we get the expression for delta v, that is v at the change point minus v at the original point. So, that gives us this expression for x 2 y 2 being a point joining point on the line segment joining this point to that point.

Now, x 1 y 1 and x 2 y 2 can be 2 different points, that is x y is here x plus delta x y plus delta y is here as we join these 2 we get this line segment and x 1 y 1 could be somewhere on that this line segment, x 2 y 2 could be somewhere on this line segment need not be at the same point may be at different point. So, now, this delta u and this delta v we have got in hand. So, what will be the corresponding delta f? That will be delta u plus i into delta v that is this; we take this and add to that i times this. As we do that we get we club together appropriate terms this plus i into this you will find here plus this into i into this you will find here. Now and an i has been kept outside and that is why in this case we sorry in this case in this case this plus 1 we replace with minus i square, ok.

So, that is why this term has come here i into this. So, i is outside this is sitting here and this is minus i square into this minus i square is 1. So, out of minus i square 1 i is outside the rest of it minus i. And this whole thing is here now which I have to simplify this. For simplification consider this if Cauchy Riemann conditions hold which is part of the hypotheses here if Cauchy Riemann conditions hold then del v del y can be replaced with del u del x here already del u del x is there we will find del v del x here also, ok.

So, then this first bracketed term we will find here.

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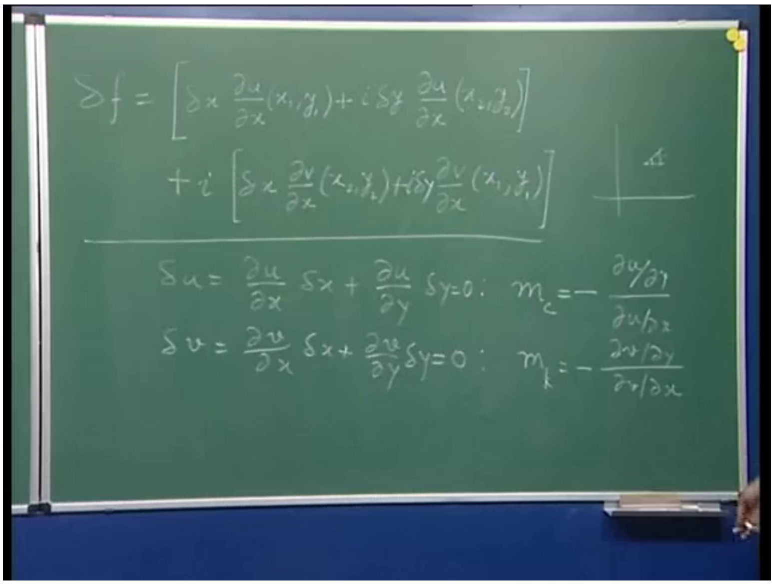

The del v del y there has been represented with del v del x. Similarly here in this case we find del u del y here which can be replaced with minus del v del x, as we replaced del u del y here with minus del v del x is minus will become plus and we will have del v del x here that is this whole thing that is this bracketed term right. Now we concentrate on this, this is del u del x these also del u del x. Now we note that this is at x 1 y 1 and this is at x 2 y 2 if this also if this were also at x 1 y 1 then this whole thing we could have taken common and what would come inside delta x plus i delta y that is delta z, ok.

So, what we can do is that for the time being here in place of x 2, y 2 we take x y, y 1 and that will mean that this term will remain and from there remain outside and from there we will subtract the same thing with x 1 y 1 here; that means, i delta y del u by del x at x 1 y 1 we add and subtract add to this and subtract from here. As we do this we get the next expression in the process of simplification. Now from here if we add i delta y del u by del x, x 1 y 1 to this term then this come common and along with that we will get delta x plus i delta y and that you get here.

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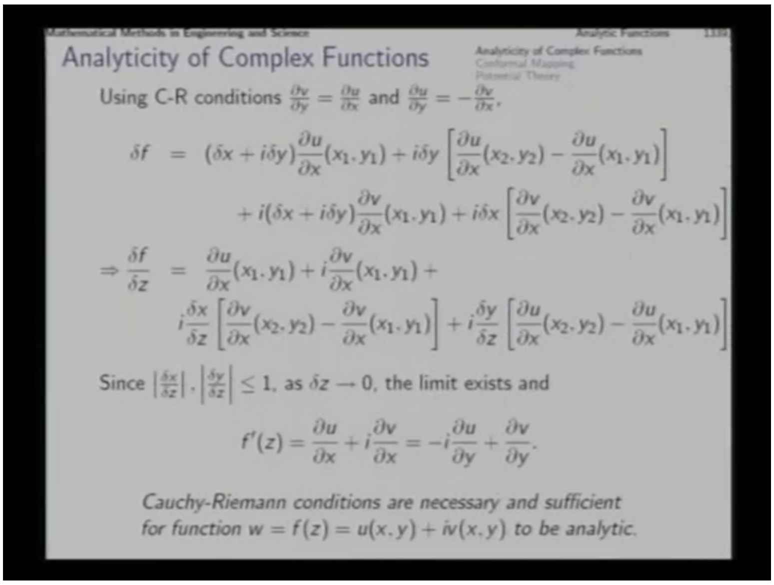

Right now then whatever we added here i delta y del u by del x at x 1 y 1 that same thing we subtract from here and for that and we will get common i delta y and inside the bracket we will get del u by del x at x 2 y 2 minus del u by del x at x 1 y 1 that is the term here a similar exercise we do on this here you see i del v by del x 1 y 1, ok.

And here there is a little mistake let me make small correction delta y delta y is missing here.

Now here we have i delta y del v by del x x 1 y 1, del i del v by i delta by del v by del x and outside there is another i. So, i, i delta y del v by del x at x 1 y 1 and along with that we would like to have del x delta x into del v by del x at x 1 y 1. So, that we add to this and we subtract from here as we add this we get the i common outside as it is already there, del v del x at x 1 y 1 we take common. And then we get in bracket delta x plus i delta y that is here and whatever we added to this that we subtract from here, ok.

So, we added delta x del v by del x at x 1 y 1. So, that same thing we subtract from here and then the result is here right. So, we find that delta x expression has come to this stage and now we want to divide with delta z note that this delta z this is delta z. So, as we divide delta z we get delta f by delta z, as this plus i into this that is these 2 things plus a lot of things from here.

So, here what we will get? I delta x by delta z into this whole thing, plus i delta y by delta z into plus into this whole thing right; now we want to take the limit of this as delta z tends to 0 and if that limit exists then we will say that the function is analytic. Now we ask this question whether that limit exists, these derivatives are all existing that we already know. Now note this that as there del delta z tends to z; that means, z tends to z plus delta z tends to z and that means, that the 2 points between which we considered the line segment in the in which we found x 1 y 1 and x 2 y 2 as 2 points and; that means, that as delta z tends to 0, this line segment shrinks and we do not get too much space to get 2 points x 1 y 1 x 2 y 2. That means, all these point shrinks to x y, x 1, x y itself that is z.

So; that means, that as z tends z as delta z tends to 0 these points have get shrunk over a length of 0; that means, these 2 points have to collapse together and that means, these will have limit 0, but what about these 2 fellows? Now since delta z is delta x plus i delta y that means.

If this is z and this (Refer Time: 23:51) number is delta z, then this length is delta z this length is delta x this length is delta y right. So, delta x by delta z is cosine of this angle and delta y by delta z in the size sense in the (Refer Time: 24:07) sense, that is the sign of this angle. So, both cosine and sin are less than 1 in magnitude ok.

So, these 2 are less than 1 in magnitude. So, as delta z tends to 0, these bracketed terms will tend to 0 and these terms are bounded that is they do not turn to infinity. If at the same time these fellows could turn to infinity then this indeterminate form of remain. So, as delta z tends to 0 these are anyway bounded, bounded by 1 the magnitude of these 2 and these turn to these approach 0. And therefore, in the limit these terms will vanish and this will be remain, ok.

So, therefore, the limit exists and that limit happens to be this at x y itself. So, then whether you write it like this or you write it like this using the Cauchy Riemann condition, it is the same and the limit exist. So, we find that Cauchy Riemann conditions are not only necessary, but also sufficient for analyticity. That means, the moment some function is known to analytic, you can immediately use Cauchy Riemann condition on the other hand the moment you can establish Cauchy Riemann condition a for a function you can immediately conclude that the function is analytic and all properties of analyticity you can assume immediately.

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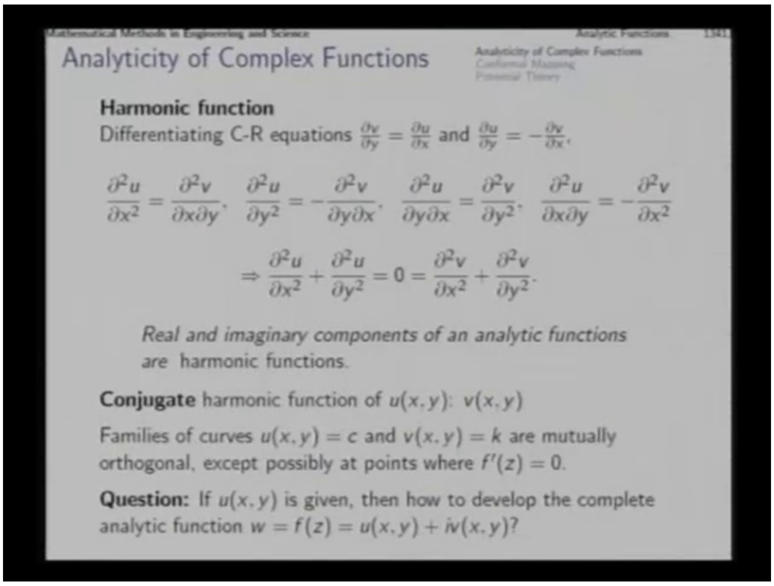

Now, if we take the Cauchy Riemann conditions and if the function possesses secondary derivative also it will possess, because if the first order this Cauchy Riemann condition is satisfied then analyticity is established and we have already discussed that analytic function is again its derivative is also differentiable. That means, derivatives of all order will exist. So, if we differentiate this, then you find del 2 you by del x square is equal to del 2 u by del x del y. Similarly del 2 u by del y square is minus del 2 v by del y del x then from here you can also find del 2 by del x del y or del y del x as equal to del 2 v by del y square and similarly from here you get this.

So, the second order derivatives will satisfy these requirements and then you can note that if we add these 2 then we get to del 2 u by del x square plus del 2 u by del y square and that is this plus this is 0. Similarly if you add these 2, if you add this and this if you add these 2 then you will find or subtract rather in this case you have to subtract from here you have to subtract this. Del 2 v by del y square plus del 2 v by del x square then as you subtract this minus this you will get 0 that means, u and v the real and imaginary components of f in that case both will satisfy the Laplace equation that is both will be harmonic functions.

So, this is a great property of analytic function that both the real and imaginary components of analytic function satisfies the Laplace equation, that is their harmonic functions and in that case the 2 harmonic functions are also called the conjugate harmonic of each other that is conjugate harmonic of u is v. Now we already know that families of curve curves a family of curve curves u equal to c. And another family of curves v equal to k are 2 mutually orthogonal families of curves in the x y plane, except possibly at point where the derivative turns out to be 0. This you can see because if you take the function u of x y equal to c and then from there you try to find out the slope of a curve from this family then.

We will consider delta u as del u by del x delta x, plus del u by del y delta y and the slope of this will be given as minus this derivative by this derivative. Similarly if you take the family if you take the take a curve from the other family, v equal to k then this will give you its slope as m k which will be this.

Now, if you multiply these 2 you will note that del u by del x will exactly cancel with del u by del y, and del v by del x will cancel with minus del u by del y leaving minus 1 in the product. In all those cases where the 4 derivatives only 2 of which are unequal, because Cauchy Riemann conditions are satisfied in the case that both of them are non zero this is obvious even say if one of them is 0, that is suppose del u by del x is 0 in that case this will be infinite; that means, the curve of the u equal to c family will be vertical that is the tangents will be vertical, but in that case if del u by del x is 0 then del v by del y is also 0; that means, the slope here is 0; that means, the curves of the other family are horizontal at that point, ok.

Curve of the other family at that point is horizontal. So, this vertical and this horizontal is again orthogonal are again orthogonal with respect to each other. Similarly if del u by del y is 0 then the curve of the first family is horizontal and correspondingly del v by del x 0 is 0 in that case the curve of the other family through that point is vertical again orthogonal. The only problem will arise if both the derivatives are 0 that is del u by del x as well as you del u by del y is 0.

In that case this slope is undefined and this slope is also undefined. So, it may happen that you may not be able to figure out that the product is minus 1 or 1. And therefore, we say that kind of a situation can arise only at a point where del u by del x and del v by del x both are 0. In that case del f by del z is actually 0 and that is what we say here that families of curve u equal to c and v equal to k are mutually orthogonal at all points except possibly at those points where this derivative turns out to be 0.

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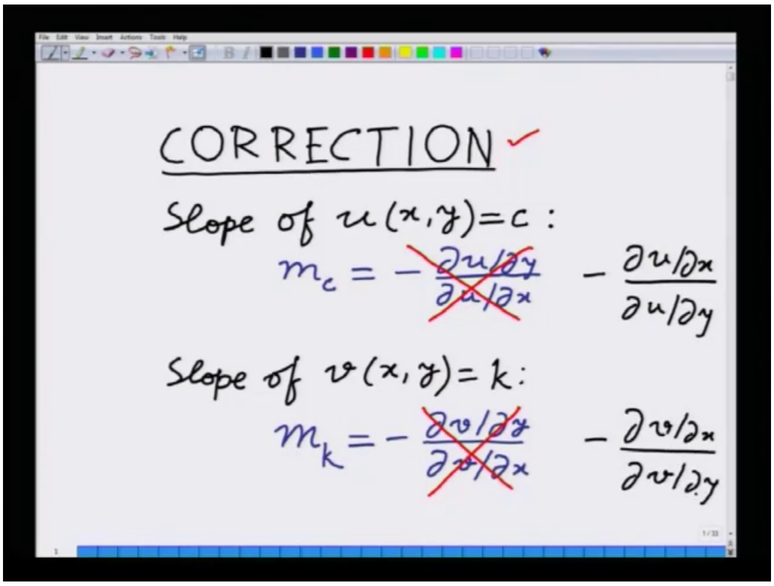

Before proceeding further please note this correction. The slope of the curve u of x y equal to c was written on the board as m c is equal to minus del u by del y by del u by del x this is not right, it should be minus del u by del x by del u by del y. Similarly the slope of the curve v of x y equal to k was written in the board as m k is equal to minus del v by del y by del v by del x, this will be corrected to minus del v by del x by del v by del y. Thank you now you can continue further in the rest of the lecture.

Now, a good question a very important question is that if u of x y is given, then how to develop the complete analytic function. This is actually an exercise the basic work regarding which we did much earlier when we were solving the first order differential equation. When we were studying first order differential equation in that context we have actually studied this particular problem, and what we do for that is that we construct del u by del x and del u by del y from the given u and using Cauchy Riemann conditions we get del v by del y and del v by del x and using del v by del x del v del y we construct v of x y.

So, that way after constructing v of x y we get the complete analytic function; that means, if one of the components real or imaginary of the complex analytic function is given, and then the other one can be derived using Cauchy Riemann conditions and integration. Now another important concept in the case of analytic functions is conformal mapping.

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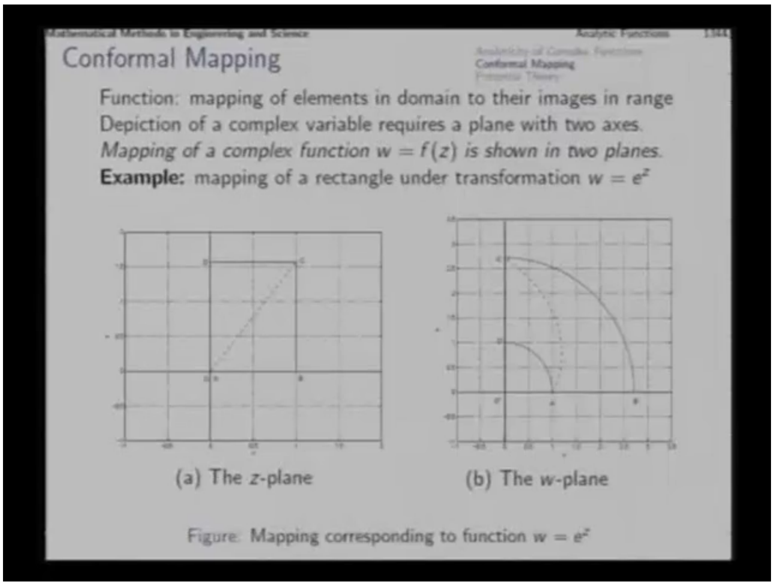

Conformal the word means of similar shape conformal; conformal mapping means shape similar mapping. So, a conformal mapping is defined by a analytic functions except that those points where the derivative is 0, f prime z is 0. So, a function a now a function will give you the mapping of elements in domain to their images in range, the domain is the z plane and the corresponding co domain is the w plane.

So, from points in the z plane as you map the points to the w plane, you get the mapping. Now here depiction of the comp in the case of real variables you plotted the independent variable x in the horizontal axis and dependent variable that is the function y along the vertical axis, you cannot do this here because the depictions of a variable itself over its domain will require a full plane. So here, how you show the mapping? You take 2 planes z plane and w plane. So, depictions of a complex variable will require a plane. So, depictions of mapping will require 2 planes together, ok.

So, in this manner, this is a z plane in which we take the domain and this is the w plane and between z plane and w plane we consider this function w equal to e to the power z. Now every point here will give you a corresponding point here, let us consider 4 points here a b c d a rectangle. So, the point a from here which is origin. So, that will you give you e to the power 0. So, there you will get one. So, 1 plus i 0 the point b that will give you that is here one. That means, 1 plus i 0. So, e to the power 1 will give you e this is 2 point 7 1 8 and so on that is p prime c is 1 plus i into pi by 2 say 1 point 5 7 pi by 2 c is 1 plus i pi by 2.

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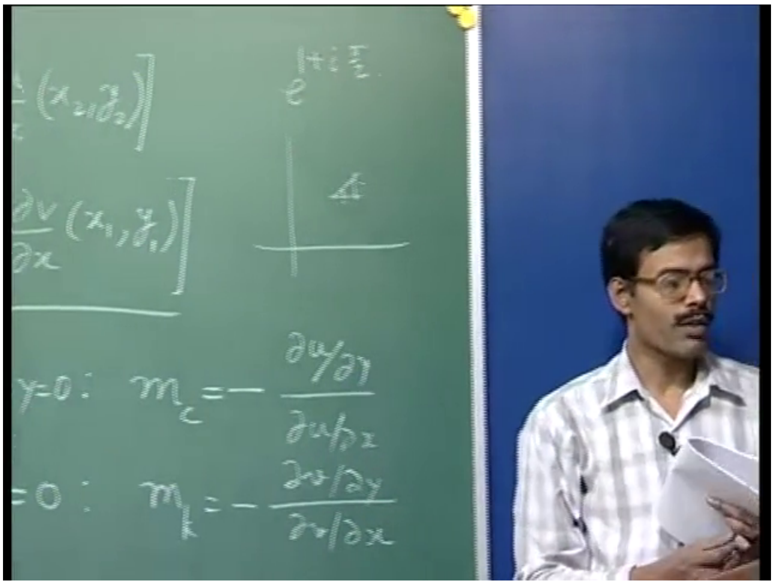

So, as you write e to the power 1 plus i pi by 2, e to the power 1 is i sorry e to the power 1 is e into e to the power i theta is cos theta plus i sin theta. So, cos pi 2 is 0 and i sin pi 2 is 1.

So, you get e into i. So, that is why you get e magnitude i that is in the vertical direction c prime comes here similarly d prime is simply i pi by 2. So, that will be e to the power 0 into i pi by 2 sin i sin pi by 2 that will bring you here. If you try to draw the diagonal you will find that diagonal a c will come like this, now this line segment a b comes like this line segment b c will come like this, c d will come like this and d a will come like this. The shape of this rectangle has changed, but you will note one important issue a b and b c were orthogonal mutually perpendicular at b, here also a prime b prime and b prime c prime the curves are perpendicular to each other here similarly b c c d are perpendicular here also b prime c prime and c prime d prime meeting at c are perpendicular.

So, all the edges have gone to the w plane in such a manner, that these between the tangents you are all getting you are getting all the right angles. This diagonal a c has been mapped to this curve a prime c prime, but note emerging from a whatever angles u are getting here a b a c a d, similar same angles you get here a prime b prime a prime c prime a prime d prime, that is along the tangents you will get the same sector here and that will happen everywhere that is because this happens to be a conformal mapping.

That is it is same shape mapping similar shape mapping, that and that similarity of shape is in terms of the local shape only make that point very clear. We can very easily establish this fact the demonstration of which we just saw through these figures.

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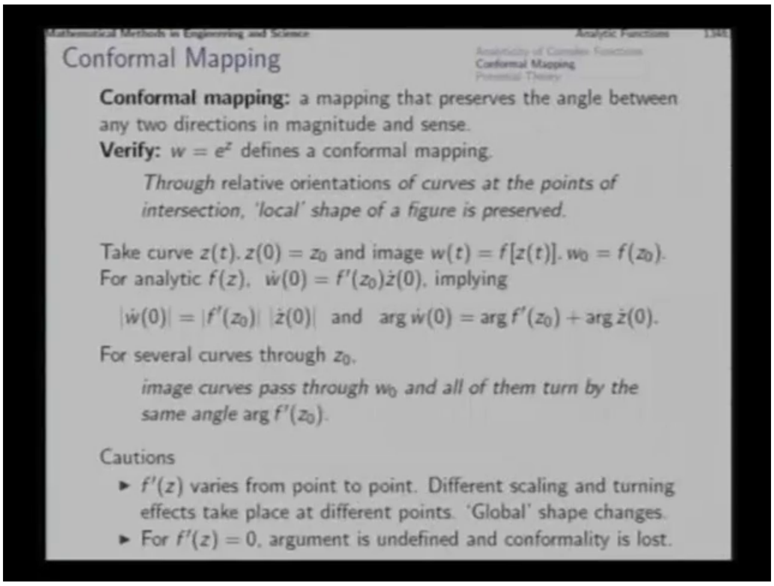

The conformal mapping is a mapping that preserves the angle between any 2 directions in magnitude as well as sense. Just now we verified this fact for this particular mapping w equal to e to the power z. So, this analytic function defines a conformal mapping. So, we find that through relative orientations of curves at a point at points of intersection, the local shape of the figure is preserved. At every point whatever rays we draw here and the corresponding rays we map to the target plane, the co domain the range we find that the relative angles here and relative angles there are preserved.

So, why should this happen? We take the curve we take a curve apart from rays we were taking rays earlier, now we take a curve z of t in the z plane passing through this point z 0 at t equal to 0. Corresponding image is w of t, which is f of z of t, because w is f of z and passing through w 0 which is f of z 0 at t equal to 0. Now if the function f is analytic then we can have its derivative, then w dot from here through chain rule will be f prime z evaluated at that point into z dot that is this right.

So, w dot evaluated at t equal to 0 will be f prime at 0 into z dot evaluated at the corresponding t equal to 0, and this will imply that this side and this side these 2 are equal in magnitude as well as direction. The magnitude equality is here and direction equality will be here that is argument of this is equal to argument plus argument of this right. Now as we draw several points through the same point z 0, then their directions will be different here 5 curves through z 0, will have 5 different angles here, but for all of them this is same because this does not depend on the those curves is a property of the function itself f itself. And therefore whatever are the differences of angles among the curves here as we map them as we map those curves to the w plane, the differences here will be that is every curve from this plane to that plane turns through this angle and this angle is same for all the curves because all of them are passing through z 0.

So, for several curves through z 0 image curve pass through w 0 and all of them turn by the same angle and turning is this. So, through z 0 if in the z plane we draw 4 curves call them 1 2 3 4. So, if curve 1 turns through the mapping through an angle 30 degree; that means, this argument is 30 degree. So, curves 2 3 4 also have to turn to the same 30 degree which is this and this depend only on the function and not the curve that we are drawing through z 0.

And this shows that the local shape get preserved if one of them turns by 30 degree then all of them turn by 30 degree through the mapping and the magnitude changes like this. So, magnitude changes direction also changes, but all the directions these curves from here say these are 4 curves drawn from a particular point in z plane. Now as they as these rays go to the w plane their lengths may all change by this factor and they all may turn by this angle. So, as all of them turn they look like now this; that means, all of them turn together. So, their shape does not change, but the important points to note in this regard is that this will happen only at those points where this magnitude is non zero, because if this magnitude is 0 then this will collapse. So, this analyticity is must apart from that for formality of the mapping the rally value of the derivative should be non zero.

Now, if f prime is 0 at that point then the argument is undefined and conformality will be lost or may be lost, now one point to another point to notice that the derivatives varies from point to point. And therefore, we say that the shape does not change locally. So, local shape is preserved. So, as around this point all of them turn by 30 degree around another point where f prime may be something else the rays may be turning by 35 degrees another point rays may be turning by 45 degrees and so on.

And therefore, the scaling and turning effects at different points are not the same, at different points of the z plane are not the same. And therefore, the local shape at every point is preserved through the conformal mapping, the global shape is not preserved the global shape may change in general that does change and that is what we saw here even though locally the collection of every rays through a collection of all rays through every point preserves their mutual angles, but the overall shape of the region defined by a b c d is not preserved, here it was a rectangle here it turns out to be a part of and a sector of an annulus. So, global shape may change because f prime z 0 at different points z 0 will be different in general.

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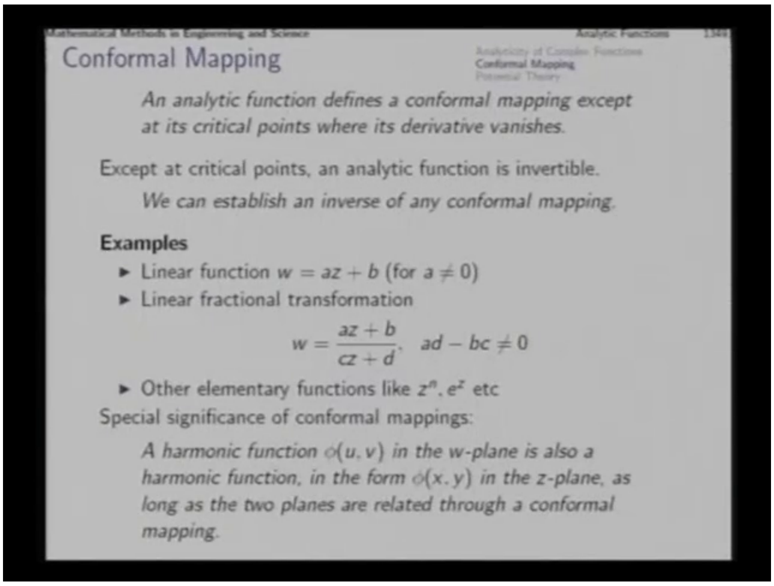

So, from the foregoing discussion we can conclude that an analytical function defines a conformal mapping at all points except at its critical point where its derivative is 0. Now except at critical points we find that analytic function is invertible also so; that means, that for any conformal mapping we can establish an inverse and this fact is of enormous practical importance. So, we are coming to that practical point later first let us see a few examples a few quick examples of conformal mappings. Linear functions like this will define conformal mappings for all non zero a, linear fractional tranfa transformation like this will define conformal mappings except for the case when a d minus b c is 0 why so, because if you try to differentiate this you will find that in the case a d minus b c you will have 0 derivative.

Now, other elementary functions like z to the power n e to the power z etcetera though they have completely different meanings in the case of complex functions as we put e to the power x plus i y, we find that turns out to be e to the power x into cos y plus i sin y. So, that turns out to be complex function in which the real part is e to the power x cos y and imaginary part is e to the power x sin y. So, it is quite different from the real function e to the power x which is all through exponential. So, even then these elementary functions with similar expressions similar meanings that we defined in the case of real calculus. Now as we put those same formulas here we get quite I mean similar formulas will yield different meanings here, yet all of these will define conformal mappings except for those situations where the derivatives vanishes.

Now, these analytic functions and you can show that in whichever case the expression of f of z you can put in terms of z only, after collapsing after collapsing all the x y terms such that x and y do not appear alone separately such functions you can always show that they define they satisfy Cauchy Riemann conditions as long as the derivative expression does not become undefined.

So, these will establish conformal mappings and special significance practical significance of conformal mappings is that a harmonic function phi of u v in w plane that is in the w plane a function which satisfies a Laplace equation is also a harmonic function in the form phi of x y in the z plane. As long as the 2 plane z plane and the w plane are related through a functional relationship which itself defines a conformal mappings, ok.

So, this fact gives us an advantage in solving a lot of inters important problems, underlying the solution in such cases is the famous Riemann mapping theorem.

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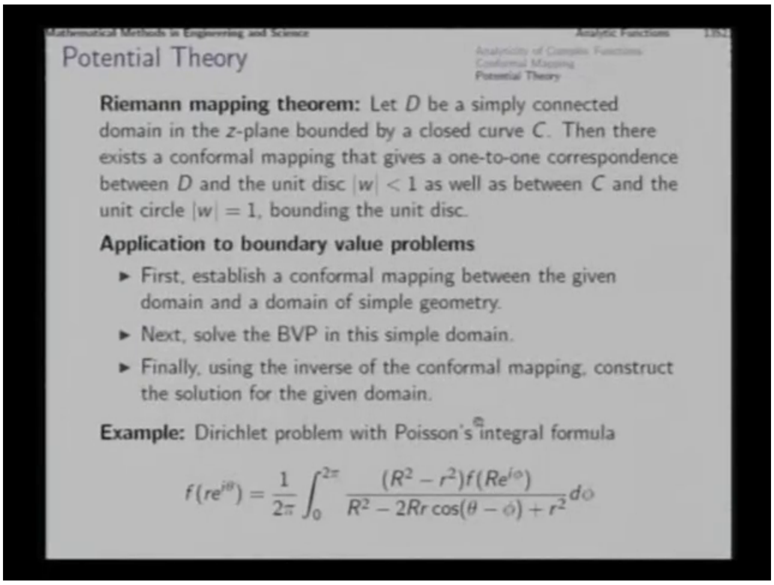

And that tells us that if D is a simply connected domain is at plane in z plane, in z plane you take a domain simply connected domain which is bounded by a closed curve. Now whatever closed curve it is? Whatever is its shape as long as the region that it encloses is a simply connected domain then there will exist a conformal mapping that will give you a 1 to 1 correspondence between this curve e and a unit circle. That means, also in between the interior of this curve interior of this region with the unit disc that is interior of the unit circle ok.

So, such a conformal mapping will give us this 1 to 1 correspondence there will be a conformal mapping, which will give us 1 to 1 correspondence between this domain d and the unit disc which is this as well as between the boundaries now this important fact gives us a very handy tool to solve boundary value problems. For example, suppose we have got a boundary value problem in which the domain is of a very complicated shape, but as long as it is simply connected what you can do we first establish a conformal mapping between the given domain and a domain of simple geometry for example, the unit disc.

Next solve the problem in this simple domain and then in the case of the conformality the mapping will also have an inverse. So, after the solution is available in the simple domain we use the inverse of the conformal mapping and thereby we construct the solution for the original domain. Now one particular advantage one particular application of this is through the poissons integral formula which is this now let us first see what this integral formula tells us, r e to the power i theta is a point z in the z plane expressed in the polar coordinates, you see x plus i y in polar coordinate will mean r cos theta plus i r sin theta right

So, if u take r common then you get cos theta plus i sin theta which is this e to the power i theta. So, this r e to the power i theta is nothing, but x plus i y in polar coordinates, and the formula tells you that this value of the function at z can be found through this integral 1 by 2 pi into integral from 0 to pi over the full circle this integral if we evaluate then we get the value of f of z. Now what is this integrant and what does this involve it involves capital R that is radius value small r the radial coordinate of z and it involves theta that is the polar this theta coordinate of z and it involves phi now phi.

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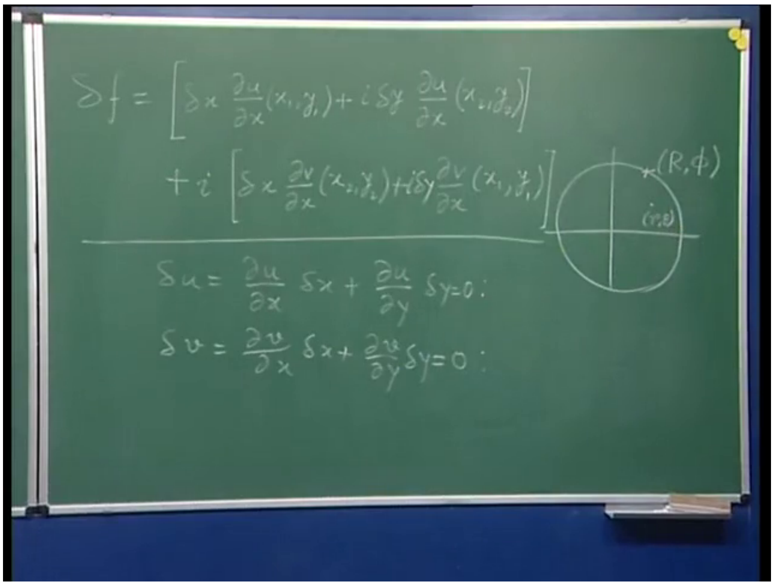

This circle at a point at the circle is r phi and a point in interior is r theta

So, this Poisson's integral formula tells us that a the function can be evaluated at an interior point here through the cyclic integral 0 to 2 pi of this integrant over this circle over the circle and for that the function value is required only at the circle right. So that means, if we know the boundary values all the boundary values then by using those boundary values here for different phi running from 0 to 2 pi for constant R we can evaluate this integrant at every point for any interior point r.

So, interior point r where we want the function value gives us the value of R and theta small r and small theta small r and theta the small r and theta and point here has radial value R capital R and the value of phi changes to 0 to pi. That means, if we know all the boundary values then we need these function values. So, by using the boundary values through this integral we can find the value of the function at any point in the interior practically. That means, that we can solve the Derichlet problem for the function f that is boundary point value we know and in the interior we want to find out the function. So, this formula itself we will be able to establish after we study a little interior calculus of complex functions also.

Now apart from that what else is the application of conformal mapping, we have already seen that the relationship between one family of curves u of x y equal to c. And another family v of x y equal to k is established through Cauchy Riemann conditions.

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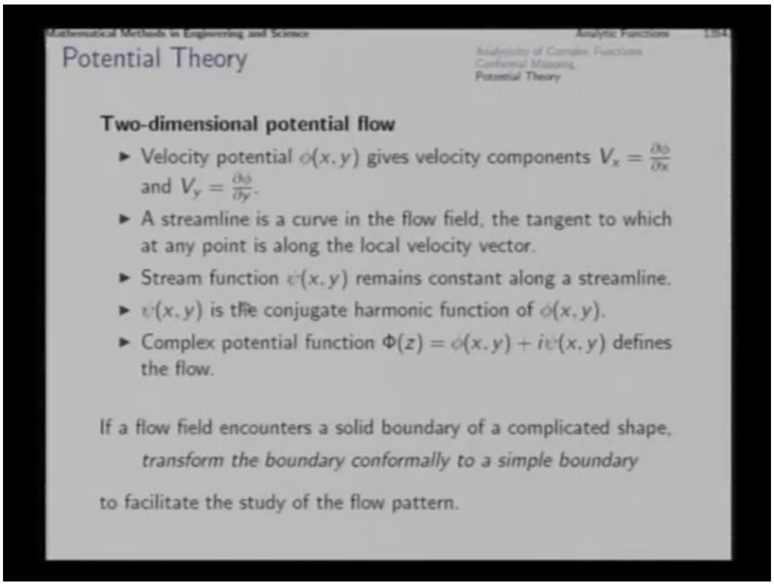

So, in the case of the analysis of two dimensional potential flows, if we have the velocity potential phi of x y that gives us the velocity components in this manner and we know that a streamline is a curve in the flow field, the tangent to which at any point is along the local velocity vector. So, stream function is a function remains constant along a streamline. So, psi of x y that is the stream function turns out to be a conjugate harmonic function of the velocity potential function and the complex potential function consisting of phi and psi together defines the flow completely.

So, in the fluid flow problems if we encounter a solid boundary of a complicated shape, conformal mapping allows us to transform the boundary conformally to a simple boundary a boundary of a simple shape and this transformation helps us facilitates us facilitates the study of slow pat pattern through analysis of the simple boundary. This is what we do in the case of complicated stream line shapes and also in the case of the airflow studies.

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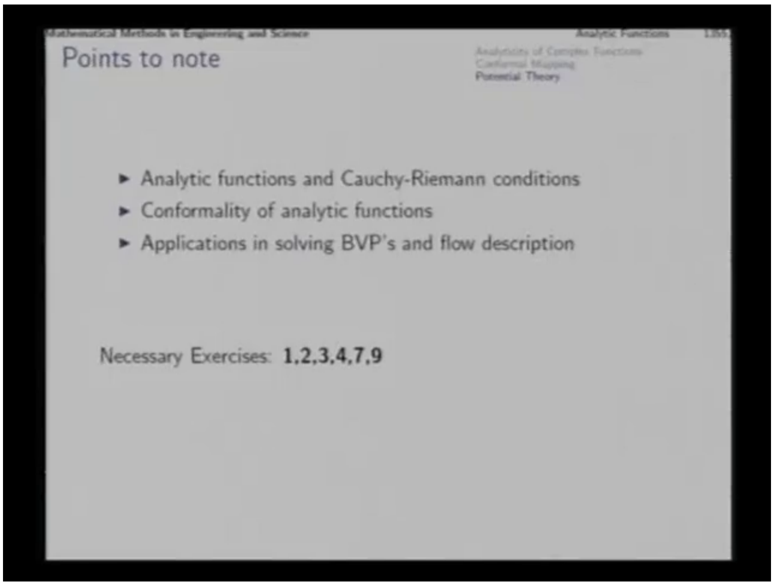

So, these are the points which we studied in this particular lesson- Cauchy Riemann conditions, conformality, and applications of the complex analytic functions in the case of boundary value problems and flow descriptions. In the next lecture we will take up the question of integrals in the complex plane; integral of complex functions.

Thank you.