Mathematical Methods in Engineering and Science
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Module – VIII
Overviews: PDE's, Complex Analysis and Variational Calculus
Lecture - 02
Parabolic and Elliptic Equations, Membrane Equation

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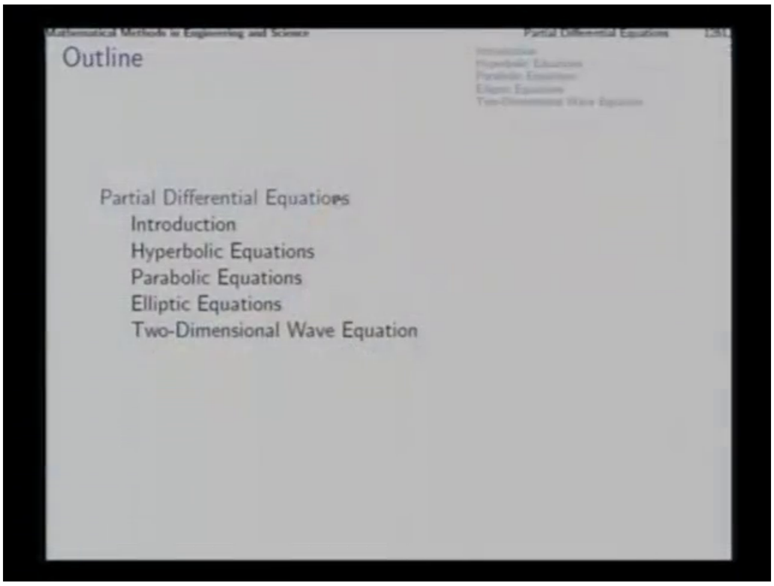

Welcome. In the previous lecture, we started our discussion on partial differential equations. And we considered the problem of hyperbolic equations. In this lecture, we will consider the two other types that is parabolic and elliptic equations.

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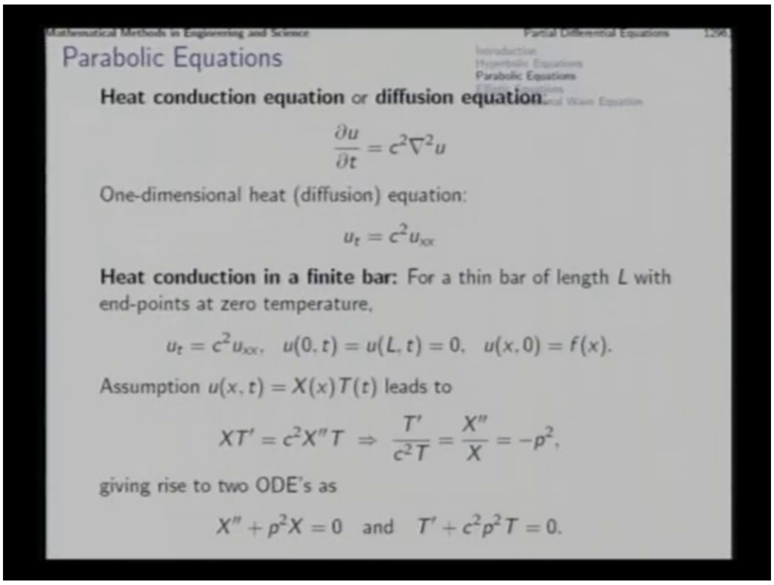

So, we take this parabolic equation del u by del t is equal to c square into del 2 u that is del square u that is the Laplacian. Now, this Laplacian over u will give you a large term del 2 u by del x square plus del 2 u by del y square plus del 2 u by del x del z square. Now, in the particular case of one dimension one-dimensional domain only del 2 u by del x square will remain and that gives us the one-dimensional heat equation which is this right. Now, we do not go into the derivation of this differential equation governing the phenomenon of one-dimensional heat equation or one-dimensional diffusion because in the chapter on vector calculus in chapter 18, you might have already encountered you might have already got familiar with the derivation of this equation through one of the exercises.

So, we take this heat equation now and try to see how we solve this for given initial and boundary conditions. Note here that the time derivative in this equation is involved only up to the first derivative that is it is a first order differential equation in time; although in the space variable, it is second order differential equation. So, for x, it will need two boundary conditions at x equal to 0 and x equal l; on the other hand, for time there will be only one initial condition that will be required and that is the initial value of the function that is here at all x. This is the only initial condition that is needed value of the function because second derivative is not involved in the differential equation only first derivative is involved. So, this is a complete description of the initial value problem in which these are the boundary conditions and this is the initial condition.

Now, in this case also to try to solve this differential equation with the method of separation of variables we will assume a solution of this type in which the two variables are separated. The function u is expressed in the form of a function of x only multiplied with a function of t only. And then we construct the first order derivative with respect to t that will be x T prime and that will put here. And then the second order derivative with respect to X that will be X double prime into T that we will put here and then we get this differential equation. And here as we divide this entire equation both sides with c square X T then here X will get cancelled and c square T will come in the denominator here c square T will get cancelled and X will come in the denominator.

Now, you see that the left side is a function of t only and the right side is a function of x only. So, sep variables are separated. So, we equate both of these that is each of these to a constant. Again the boundary conditions over X will require this constant to be negative because we want the solution to be zero at x equal to zero and return back to zero at x equal to l. So, we will need this negative constant, so that the result comes in terms of sinusoid rather than exponential. So, then when we separate the so with this negative constant here we separate the variables and get these two differential equations. This is a second order ODE, and this is a first order o d e links together through the common value of p. We first consider this differential equation the solution of which we constructed in the last problem also.

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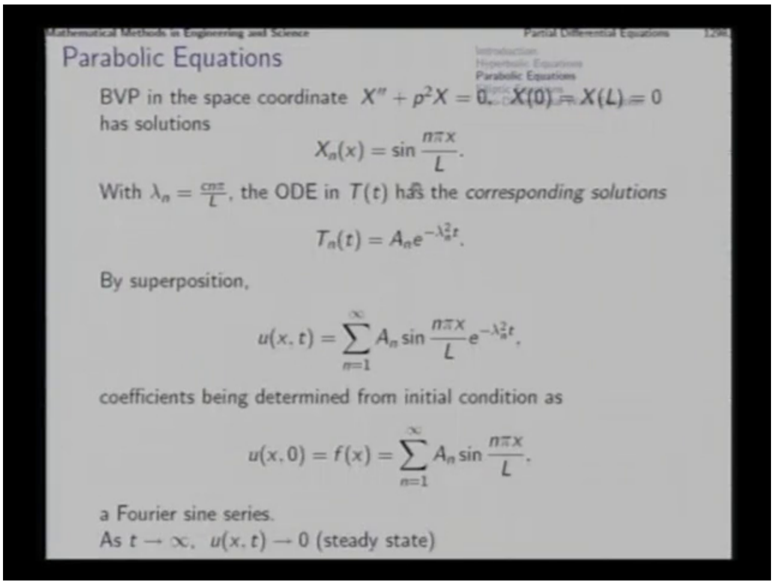

And that is this for all values of n, n equal to 1, 2, 3 and so on. And the corresponding value of p that is n pi by L we take and then the other equation gives us here c p, so c p whole square, so lambda n is c p. So, when we put that we get the solution for T which is capital T which is even simpler because that is a first order differential equation, so we get this exponential solution. And then every product of this and these for every value of n will be a solution and their linear combination is also a solution that will be the complete solution found through superposition. And this solution will satisfy the boundary conditions because each and every component of it satisfies the boundary condition. And this will certainly satisfy the differential equation also. Now, only required thing is to find the coefficient of coefficient A n, A 1, A 2, A 3, A 4 and that we find by applying the initial condition.

So, as we applying the initial condition, we find that we get this which happens to be the Fourier sine series of the function f which is the initial distribution of u over x. And as you see that in time this is an exponentially decaying term. And therefore, as in enough time elapses as t tends to infinity, we will find that this tends to 0, and that means, the steady state solution of the problem is that is after a lot of time elapses, the steady state will be reached in which the entire solution will be zero. And that makes direct sense from the physics of the problem in which both the endpoints are maintain at 0 temperature, and there is no heat source in between. So, that means, that whatever nonzero temperature values were there earlier initially over the rod or bar, so that higher temperature will reduce and that will happen through heat transfer through the boundary. Similarly, if the inter interior temperature is lower than from the outside of the system, it will be taken in and the entire rod will reach zero temperature after enough time has passed. So, this is the solution of the problem with t tends to infinity, this entire thing will become zero all over x.

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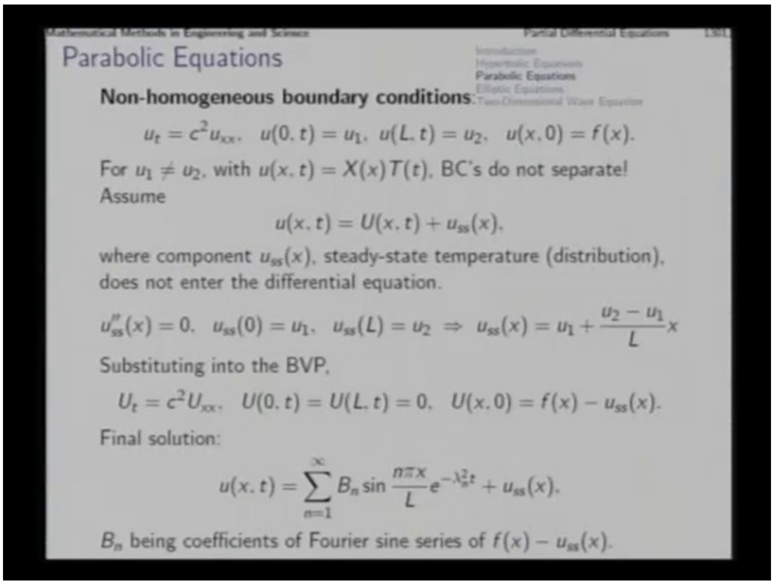

Now, in this particular case, we had homogeneous boundary conditions. Now, if the boundary condition is non-homogenous in that case we will find that say suppose at x equal to 0, this is the condition at x equal to L the condition is u 2. Even if u 1, u 2 are not zero, but equal the previous solution will still be applicable because the entire temperature scale will basically rise and rest of the thing will be same. On the other hand, if u 1 and u 2 are not same if they are unequal that is that two end points are maintained at different temperature values, in that case we will find that the boundary conditions do not separate when we try to apply the separation of variables. And that kind of a situation stops the problem from being a separable problem directly.

And in that case, we try to apply a little adjustment. And in the adjustment, we say that we assume the solution as a sum of two terms capital U plus u s s this subscript s s refers to steady state in which we say that this component capital U satisfies the homogeneous boundary conditions that is zero boundary condition at both ends, and it satisfies the differential equation also. On the other hand, this satisfies the boundary conditions. So, then this component as we differentiate it and put inside this, so del u by del t will be equal to del capital U by del t because this does not depend on t, so that we put here. And the x derivative second x derivative of this will be del 2 u by del x square that is del 2 capital U del x square plus that second derivative of this. And then we say that we will be able to separate the variable if this does not enter into the differential equation at all that is if this fellows second derivative turns out to be 0.

So, the second derivative of this is 0 and it satisfies the boundary conditions, this will give us a boundary value problem of the ordinary differential equation. And this is the ordinary differential equation that is second order ordinary differential equation in which the second order derivative is 0 that means, the solution is linear function of x. And linear function of x to satisfy, these two boundary conditions must be this. So, this is the u s s x component that is the steady state solution and the rest which is the transient solution when we put back we find that that will satisfy the earlier boundary value problem initial boundary value problem with the homogeneous boundary conditions that is this which we have studied.

So, here in the place of f x which is the initial value of small u will have the f x minus u s s which is the initial value of capital U. And this change in a initial condition with the other homogeneous boundary conditions and this differential equation will give us the same solution which we found earlier and that solution we compose with the steady state solution and get this complete solution. Now, the steady state solution will be this which will remain after enough time has passed and this will be the transient component. And in this case, the coefficients B n will be found from the Fourier sine series of not f x, but f x minus u s s.

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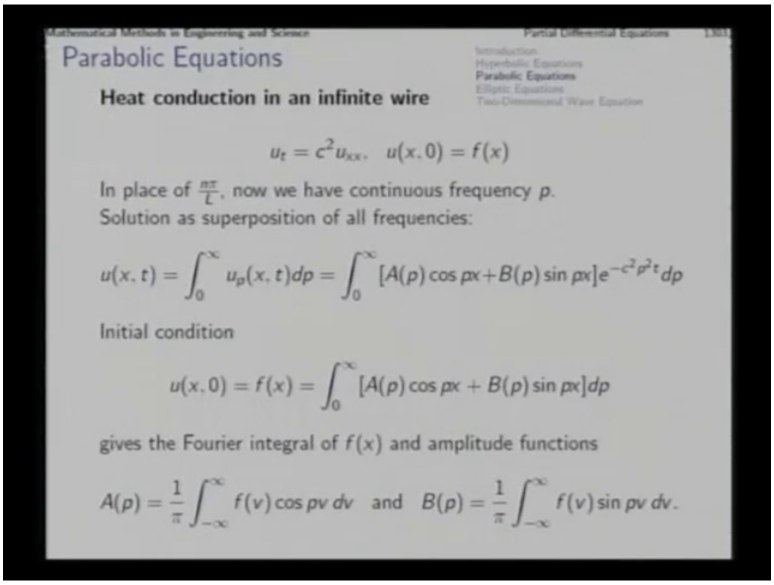

Another particular case we can consider and that is the problem of the same heat conduction problem, but not over a finite rod not over a finite domain but over an infinite wire. For example, this is suppose the infinite wire, and in this case we will not need any boundary condition because the infinite wire will have no boundary, we are trying to find out solutions which are bounded that is all, so over the infinite domain. So, this is the differential equation only with an initial conditions, there is no boundary, so there is no boundary condition.

So, in place of n pi by L, now we will have continuous frequency variable p and rather than finding the solutions in the form of a sum of discrete components, we will find the components now are continuous. So, therefore, the summation will be replaced by an integral. So, earlier we had sigma u n, now we will have integral of u p over d p that is the d p is the differential of the frequency variable p. And the solution u p in a similar manner as we found in the case of Fourier series, here we will get a corresponding Fourier integral that is the term here which earlier we equated with the Fourier series, now will be a Fourier integral in which the Fourier integral coefficients will be found in the usual manner.

Now, if we combine these two and put here, you can also put this entire term in the form of the cosine of p x minus p v that kind of a term you can put here, if you insert these A p and B p expressions here. So, the whatever role in the earlier finite case was being played by a Fourier series, now will be played by Fourier integral and everything else remains same with appropriate changes. So, the frequencies will be now continuous and this A p and B p the coefficients will be determined through these integrals. Alternatively in the case of infinite wire rather than using Fourier integral one can also attempt the solution with the help of a Fourier transform.

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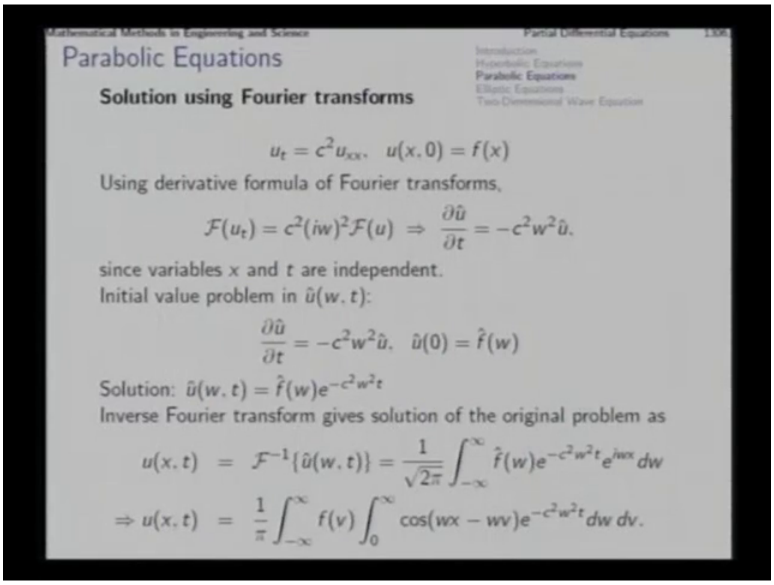

And the way to do that is this. We first use the derivative formula of Fourier transform and with respect to one of the variables in this case x we apply Fourier transform. So, here, we on the left side, we have to apply the Fourier transform of u t. Now, derivative is with respect to the variable t and Fourier transform we are taking with respect to variable x. Since, these two variables are independent then we can change the order of differentiation and integration note that the Fourier transform basically involves an integration. So, the Fourier transform of derivative with respect to t is this which we the same as the time derivative of the Fourier transform. On this side, the derivative is with respect to x and a Fourier transform is also taken with respect to x. So, we use the derivative formula of Fourier transform that is the Fourier transform of the derivative is i w into Fourier transform of the original function So, second derivative. So, twice derivative formula has to be applied, so i w whole square, and then this is u hat there is the Fourier transform of u.

Then this differential equation that we have got is a first order differential equation with respect to time in the function u hat that is Fourier transform of u with respect to x that is we have converted this differential equation partial differential equation to an ordinary differential equation of the Fourier transform of u with respect to x. Now, the initial value problem we can construct by taking the Fourier transform of the initial value function also that is both sides we take Fourier transform and that gives us the initial value of the Fourier transform. So, as we take the Fourier transform here, so we get u hat at time equal to 0 as f hat of w, so that means, for this differential equation which is a first order differential equation we have got this initial condition.

So, this differential equation with this initial condition first order differential equation. So, the solution is very easy. So, we get this solution. This is the solution of this first order differential equation with this initial value. Therefore, it is appearing here. And then we say that after we have got the solution, now we take the inverse Fourier transform of this. An inverse Fourier transform will give us this one from the directly from the formula that is f x into e to the power minus c square omega w square t in to e to the power i w x integrated over w from minus infinity to plus infinity. Which if we put this here then we can put it in this form and we have actually expressed the solution in the form of quadrature and this same solution we would get from the Fourier integral representation also. Now, how this integral how this quadrature is to be evaluated that we will consider a few lectures later after we have studied complex analysis also.

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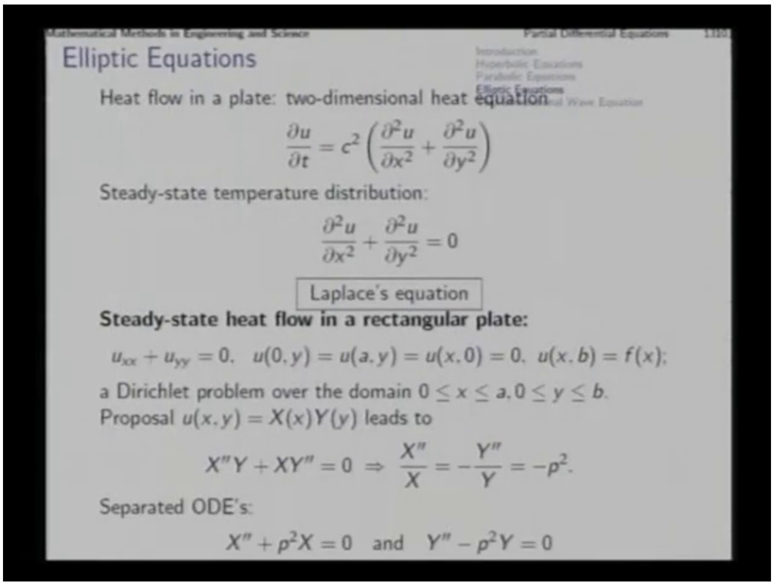

Currently we proceed to the discussion of the third type of differential equations that we are going to and do here and that is the elliptic equations. The same heat flow equation that we have been discussing that is del u by del t is equal to c square in to Laplacian of u. So, in that the Laplacian involves del 2 u by del x square plus del 2 u by del y square plus del 2 u by del z square, till now we have been discussing one-dimensional problem. Now, let us take a two-dimensional heat flow problem. So, if you consider the heat flow in a plate. So, the entire boundary of the plate will require boundary conditions. So, this will give us the two-dimensional heat equation in this manner.

And then you say that for this particular problem we consider that enough time has passed and we want to find out the final steady state temperature distribution over the entire plate if we prescribe conditions appropriate conditions over the boundary of the plate. So, then if enough time has been elapsed and the temperature distribution has reached a steady state that means, del u by del t will be 0 and then we get this equal to 0, and this is the Laplace equation. Laplace equation is an elliptic equation.

Note that in this particular case the original equation this is actually a parabolic equation in time and space variables. In time, you have got this first order derivatives in space variables you have got this term. So, between time and space, you had the parabolic nature of the differential equation, but after the steady state has been accomplished this time derivative gets removed from the differential equation and then what remains among the space variables it is elliptic. Now, when we try to solve this Laplace equation, this elliptic equation then if we apply the separation of variables method over the boundary value problem of this for a rectangular plate to keep things simple. Now, in this rectangular plate the three sides of the rectangle that is x equal to zero that is left side x equal to a that is right side and y equal to zero that is bottom and y equal to b.

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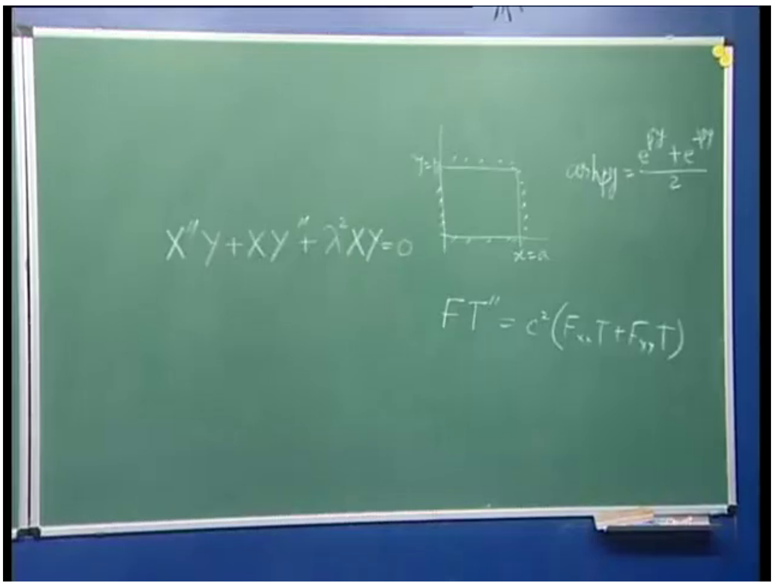

That is this is one boundary, this is another boundary, this is third boundary, and this is the fourth boundary. So, over this boundary, if we prescribe the temperature values over the entire boundary then we have got a Dirichlet problem. So, this is the differential equation and boundary conditions are here. Now, notice that on three sides we are replying zero boundary conditions for this particular case, it could be different; on the fourth side we are applying an arbitrary boundary condition in the form of f function f x.

Now, if this also f zero then we know what is the solution, then only solution possible is over all 0. So, therefore, to make a nontrivial problem we are applying and arbitrary boundary condition on the fourth boundary. So, we have got the Dirichlet the problem over this rectangular domain that is here. As we apply the separation of variable technique that is as we try to apply we get we start with this proposal, and the second order x derivative of this that is here, this one will give us X double prime into Y and del 2 u by del y square will give us X Y double prime. So, this we get and then over all as we take this on the other side of the equation and divide throughout with X Y then we get this equal to this. So, variable separation has succeeded this is dependant only on X, this is dependant only on Y.

And then as we equate it to a constant Y negative in this case because we want the x equation to have periodic solution that is starting from zero should come back to zero because at x equal to zero and x equal to a the value is zero; for y such a compulsion is not there. So, therefore, the x solution we want to be periodic and therefore, we take a negative constant here corresponding y solution will be exponential. So, then we separate the two ODEs get this. And this now the solution of this will have sin and cosine terms, it will be periodic the solution of this will be exponential.

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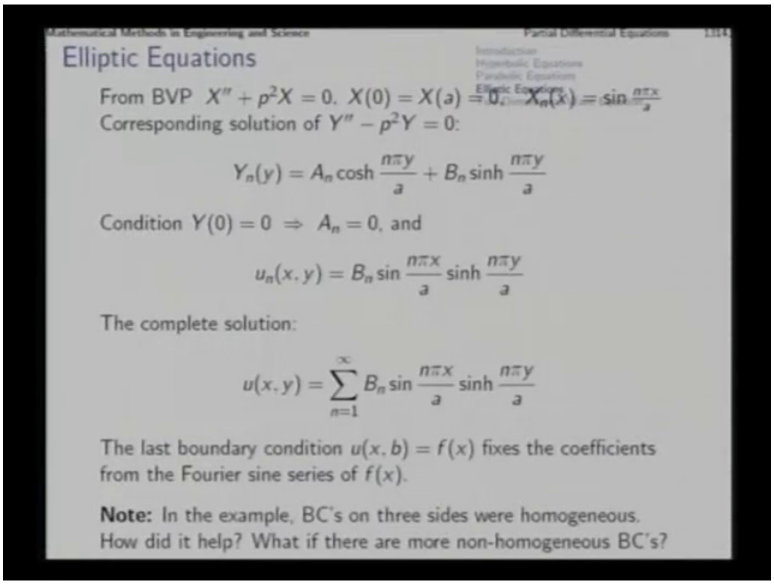

So, then this boundary conditions, we have been seeing for quite some time in the previous lecture also we saw this. So, then this gives us the solution which is the same as earlier cases sin n pi x by a. Now, this one either you can write the solution of this in the in terms of e to the power p y and e to the power minus p y or you can write in terms of hyperbolic cosine and hyperbolic sin that is obvious that the two are equivalent. Because hyperbolic cosine is nothing but e to the power y plus e to the power minus p y by 2; and similarly sin hyperbolic will have a minus sign here. So, since we are talking about linear combination of that two. So, whether we talk in terms of two exponential functions or sin hyperbolic and cos hyperbolic result is equivalent.

So, now, with this Y n and this X n when we put them together back into the proposal then every X n into Y n will be a solution of the differential equation each of them satisfying the boundary conditions. So, infinite some of such products such products this kind of products will give us the solution in which the initial condition that is Y of 0 equal to 0 that we get from here that is as we put small y equal to 0 in the boundary condition here as we put small y equal to 0, here we get u equal to 0, so that means, in the proposal the corresponding capital Y has to be 0. So, that gives us A n equal to 0 that means, this part goes off because sin hyperbolic is anyways zero, so A n coefficient has to be 0, so that gives us only this term. So, then we have got the solution here like this. So, the complete solution will be an infinite sum of such terms for n equal to 1, 2, 3, 4 and so on that is this.

Only remaining thing now is the determination of these coefficients b n and that we do by imposition of the last boundary conditions that is over this boundary y equal to b. So, as we impose that condition we get the this as a constant and that value into b n into sin of this. So, that gives us Fourier series so; that means, as we apply this last boundary condition we get the coefficients B n in terms of the coefficients of the Fourier sine series of the function f x and that completes the solution.

So, in this particular case, we found that the boundary conditions on the three sides were homogeneous that helps us a little that is finally, we had to track only one Fourier series. On the other hand, if the boundary conditions over all the four sides are nontrivial then we can split the problem into four different parts in each of the parts one boundary conditions is taken as nontrivial others are zero. And then finally, when we combine the four solutions we get the complete solution we get the correct solution now till now we have seen cases where the differential equation is separable, but the boundary conditions are not sometimes boundary conditions are not separable. And then we took the steady state solution separately and constructed a solution there could be also situation where the differential equation itself is not separable.

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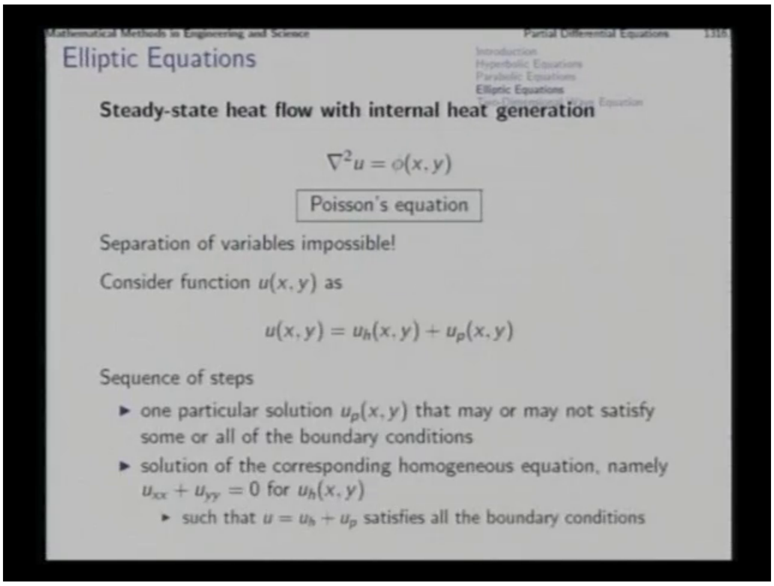

For example, if there is a heat generation in the same problem then you may get. a phi x y like this that is Laplace equation will change to for poissons equation is rather than zero on the right side you have got a function of x, y. Even a constant will create a problem with the separation of variable; in this case the variables will not be possible to be separated. So, separation of variables will be impossible with a non zero term here. So, while solving the Poisson's equation, we need to do something else something more. So, for that we can construct the solution in two steps, we consider a solution in this manner in which we have got a u h component which is actually the solution of the corresponding Laplace equation del 2 u equal to 0, and take u p, a particular solution of this differential equation.

And then the way we solve the ordinary differential equations as summing up the complementary function and particular integral particular solution in this in that manner we can find the solution of this Poisson's equation. For that the sequence of steps is also important. In this case, in the solution of Poisson's equation we first find out a particular solution of this differential equation by some method by some means is if we can find one solution of this which perhaps does not satisfy the boundary conditions then we keep that. And after finding that particular solution which most value does not satisfy all of the boundary conditions may be does not satisfy any of the boundary conditions, but still that having a solution the differential equation itself helps. Because then we consider the solution of this Laplace equation for u h in such a manner that u h plus u p together satisfies the boundary conditions. Then u h plus u p together this proposal satisfies the boundary condition and u h satisfies del 2 u equal to zero and u p satisfies del 2 u equal to phi and that means, the sum will satisfy del 2 u equal to phi.

So, in this manner if we can get hold of one solution of the Poisson's equation even if it does not satisfy the boundary conditions it will be good because using that solution we will apply the boundary conditions on u h appropriately. So, that whatever boundary conditions u p gives that component gives the other component u h which is in our hand which we know how to solve with homogeneous boundary conditions. So, we find out that what boundary conditions u h should satisfy which will satisfy the given boundary conditions and also compensate for the boundary values of u p, so that way we can construct the solution of a boundary value problem of this also.

So, in this case only separation of variable did not help, but through one particular solution we could reduce the rest of the job to the solution of a Laplacian equation that is this, so one example in the textbook that we are considering in a state this particular situation. As our next example, let us consider a three variable problem. Till now all problems that we have considered have been two variable problems.

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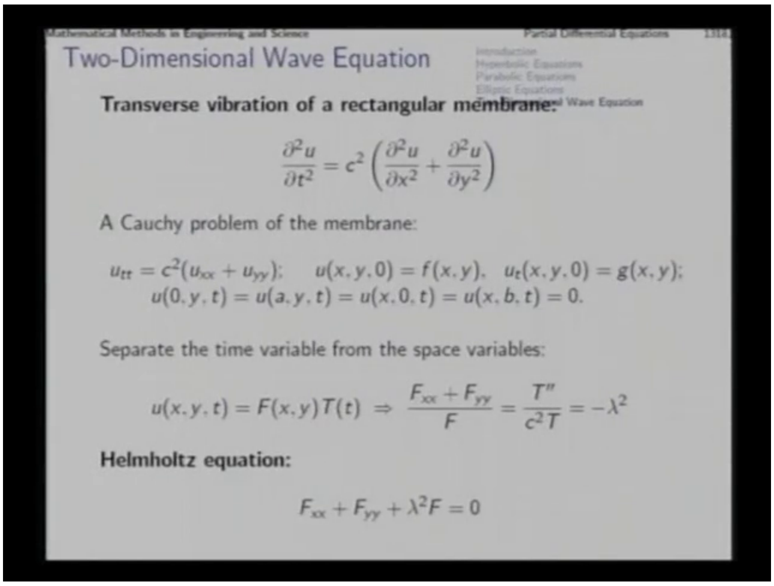

Suppose, we have got rectangular membrane rectangular because we want to keep the discussion simple otherwise the member could be of any shape. So, for a rectangular membrane suppose a boundary the rectangular boundary of it is bound and fixed and then the membrane can vibrate. So, earlier we have seen that the one-dimensional analogue of this the one-dimensional version of that is the vibration of string problem in which we had del 2 u by del t square equal to c square del 2 u by del x square that is all. Now, rather than are string, which is a one-dimensional medium. Now, I have got a membrane which is a two-dimensional entity two-dimensional domain. So, we have got this.

Now, this equation is actually a hyperbolic equation, as any vibration problem will entail. When we say that it is a hyperbolic equation, we mean that hyperbolic nature is manifest between the time and the space variables, that is between this and this it is a hyperbolic relationship. So, the Cauchy problem of the membrane will be this that is the difference equation along with these initial conditions initial position over the entire x, y domain and initial velocity over the entire x y domain. And these will be the boundary conditions u at u 0 y, t that is x equal to 0, x equal to a and y equal to 0, y equal to b this same rectangular domain over this rectangular domain around the boundary we have got this boundary conditions.

Now, how to solve this particular initial boundary value problem? So, first we try to separate the time variable from the space variables. So, for that we propose a function of x, y of the space variables multiplied with a function of time variable only in this manner we propose the solution in this manner. Now, as we propose this solution and affect the derivate differentiation, so this will be the del 2 f by del x square that derivative that is F x x this will be F y y that is this is will be F x x into T and this will be F y y into T. And here this will be F into T prime by T double prime that is second order derivative with respect to small t. So, then we will have this differential equation as f t double prime is equal to c square into F x x T plus F y y T.

Now, to separate the variables x, y from the variable T overall we will have to divide with f and t and if you want with c square also. And as you do that then here we will get as we are dividing with c square F T, so c square will go off T will go off F will come in the denominator, so F x x plus F y y by F, so that is here. On this side F will go off and c square T will come in the denominator that is here. Now, again we say that this side depends only on the space variables and this side depends only on the time variable. Now, for both of them to be equal they have to be equal to a constant.

What kind of a constant. So, for answering this what kind of a constant we will again see that if we take a positive constant here then F x x plus F y y minus that positive content constant into F will come and that will require that in space the solution increases continuously or decreases continuously. And in that case this kind of homogeneous conditions the boundary will not be able to fulfil. Therefore, here we need a negative constant. So, we put minus lambda square as we put minus lambda square then this equal to minus lambda square will give us this equation because f will go here minus lambda square brought on this side of the equation it will give us F x x plus F y y plus lambda square F equal to 0. And this differential equation will give us the periodic sinusoidal solution of x and y which will be able to satisfy this boundary conditions, this equation is called the Helmholtz equation.

The other part will give an equation in the function capital T. First we consider this these again a partial differential equation because it has got two independent variables involved x and y. Note that from this differential equation as we applied separation of variables between time and space variables. So, this time variable was separated in the other differential equation that we will get and this is an equation in the space variables only. And among the space variables this happens to be an elliptic equation in this elliptic equation involving the space variables only we apply one more round of separation of variables and for that.

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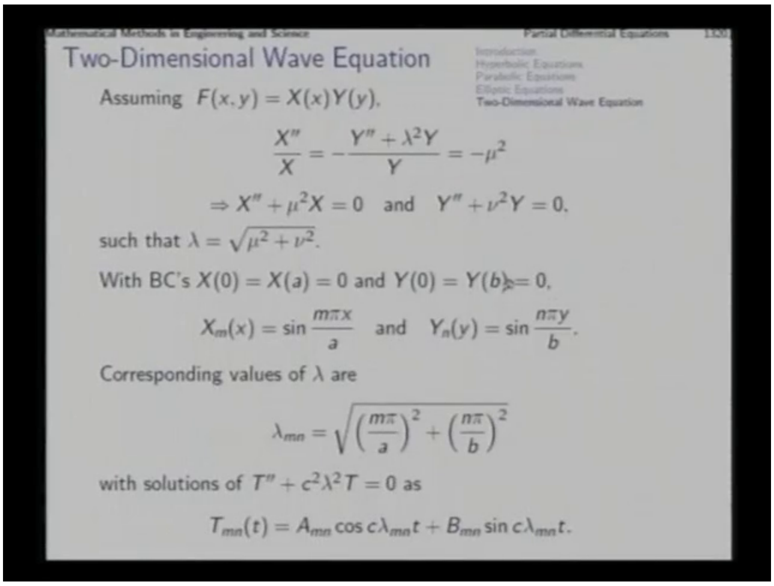

We assume F x, y in this manner as we do that the first term F x, x will involve X double prime into Y, this will involve X into Y double prime. So, we can write this in this manner x double prime into Y plus X into Y double prime plus lambda square f which is x y equal to 0. Now, in order to separate x and y terms we can do it in several ways. So, one possible way to do it is to divide overall with X, Y capital X capital Y.

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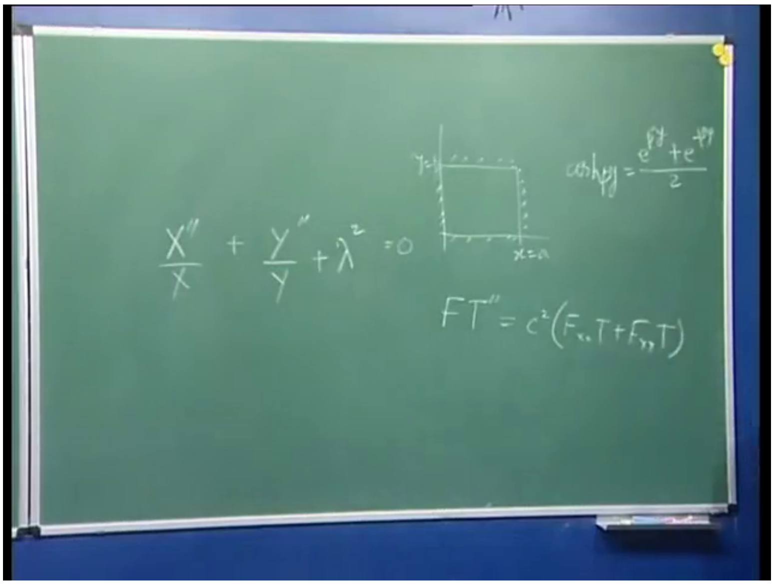

So, as we do that from here y will go off X here from here X will go off Y will appear here and from here both will go off. So, we will get this right and then we can take these two terms on the other side or we can take only this term on the other side. So, suppose we take both of these terms on the other side the equation, and then we will get this entire term minus Y double prime plus lambda square Y by Y. So, this whole thing depends only on Y component and this depends only on X component. So, this again we want to equate to a constant.

And here again this equated to the constant of separation will give us a straight forward differential equation in x which we have earlier also seen and that will have X double prime plus mu square equal to X. And therefore, this is necessary to be taken as a negative constant minus mu square, because otherwise in X the solution would not come back to 0. On the other side, when we equate these two, we find that negative sign gets cancelled and we get Y double prime plus lambda square Y equal to mu square Y that means, y double prime plus lambda square minus mu square into y. And that lambda square minus mu square has been mentioned has been denoted here as nu square which basically gives us a representation of lambda in terms of mu and nu that is the lambda that we chose here.

Now, gets represented in terms of the two new constants mu and nu that we are choosing now. So, in mu and nu a that is in capital X and capital Y, the two differential equations are similar. And for the those with homogeneous zero boundary conditions, we know the solutions already and we construct those solutions. This homogeneous boundary conditions over x and y which we derive from the homogeneous boundary conditions given for the variable the function u itself we get the solutions in this manner.

So, we find mu equal to m pi by a will give us solutions for x nu equal to n pi by b will give us solutions for y. And for every value of m and n that mu and that nu will give us different solutions for x m and y n. And corresponding value of lambda will be found from here that is mu square plus nu square under root, so that is for every value of m and for every value of n we will get a value of lambda. That means, we take large number of m equal to one two three four and n equal to one two three four then for that we will get a set of sixteen values of lambda, lambda 1 1, lambda 1 2, lambda 13 and so on. So, for every pair of values of m and n we will get a value of lambda.

Corresponding lambda will give us from here the corresponding differential equations for the function T of time and that will be T double prime plus c square lambda square capital T that is this T double prime plus c square lambda square capital T equal to 0. Now, we know the solution of this also will be found in this manner itself, but here we write the complete solution because on this it is time variable involved and we do not have so straightforward boundary conditions as we had in the case x and y conditions. So, here we write the complete solutions in terms of lambda m n and coefficients A m n B m n that means, for every pair of values of m and n we will get coefficient different coefficient here corresponding to different lambda m n values. So, this will be denoted as t m n. Now, we find that X m of x into Y n of y and corresponding T m n of t the product of these three terms will give us one solution of the original differential equation that will satisfy the differential equation and all the boundary conditions.

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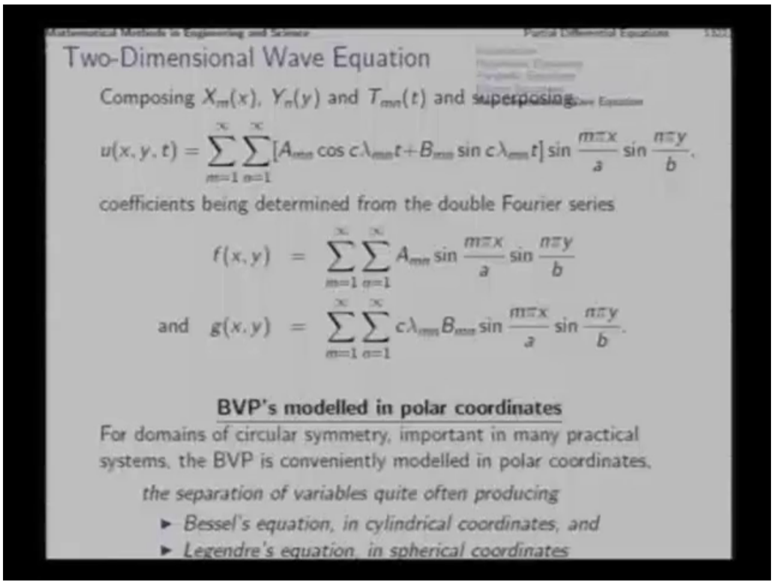

So, we get the complete solution through superposition of all these solutions for different values of m and different values of n that means, we get the solution in the form of this double summation the outer summation is with respect to m the inner summation has been shown with respect to n. So, this is T of t this is that is T m n of t this is X m of small x this is Y n of small y this is the complete solution which will satisfy all the boundary conditions and certainly the differential equation.

Now, as we put the initial conditions we put T equal to 0. So, at t equal to 0, this sin term will vanish and we will get a m n this cosine will be 1. So, A m n sin m m pi x by a into sin n pi y by b that is this which will be the initial configuration of the membrane that is given and when we differentiate it then this will give us sin. And upon substitution of t equal to 0 that will be 0, and this will give us cos along with a coefficient c lambda m n that is here and upon substitution of value t equal to 0 that cosine will be 1. That means, c lambda A m n into B m n into sin and sin that will be equal to the initial velocity of every particle over the membranes that is for all values of x and y.

So, now you find that the coefficients A m n and B m n can be found from the coefficients of the double Fourier series of f x, y and g x y. One simple case of this is given in the exercises in which case you then go ahead and continue to find the values of A m n and B m n from expansion of the Fourier series of the initial condition functions.

Now if other than this rectangular domain the domain is of some other shape. So, then you find that sometimes the differential equation that you get are not so straightforward in which case we get the solutions by solving a simple solution like x double prime plus lambda square x kind of things, but for domains of circular symmetry which is important in many practical systems say cylinder. So, in the case of cylindrical domain quite often we find that the b v p that we construct gets modelled more conveniently in terms of cylindrical polar coordinates and sometimes in domains which are spherical in shape we similarly use spherical coordinates.

So, use of cylindrical polar coordinates and spherical polar coordinates quite often leads to individual separated equations in the form of Bessel's equation or Legendre's equation. And in that case the component ODE that we solve to find out the solution with respect to one of the variables turns out to involve the Bessel's function and Legendre's function also. So, depending upon the shape of the domain you may find that the solution of the resulting ordinary differential equation that is resulting from the process of separation of variable may turn out to be the straight forward differential equation that we have been solving with constant coefficients or depending upon the shape of the domain with circular symmetry quite often Bessel's equation or Legendre's equations also evolve. And that actually is the starting point of the study of Bessel's equation Legendre's equation in the ODEs.

So, now that we have already discussed the series solutions and Bessel's polynomials. So, these since we have already studied the solution series solutions of ordinary differential equations and we are already acquainted with the Bessel's function as arising out of this kind of equation and Legendre's polynomial as arising out of the solution of this kind of equations. So, as we consider such symmetries circular symmetries, we can get the component solution and construct the complete solution and then apply the initial conditions to these situations also.

In the exercises in the textbook in this chapter, you will find some examples which involves such cases. However, if the domain in the x y plane or the x y z space does not have any such symmetries neither rectangular nor cylindrical nor spherical then that is if the domain is general shape in that case such analytical solution will not be possible. And in that case typically we try to solve the differential equation in a with the help of numerical means and numerical solution of partial differential equations is in itself a large topic and then we do not go into that as part of this course.

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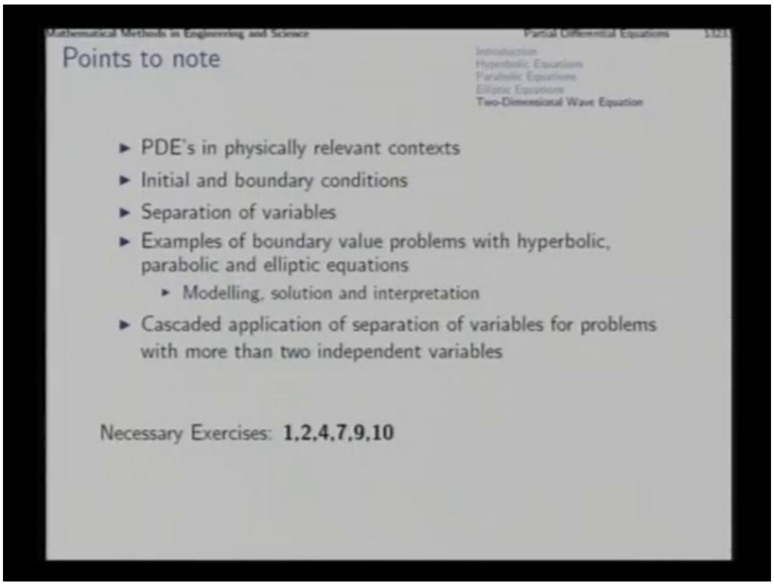

So, let us summarise at this point, what are the things that we have discussed in these two lectures on partial differential equations. First is that the partial differential equations appear in several physically relevant contexts. And some of the very important governing equations of phenomena we have tried to study through some analytical solutions with the help of Fourier series Fourier integral and Fourier transform based solutions through the method of separation of variables. In that we have discussed the initial and boundary conditions their meanings and their different situations in the different kinds of problems that we apply them and we have studied the separation of variables method.

And examples of such boundary value problems or initial boundary value problems in with hyperbolic parabolic and elliptic equations of a few cases we have discussed in some cases we have conducted the modelling and then the solutions and then interpreted the solutions in these two lectures. And in the last example, in the case of two-dimensional membrane problem, we have also seen one particular case of cascaded application of the same separation of variables twice. In the case of a three variable problem the first application of the separation of variables technique produces one ordinary differential equation in one variable and another new partial differential equation in two variables. A second application of the same separation of variables over this partial differential equation splits so, equation into three different differential equations that is complete separation. And that kind of a cascaded separation of three variables will finally, involve a double Fourier series for the determination of the coefficients the Fourier coefficients.

Now, this gives us a very brief overview of the partial differential equations, which appear most often in practical situations. And in the course as we have throughout tried to emphasize on interconnections of several different areas of applied mathematics this particular topic puts those interconnections in a very intricate manner. And some of the solutions which we have got here will be completely reduced after we develop some integrals based on the theory of complex analysis.

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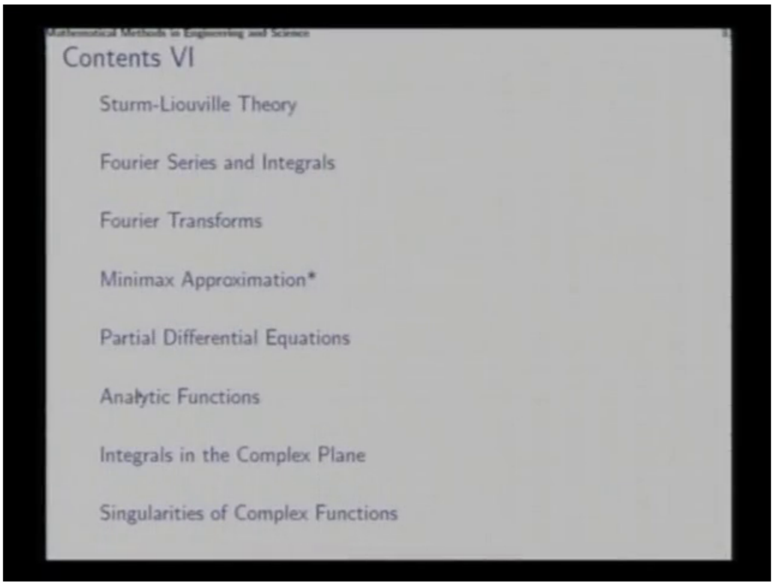

Our next module of this course is on the complex analysis in that we will have next three lectures devoted to the area of complex analysis. And after the third of those three lectures, we will develop some tools by which we develop quadrature formulae for some of the quadrature some of the integrals that appear as a result of the solution process of some of the partial differential equations that we have studied in this lecture. So, next lecture, we start with the first topic of complex analysis and that is the topic of analytic functions.

Thank you.