Mathematical Methods in Engineering and Science

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Module – VIII
Overviews: PDE’s, Complex Analysis and Variational Calculus
Lecture - 01
Separation of Variables in PDE's, Hyperbolic Equations

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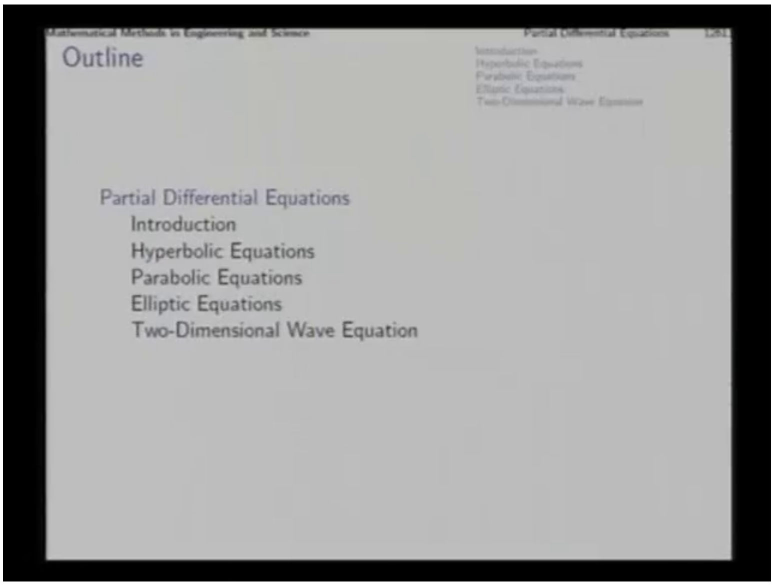

Good morning. In this lecture and in the next within these two lectures, we will have a quick cursory overview of a few very important partial differential equations which have a lot of importance in physics and engineering. Now, also this course we have been stressing the issue of interconnections of several different areas of applied mathematics. In this lesson, partial differential equation spanning over a couple of lectures, you will notice a lot of such interconnections.

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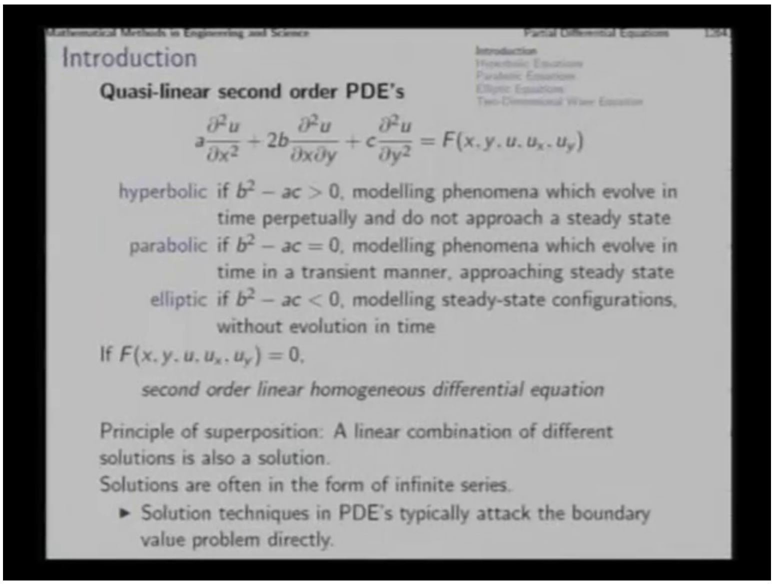

So, first we fix the special class of partial differential equations which we will be studying because as you know the field of partial differential equations is so fast that a full course itself is needed to do justice to that topic. But here as a part of the course on applied mathematics, we will concentrate on a very limited class of partial differential equations and that is this class which is called quasi-linear second order partial differential equations by second order because the largest highest order derivatives that is involved in this differential equation is of the second order. So, these are second order partial differential equations.

Now, why quasi-linear, this is linear part, but then why quasi. So, the differential equation can be non-linear in u del u by del x and del u by del y that class of differential equation that class of partial differential equations in which the unknown function u and its first derivatives can appear in any manner. But the differential equation is linear in the second order terms that kind of differential education is called quasi-linear, that is in the second order sense it has the linearity. Now this kind of quasi-linear second order PDE's are classified in three classes as hyperbolic, parabolic and elliptic. The criterion is actually very straightforward. If b square minus a c is greater than 0, then we call it a hyperbolic equation in analogy with the equation of conic sections in the x y plane. In the x y plane, the equation of a conic section has this form a x square plus 2 b x y plus c y square is equal to some linear function of x y and constant, so that is low order terms.

Now, you know that if this turns out to be a perfect square then the corresponding equation is the equation of a parabola. On other hand, if you find that that will happen when b square is equal to a c; in that case that corresponding curve will be a parabola. On other hand, if this is satisfied b square is greater than a c that is this term is more compared to a c, then you have the equation of hyperbola. On other hand, if this b square is less than a c then you have an ellipse. So, in analogy with the classification of conic sections in the x y plane, we classify the differential equations these partial differential equations into hyperbolic, parabolic and elliptic classes.

Why do we do that we do that not for simply this particular analogy with the ellipse, hyperbola and parabola, but in order to put different phenomena into different classes. Hyperbolic equations typically model those phenomena, which evolves in time that is one of the variables is time and with time the system evolves. So, such phenomena are typically modelled with hyperbolic equations. And as we study the solutions, we will see that the nature of solutions confirms that. And these phenomena these systems perpetually evolve in time and they do not approach a steady state. So, such equation are typically modelled through a hyperbolic differential equation.

On the other hand, parabolic differential equations parabolic partial differential equations model those phenomena which do evolve in time, but in a transient manner and as enough time ellipses they approach a steady state; and that forms a particular class of partial differential equations. Now, elliptic equations do not evolve in time at all, they typically model steady state configurations or steady state phenomena in which there is no time evolution. So, in the differential equation time variable will be missing in the case of an elliptic equation. The equations will be in terms of space variables x, y, z and so on. Now, you can consider that if there is a say in this particular case there is a very direct link between a parabolic problem and an elliptic problem in one less dimension.

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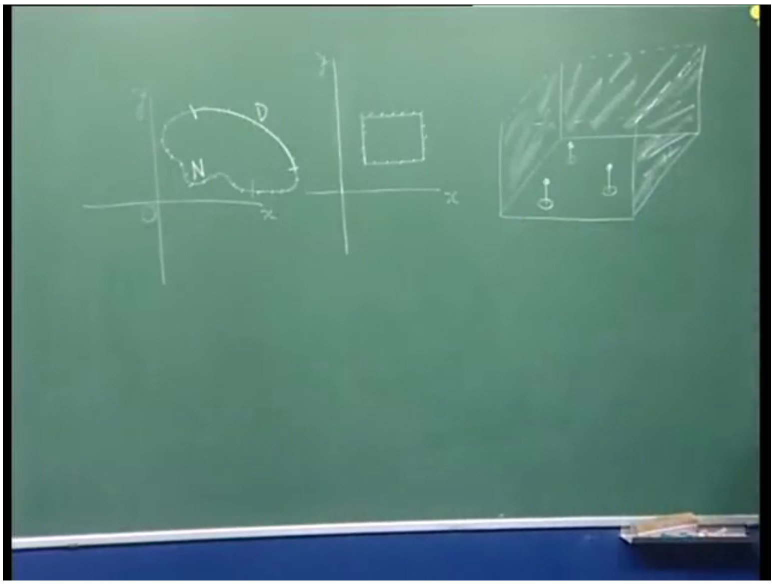

For example, suppose this room has got if you heating equipment in which are designed to maintain these balls at a constant temperature high temperature say 60 degree. So, these balls are maintained at 60 degree through some gadget that those three pieces of that gadget are placed in this room. And the walls of this room are all insulated, this wall this wall, this wall and the front wall also, these walls are insulated. And then above and below you have got constant temperature condition then you say that as we get from somewhere the three gadgets which are already maintained at 60 degree Celsius and put them here. Right at that moment, suppose we have measured the entire temperature distribution in the room, but these three points these three balls are maintained at high temperature.

Now, with these insulation and other conditions, we leave this room for enough time. Now, as we leave this room for enough time, so what happens, after a long time, after a long time what happens this question when we ask then initially it is a parabolic problem, if we ask with time how does the temperature distribution in the room change. So, that is a transient problem and that is a parabolic equation.

And then if enormous time elapses, and then we find that the steady state has been reached that is at no point in the room the temperature is varying further with time then that is a steady state solution. Now, if rather that is also a computationally important problem. And it is you say that with these at such and such temperatures with the boundaries maintained as required what will be the temperature distribution in the room and that is a problem in the x, y, z variables. This will be modelled with an elliptic equation, in which you are actually not looking for the simulation or prediction of something, but you just want to find out a configuration, which is the result of maintaining the boundaries and given points at specified values of the unknown function u, so that kind of an equation is elliptic.

So, typically elliptic equations will model such phenomena. It is also possible that the same differential equation is elliptic in some part of its domain parabolic or hyperbolic in some other parts of the domain that will happen if a, b and c are not constants, but functions of x and y; in such situations different segments of the domain can satisfy different conditions among these three. Now, in this particular case, if the right side is 0, then we have what is a special case of even this and that is second order linear homogeneous differential equation.

Now, in this particular case if this is zero then a non-linear part vanishes, so you do not have to say quasi-linear then it becomes actually linear; and this zero gives you linear homogeneous differential equation. So, with f equal to 0, you will have a linear homogeneous partial differential equation. And in that case principle of superposition will hold in the sense that if you have got a good number of solutions of the corresponding differential equation then any linear combination of them also will be a solution to the differential equation because of the homogeneity.

And as we see further quite often the solutions that we will get will form will be in the form of infinite series that is we will find infinite solutions for the homogeneous equation, and then as we linearly combine them we will find an infinite series. And these series are then completely determined that is their coefficients are determined based on the initial or boundary conditions that are given with the differential equation. Now, in the case of ordinary differential equations, we found that we first look for the complete solution in terms of arbitrary constants, and then we can fix we can put the initial and boundary conditions to get the particular solution which completely satisfies the initial value problem or boundary value problem.

In the case of partial differential equation, it is most of the time extremely complicated to do such an exercise, because here the arbitrary constants quite often are not just constants, but functions of some variables. In a sense that at a particular step, while solving a partial differential equation if integration with respect to x is involved then the arbitrary constant that gets into the solution is not just a constant, but could be a function of y. Similarly, integration with respect to y which involved an arbitrary function of x; and in that manner, lots of arbitrary functions may get into the situation. And therefore, in the case of solution of partial differential equations, most often we do not follow the strategy of first finding the complete solution and then applying the boundary conditions at the end.

Quite often we considered directly solution of the boundary value problem itself boundary value problem or initial value problem quite often we call certain problems as IBVP that is initial boundary value problems. And then we apply whichever condition or whichever equation we find easy to apply at any stage, and the final expectation is to find a solution which at the same time satisfies the partial differential equation as well as the conditions given.

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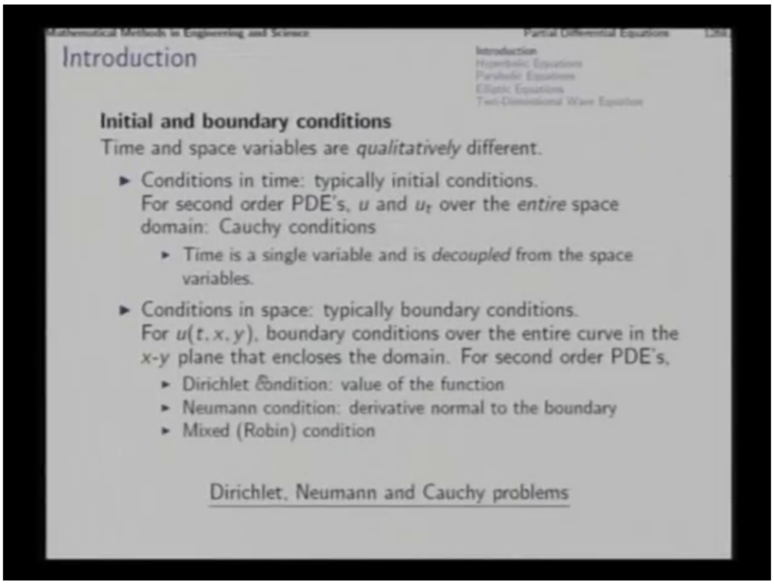

Now, what are these initial and boundary conditions. As we discuss this we need to make note of a particular fact that time and space variables are qualitatively different. Now, the kind of problems are going to discuss, they have direct physical relevance. And there according to the physics of the problem, you will notice that time is one kind of variable and space is another kind of variable in the space variables you have got three variables x, y, z or in simpler phenomena you have got two variables x y or even simpler phenomena will have only one variable x. So, in those situations, where you have got say two variables x and y of space and one variable of time.

So, when you consider this kind of variables some time variable and a few space variables. Then typically in the important problems initial conditions are supplied with the time variable and boundary conditions are supplied with respect to the space variables. And there you find that for example in a second order partial differential equation, for time you will require u and del u by del t to be supplied exactly the way we considered initial value problems in ordinary differential equations if the equation is second order then the function value and the first order derivative should be supplied. But now here u and del u by del t needs to be supplied over the entire domain in space such conditions are called Cauchy conditions and the corresponding initial value problem will be called a Cauchy problem.

Now, as time is a single variable it is decoupled from the space variables. So, for example, suppose in this case, we want to solve a Cauchy problem in which the initial conditions in time are given. So, what will constitute in initial conditions in time that is function value at time equal to 0, and its first derivative del u by del t at time equal to 0, but where over the entire x y domain that means, in the entire interior of this domain, we will need to supply the initial condition. So, you will actual need the functional form of u of 0 x, y that is at t equal to 0. So, if the unknown function is u of t x, y then as initial condition you will need u of 0 x y and del u by del t at 0 x, y over the entire domain.

So, these will be two bivariate functions that will appear as the initial conditions. And note that time is a single variable and his decoupled from the space variables. On the other hand, conditions is in space are typically boundary conditions. So, for example, for u of t x y boundary conditions will be needed over this. You cannot say that I will supply boundary conditions over x and over y. If the geometry of the domain is like this then over this entire boundary all points entire boundary curve you have to supply the boundary conditions.

Now, here you see that the variables x and y have been combined. So, you are talking about supplying the boundary conditions over a curve in the x, y plane such a thing you cannot talk about in terms of t and x. For example, you cannot say that we will supply boundary conditions over a curve in the t, x plane you cannot say that. The time variable t and the space variable x are essentially and conceptually decoupled you cannot make a boundary curve like this and supply boundary conditions there. So, in the case of x, y plane you can consider these points in which x and y are not decoupled quite often that will be the case and in such situations even describing the domain will require quite a bit of nontrivial work.

So, for second order conditions for second order partial differential equations there are two basic kinds of boundary conditions that you can give one is called Dirichlet condition in which you supply the value of the function over the entire boundary. Another condition is Neumann condition in which you supply the derivative normal to the boundary. Now, it is possible that for a particular problem you supply Dirichlet condition over part of the domain part of the boundary and the rest of the boundary you can you supply the Neumann condition that kind of situation is possible. So, in this kind of a situation over this part of the boundary you supply the value of the function which will be maintained; over this part of the boundary you may supply the value of its derivative which derivative normal.

So, the derivative of the function at this point will have two come components one is del u by del x the other is del u by del y. A linear combination of that will be able to form the derivative along any direction directional derivative. And then you will say that we will consider the derivative value normal to the boundary, when you supply that that becomes the Neumann condition. A special class of that is Neumann condition that is a special class of Neumann condition is the homogeneous Neumann condition when you say that derivative across the boundary that is directional derivative along the normal to the boundary is 0. In that sense, in that case you basically say that there is no flow across the boundary.

And supplying the value at the boundary which is the Dirichlet conditions, you basically say that these temperature is maintained or this potential value this u value is maintained and that means, that whatever potential change is required that does not take place because there is enough flow across the boundary. So, you maintain the value of the function there.

Now, you have the third kind of a condition also which is called the mixed condition or robin condition and is over the entire boundary you supply the function value u value then you get a Dirichlet problem. Similarly, over the entire boundary if you supply Neumann conditions at the same point you cannot supply both conditions you cannot say that u is also 0, u is also specified and normal derivative is also specified both you cannot specify that will over constrain the problem. So, if over the entire boundary u supply Neumann condition then that particular problem is called a Neumann problem. Most problems could be neither Dirichlet nor Neumann because both of these problems are special problems. Now, if in a particular situation, you have part of the boundary where Dirichlet condition is specified, you can have the rest of the boundary over which Neumann condition will be specified. Cauchy problems are of a different class in which the time variable gets involved and the most direct problem is simulation is to predict what will be the evolution of the system in time that means, that here time evolution will be studied.

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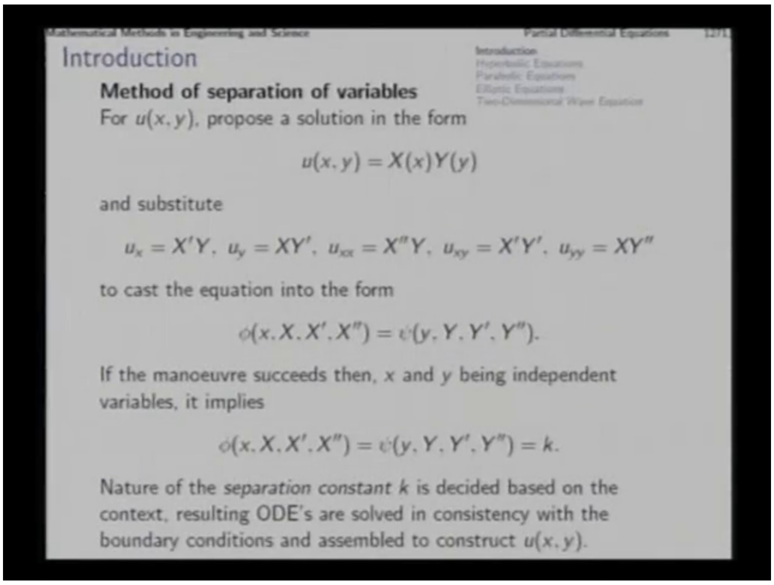

Now, we will consider in detail one particular method of solving such partial differential equations and that is called the method of separation of variables. Note that this method will not always work that is not guaranteed to work in all quasi-linear second order differential equations, but in many cases in many practical problems it does work. So, what is the idea of separation of variables for a problem in which you have to determine u of x, y from that partial differential equation. We try to propose a solution in this form that is if it would be so nice if the unknown function of x and y can be put into the form of a product of two functions. One a function of x only and the other is a functional y only. If this manoeuvre succeeds, then we will be able to separate the two variables and solve the partial differential equation through solutions of two ordinary differential equations.

So, for that purpose we assume it in this form we propose it in propose the solution in this form and then differentiate with respect to x that this will the derivative because this Y of y, Y of small y part will not depend on x. So, its first order derivative with respect to x will be this similarly first order y derivative will be this then differentiating this further with respect to x, y and differentiating this with respect to y will find these right. So, these are up to the second order derivatives.

Now, we take all these differential coefficients and insert them into the given partial differential equation. And try to organise the terms try to simplify the equation in such a manner that all terms containing the variable x the function capital X and its derivatives go to one side and all terms dependent on y go to the other side there is phi is free from small y capital Y and its derivatives. Similarly, psi is free of small x capital X and its derivatives if we can cast it into this form, which is not always possible, but quite often in many practically significant practical relevant problems this turns out to be possible.

So, then we will say that since the variable x and the variable y are independent of each other. So, these two combinations being equal means that each of them is equal to a constant because x and y themselves are variables, but they are independent. So, as we equate that to a constant, this constant is called constant of separation and is a what kind of a constant of separation we will get here that kind of constant of separation is determined from the context of the problem. And then as we equate this equal to a constant we get differential equation in capital X which we solve with small x as a independent variable, similarly this. And then we for every individual solution of X of x and Y of y we insert them back here and then construct the solution for the partial differential equations in this manner after getting capital X of small x and capital Y of small y that is after solving two ordinary differential equations.

We collect the corresponding solutions and put them here to form the solution of the partial differential equation. The crucial point in this entire method is to be able to separate the two variables like this. It will work quite often; sometimes it does not work. When it does not work then we have to take some special measure. Now, in this these two lectures we will see quite a few cases where this separation directly works. And in a cases of the other kind of also in which it directly it does not work, but we can do something about it to make it work to a modified equation.

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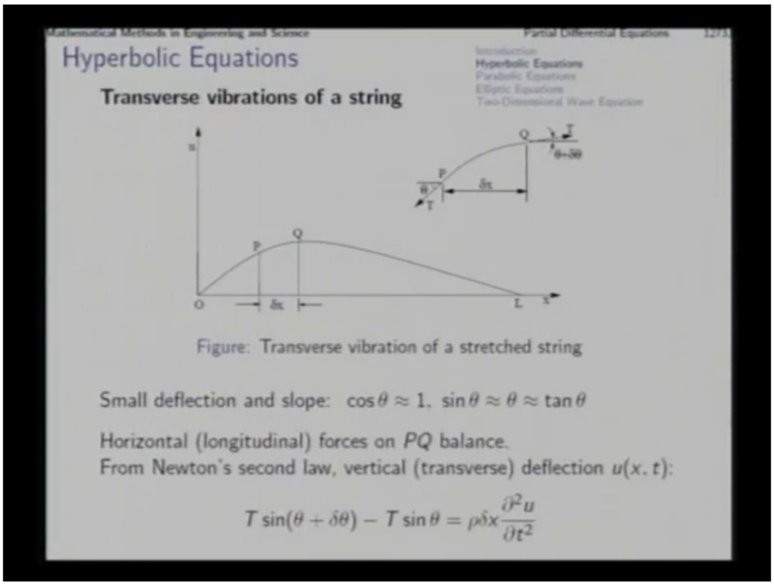

So, first we consider this phenomenon that is transverse vibration of a string. Say the string is of length L with which is held with a tension T all over and we first try to develop the equation of the string. Now, every point on the string refers to a value of x. So, value of x refers to a particular point on the string. And y there is no y, u; u refers to its displacement at a particular instant from the x-axis which is its equilibrium position. So, and as we are talking about the position at an instant that means, the variable u also depends on time. So, here u will be the unknown function, u will be the dependent function which we will try to solve and the independent variables are position x and time t.

So, here if this is the shape of the string at a particular instance then at x the position is P at x plus delta x the position is Q at this particular instant, and then the assumption of small deflection. And slope will mean that whatever angle at this point the string is making with the positive x direction that angle theta is small. If the angle theta is small then we can say cos theta is close to one and sin theta and tan theta both are close to theta and therefore close to each other.

Now, this particular small element P, Q, if we see in exaggerated form here then here we have got the tension t at an angle theta here the tension t is at an angle theta plus delta theta the angles at the two points could be different slightly. Then we find that t cos theta is working on this side and that will balance the horizontal force on this side. So, horizontal forces are balanced. What about the vertical force as we equate as we write the vertical force balance then we will find T into sin theta plus delta theta upward minus T into sin theta downward. So, this is the resulting force. And this resulting force is giving this particular element of the string its acceleration.

So, then this resulting net force is going to be its mass that is linear density rho mass per unit length into delta x that will be the mass of this P, Q segment multiplied with the acceleration of this particular segment that is second derivative of u with respect time. Now, here you see T is common here, which is a constant positive constant; rho is another positive constant. So, we can take the ratio and that will be a positive number.

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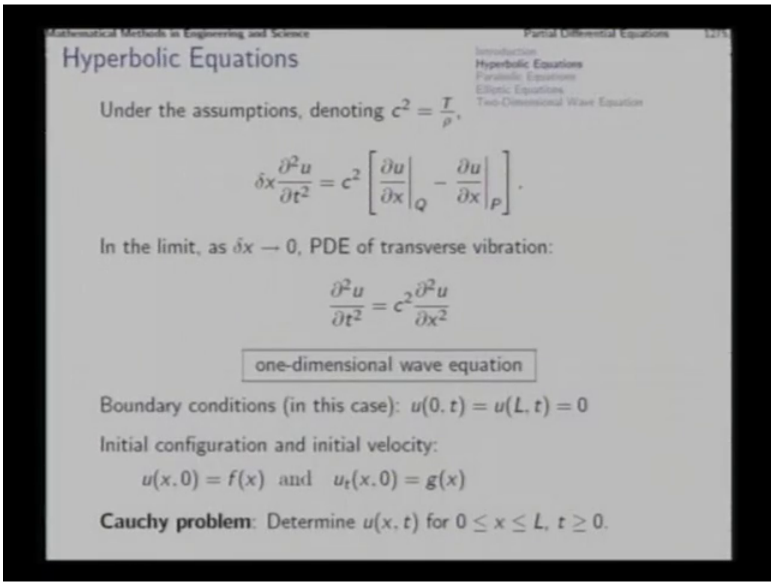

So, currently that ratio T by rho that positive number we denote by c square positive number so we can say c square. Later as we proceed we will find that this quantity c has a great relevance that is the speed of the wave travelling in the string, but currently just because it is a positive number constant so call it something square. So, say this is c square. Now, we will find that as we put that c square here, now what is this sin theta plus delta theta that will be equal to tan theta plus delta theta in a way because for small theta this three are equal.

Similarly, this will be tan theta in a way close to that. So, tan theta is the slope here and what is slope del u by del x; that means, this sin theta plus delta theta is the value of the slope here, and sin theta is the value of the slope here. So, other than T by rho which is c square these terms will remain on the left side. So, that T by rho is c square. So, other than that we have slope at Q minus slope at P right. And x difference between the positions P and Q is delta x that is here.

Now, you see that if you divide this side with delta x then what you have, slope at Q minus slope at P divided by the x distance x difference between P and Q as delta x tends to zero that is infinitesimally small then this ratio this bracketed term divided by delta x this ratio will tend to the derivative of del u by del x which is the second derivative of u. So, then you have got this which is the one-dimensional wave equation. And this is equation that governs the transverse vibration of a string.

Now, in this particular case, if both the points both the end points we have tied firmly then u at the two endpoints at x equal to 0 and x equal to L will be zero for all time these will be the boundary conditions. What will be the initial condition and initial configuration initial velocity, because it is second order differential equation in time, so you will need to give initial position and initial velocity that is u at x 0 and del u by del t at x 0. So, these two suppose are specified as functions of x that means, initially the configuration at time zero the configuration of the string is this and it is moving at such and such speed all over the x variable. So, this will constitute complete initial condition initial configuration full f x and initial velocity that is full g x. And the resulting problem of predicting the evolution, predicting the transfer vibration in time at all positions of x that is the Cauchy problem.

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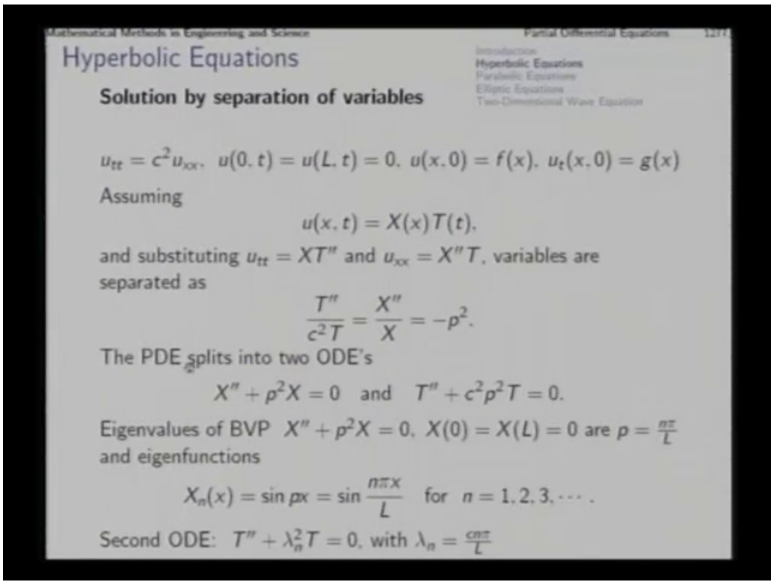

Now, how do we solve it by help a using the method of separation of variables? This is the differential equation with these initial these boundary conditions, and these initial conditions. So, if we try to apply method of separation of variables then we will be assuming the function in this manner the function that we are trying to determine that function we will be assuming in this manner. And then the second derivative with respect to time that will involve second derivative of this holding this as constant, there will be x into T double prime. Similarly second derivative with respect to x will involve price differentiation of this part, which is this.

Now, these two expressions of u t t and u x x if we insert here then we will get X T double prime is equal to c square X double prime T and now we want to get all the X terms on one side and all the T terms on the other side. So, we divide throughout with c square T X. So, on one side, on this side, we will have X will go off because we are dividing with it the rest c square twill come here. On the side c square will go, T will go, X will come here. Now, you see that this side and this side are totally separated. This entire left side is dependent only on small t and its function capital T and its derivative etcetera. Similarly, this will be totally free from T and it will depend only on X.

Now, since these two are equal and T and X variables are independent therefore both of them should be equal to the same constant. Now, what kind of a constant that will be determined based on the context of the problem. See, in this case, if we put a constant which is positive; in that case we will get X double prime is equal to some positive value into X which will after you put it in this manner, you will have X double prime minus something into x as we take it on the other side. And that kind of a differential equation will typically not have a solution which starts from 0, and comes back to 0, because it will give rise to exponential solution.

So, in order to be able to satisfy the given boundary conditions it is important that we look for that kind of a value of the constant of separation which will give us periodic solution which after starting from 0, will come back to 0. And therefore, we must take a negative, non-positive constant here. As we take that minus p square then as we equate this to minus p square, we get this differential equation which is an ordinary differential equation in X of x. And when we equate this minus p square to this then we get T double prime plus c square p square T equal to 0 which is another ordinary differential equation over capital T of small t. Now, these two differential equations we try to solve and in that the connection between the two ordinary differential equations is through this p it will be the same p here and here

Now, first let us attempt this partly because the conditions corresponding this are simpler. As we solve this, this is our old friend, we have already solved this kind of ordinary differential equations. So, you will notice that if we supply the boundary conditions to this, as see till now we have seen that the differential equation has been separated in T and X. What about the boundary conditions here if we consider the boundary conditions that is u at x equal to 0, for all t is 0, that means, if you put 0 here then for all t, x of zero into T of t will turn out to be 0. Now since we do not expect T of t to be zero for all t, so this part must be zero that gives us this equal to zero as the boundary condition on this differential equation.

Similarly, at x equal to L again the same condition applies so these two we have found to be 0, at X equal to 0, and small x equal to 0 and small x equal to L. And that will mean that we have got and a boundary variable problem in this ordinary differential equation itself, so that will tell us that the solutions of this will turn out to be in this manner. Now, cosine term vanishes because of the particular zero boundary condition that we have got and its solution we have seen earlier. So, and it will have solutions for quantised values of p, this is to be assumed probably note that. So, P equal to n pi by L for that it will have solutions. We are omitting the amplitude here because something else will later come in that place anyway and after we find that solution for this are available for this value of p then that same value of p we insert here to write the second differential equation in this manner T double prime plus c square p square t c square p square is lambda n square lambda n this c p, p is this. So, this is the second PDE, which we now go to adopt.

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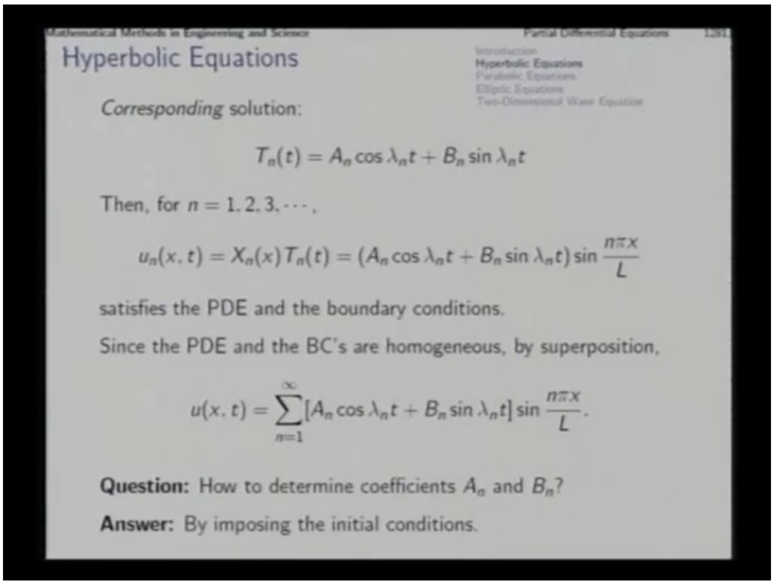

In a similar manner the solution of this differential equation is then here like this. In this case, we consider the complete solution including both the coefficients. Now, if we combined the X n and T n as we got now and put in this proposed solution then we get the solution u n as the product of X n and T n for every n. So, this is the T n part, and this is the X n. And this is the reason why in the case of finding the solution X n, we did not include the amplitude the constant on this side because now we have got the constant anyway.

Now, we know that this solution will satisfy the partial differential equation and boundary condition for every n that is u 1, u 2, u 3 which are correspondingly X 1 T 1, X 2 T 2, X 3 T 3 each of them will satisfy the boundary conditions because of this kind of a factor and they will solve the PDE also. However, each of them independently is unlikely to satisfy the initial conditions. And therefore it is important to form a form the complete solution, which will be the sum of a linear combination of all such solutions. And as we sum them up, now this is a homogeneous equation. So, if individually these are solutions of the differential equation then a linear combination will be also a solution the differential equation. So, we frame the complete solution for all n's. So, this is the solution complete solution of PDE satisfying the boundary conditions.

And then if we can determine a n and b n for all ns then that will complete the solution process of the initial value problem the complete Cauchy problem. Now, the how to determine the coefficients A n and B n that is by imposing the initial conditions. So, as we try to impose the initial conditions at time equal to 0, at t equal to 0, we will supply the u x zero and del u by del t of x 0. So, as we put t equal to 0, this part will go off and we will get u x zero as only this part into this. So, there will be this, A n sin n pi x by L.

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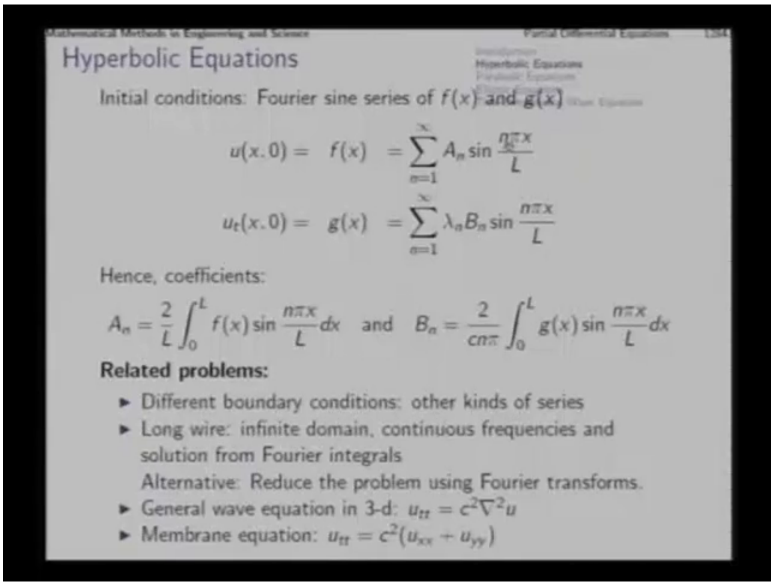

So, as we put t equal to 0, this is 0 and this is 1 the form this bracket we will get only A n. So, A n sin n pi x by L sum of that for all ns. So, this will be equal to the initial u that has been supplied which is f x. Similarly, we differentiate it with respect to t in that case this will turn out to give a sin term and that will turn out to be zero at t equal to 0 and this will give a cos term along with a factor of lambda n. So, we will have lambda n b n into cos zero that is one. So, lambda n B n into this will be the first grade. So, lambda n b n into this the sum of all at will be the first grade which is have which has been given as del u by del t at equal to 0.

So, now, we notice that A n happens to be the Fourier coefficients of the initial shape function of the string and that means, if we can construct the Fourier series of f x then the coefficients that we get out of that Fourier sine series, because no cos term is here. So, Fourier sine series of f x will have coefficients which will be the same A 1, A 2, A 3 which we need to determine. Similarly Fourier sine series of g x will have coefficients which will give us lambda 1 B 1 lambda 2 B 2 and so on. So, since we already know lambda 1, lambda 2 etcetera. So, by dividing with those we get that corresponding B 1, B 2 etcetera and that means, the A n and B n coefficients we determine by applying the initial conditions and working out the Fourier sine series of this f x and g x functions which are these. Here for determining B n we have divided by lambda n, which is c n pi by L. Therefore, L has gone from here c n pi has appeared. So, this way we can find out the coefficients and that completes the solution process.

Now, there are a few related problems. If you have different boundary conditions in this case both boundary conditions were 0. If you have different boundary conditions then you may have to look for other kinds of series not sine series maybe cosine series. If the wire, the string is enormously long very long wire in that case the domain can be more appropriately represented within infinite domain. And in that case these frequencies n pi by L will turn out to be continuous. And the solution will come not from a Fourier series, but if Fourier integral. Alternatively one can also reduce the problem to an ordinary differential equation in terms of Fourier transforms.

General wave equation in 3-d is this in which the complete Laplacian appears in which you will have x, y and z variables all of these variables that will be a four variable problem completely. In particular case of that we will have membrane equation, which is there in the textbook and towards the end of this topic we might have a brief discussion of that. Now, after determining all the coefficients A n and B n, now we know that we can find out the solution solutions like this and an infinite sum of these will actually construct constitute the solution.

However, we are talking about the string. Now, what kind of initial condition has been given suppose at one point the string has been plugged and then reduced. The shape of the string that you get out of the plucking is the initial configuration f x. And for example, considered g x as 0 that is initial speed is 0, and then as you release then you find that at different time and at different x values you can predict that this is the way the string should continue to vibrate.

Now, one question arises that as one part of the string is plugged and then released, after release how do the other parts of the string know in which way to vibrate in which way they are supposed to vibrate that transmission of information from the plug point to or disturb point to other places is through the stiffness of the medium. In this case that stiffness is appearing into the equation in the form of T, which we initially considered and T by rho is the value c square. So, now the physics inside of the problem becomes clear when we consider, another alternative method of solution of this particular problem.

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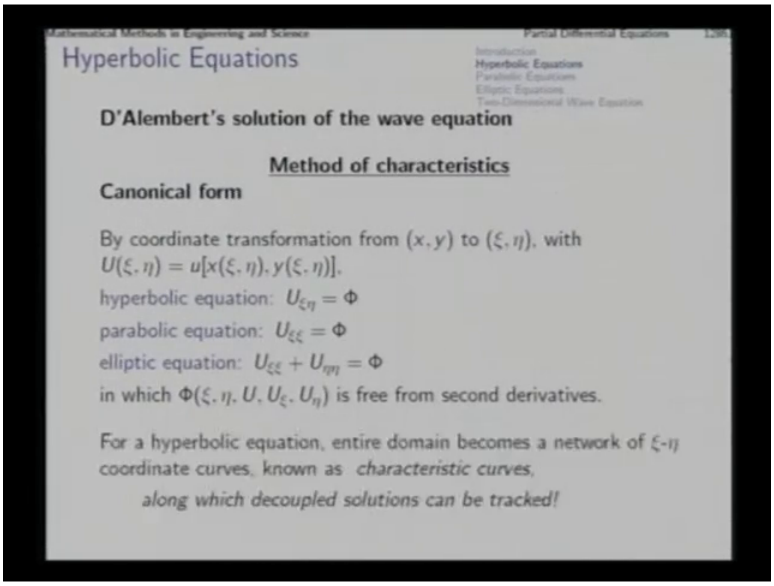

And that is called that is the D Alembert's solution of the wave equation method of characteristics is a competing method to solve partial differential equations and that is most useful that has greatest ability in the case of hyperbolic equations. Now, every differential every PDE of the quasi-linear type can be converted to a canonical form a very simple form through a coordinate transformation. Say from x, y, if you change to pair of variables xi eta through some coordinate transformation like this then you can convert the problem to these two new variables. And the canonical form of a hyperbolic equation in the new variables turns out to be this del to u by del xi del eta equal to lower order terms.

Similarly, for parabolic equation the corresponding canonical form is del to u by del xi square equal to lower order terms. And similarly, this is the corresponding canonical form for an elliptic equation. You will note that any hyperbola in x, y plane can be converted to the form x, y equal to constant through rotation of the axis and so on that is equivalent to this. Similarly, every x y equal to some linear terms every parabola equation of a parabola in x, y plane can be reduced to x square is equal to for a y lower order term. Similarly, all the any ellipse equation of an ellipse in the x, y plane can be reduced to the form x square by a square plus y square by b square equal to constant. So, in analogy with that you can reduce the quasi-linear series also into these canonical forms.

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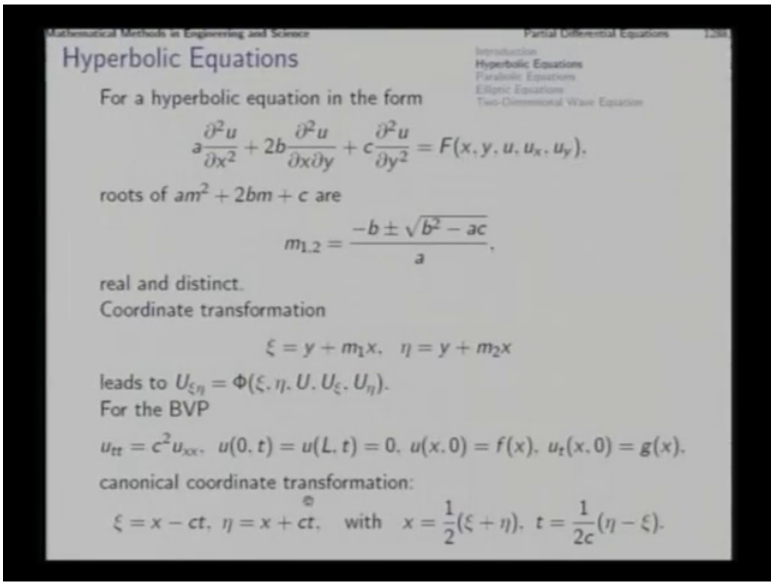

And in this case in the particular string problem that we have been considering you can reduce it to the canonical form that is from here, you can reduce into canonical form through these coordinate transformations in which x and t will be found in this manner.

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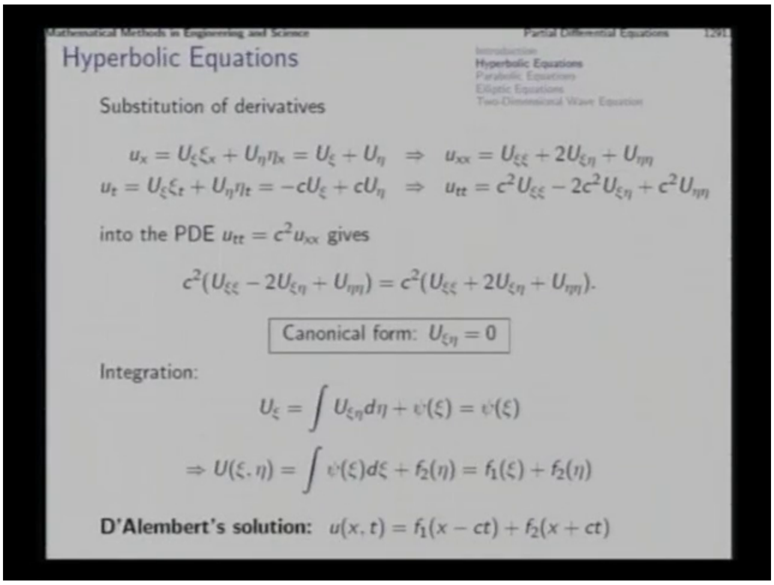

And as you do that as you conduct that reduction you will find that the canonical form turns out to be u del u del to by del xi del eta equal to 0. Now, you will find that solution of this is extremely simple. How, you will just integrate it once with respect to xi and once with respect to eta. The first round of integration with respect to eta will give you u xi on this side and here this is 0, so this will be just a constant. In case of constant, you can now have an arbitrary function of xi because an integral is partially with respect to eta. So, you call it psi of xi this is an arbitrary function of xi. Then you integrate it once more with respect to xi and that will give you the function capital U and that is integral of this plus another arbitrary function this time arbitrary function of eta.

Now, call this integral as f 1 then you have got this as the general solution. And any two functions f 1 of xi and f 2 of eta will fit the purpose for the solution of the differential equation PDE only here also there is a kind of separation of variables u in a way. Now, if you put xi and eta in terms of x and t then you get this as the famous D Alembert's solution. Now, what does this mean this means that here you have got this as one solution and this is one solution.

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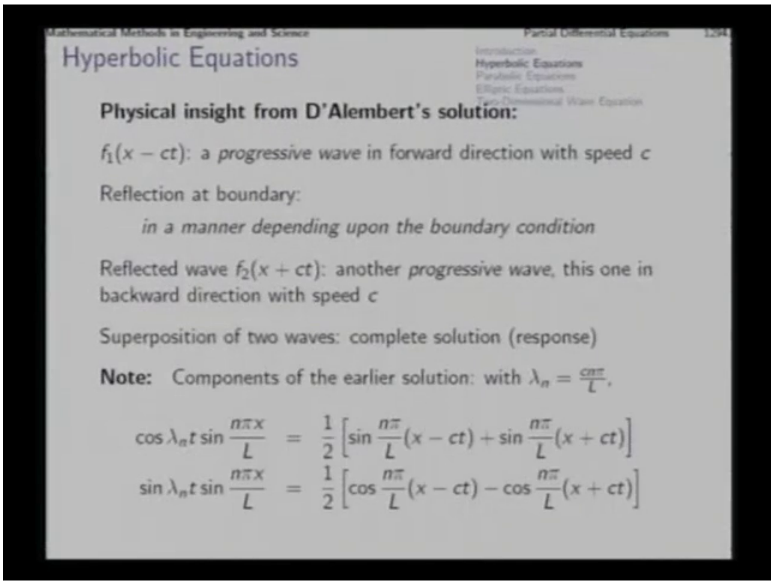

That this particular solution is actually the equation or description of a progressive wave in forward direction with speed c that you can notice if you say that at time zero at x 0, what is f 1. And then at time t and at x c t what is the situation, it will be the same f 1 because if x and t are increased together in the ratio of c then any change in t and the corresponding change in x by c into change in t will not change the argument of a f 1 at all. So, that means, that any positive change in t and corresponding positive change in x which is c times the change in t will not change f 1 at all. That means, whatever is this scenario right now here will be the scenario at a distance c t after t time has been elapsed. So, whatever is the situation here will that situation will travel forward with the speed of c. So, this is actually the representation of a progressive wave in forward direction with speed c.

Similarly, the other solution f 2 of x plus c t will be a similar backward wave. So, the point where it has been plugged as it is released then that information will travels along the forward direction with speed c and along the backward direction also with speed c. And then as the progressive wave hits the boundary then the wave is reflected and the reflection of this wave will be f 2 of x plus c t. And the superposition of these two solutions, these two progressive waves the superposition of these two progressive waves one forward and one backward will form the entire solution.

Further if you consider the components of the solution that we found earlier here. You find that one component is cos lambda t sin n pi x by L, other is sin lambda t sin n pi x by L. If you consider this cos lambda t sin n pi x by L and sin lambda t into sin n pi x by L, then if you put a two here and for that we have put another extra half here. So, as you put half, we get twice cos A sin B twice cos A sin B that can be split into the form of two sins sum of two sins. And similarly this sin into sin and we similarly in the form of the difference of two cosine.

And now notice that this part is a function of x minus c t only that is whichever way the t and x - two variables appear they appear together in the form of x minus c t. So, this is the forward wave. And similarly on this side, you have got the x plus c t that is the backward wave. So, as you consider these two split parts and separately you can take them and you can see that these turn out to be equivalent to the solution that we got just now this is the function of wave a c 1 x minus c t this is a function of x plus c t.

Now, after finding the solution of the D Alembert's equation after finding the D Alembert's solution to the wave equation in this manner this can be supplied to the initial and boundary conditions to determine the particular function that we are going to get f 1 and f 2. And as we try to find the particular functions f 1 and f 2, if we find the same solution completely which we got in the other solution that is through the separation of variables. Now, these are a few considerations which you will appreciate in more detail when you solve a few further problems for example, in the textbook these some of these cases have been considered in the exercises of the chapter and the step by steps of the solution have been given in appendix. So, I suggest that you follow some of these exercises and presently we will proceed to the other two classes of quasi-linear partial differential equations that is parabolic equations and elliptic equations in the next lecture.

Thank you.