Mathematical Methods in Engineering and Science
Prof. Bhaskar Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Module - VII
Application of ODE's in Approximation Theory
Lecture – 04
Fourier Integral to Fourier Transform, Minimax Approximation

Good morning, this is our last lecture on the topic of function approximation. In the previous lecture we saw several ways of function approximation by the use of several families of Eigenfunctions of different Sturm-Liouville problems and in particular we also went through the Fourier series which provides some additional facility other than what any family of Eigenfunctions of a Sturm-Liouville problem could offered.

Towards the end we also discussed how to extend the idea of Fourier series to a function which is not periodic which is defined over the entire real line and it is a non periodic function for that we ask this question how to apply the idea of Fourier series to a non periodic function over an infinite domain.

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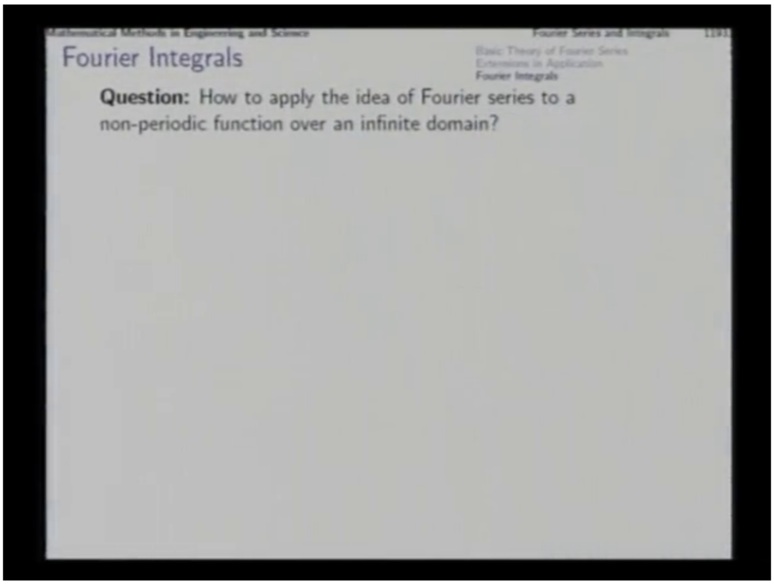

So, for that we constructed the Fourier series over a finite interval of length 2 l and then we let the variable l increase and tend to infinity and as a result we arrived at this representation of the function which is the Fourier integral.

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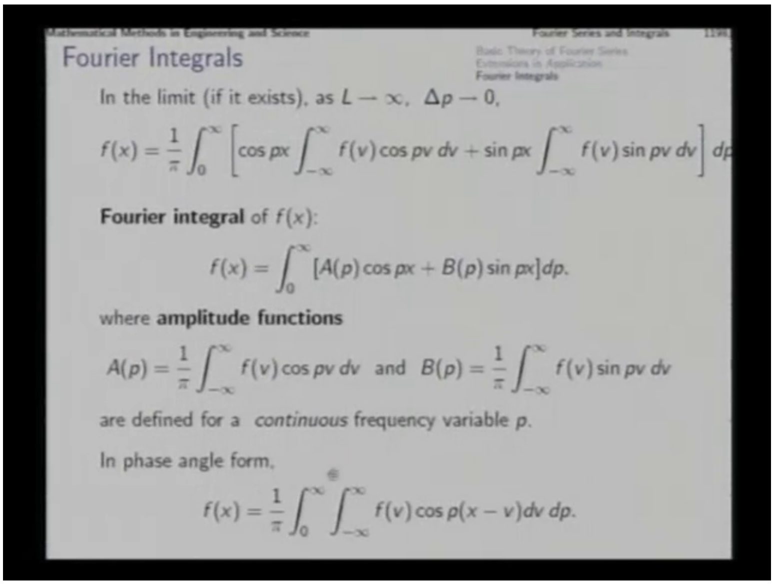

Now, in the Fourier integral this is the coefficient available in the form of an integral and similarly this integral over an infinite domain of the continuous variable v. So, these 2 integrals are the coefficients.

And the Fourier integral representation turns out to be a superposition of infinite such components with continuously changing value of p. Now in the case of the Fourier series this integral was a summation. Now here the summation becomes a sum of continuous components and therefore, you get an integral and in the place of the 2 Fourier coefficients we have got these in a similar manner in which as we got it in the case of Fourier series. So, this Fourier integral can be now represented as this integral with the coefficients as coefficient functions earlier these coefficients of these coefficients in the case of Fourier series where A n and B n that is they were indexed.

Now, these are continuous functions not for n equal to 1 2 3 4 and so on, but p which varies continuously from 0 to infinity. So, in place of Fourier coefficients indexed with an integer variable n we have got Fourier coefficients which turn out to be continuous function of the frequency variable p and these are in a way the corresponding amplitude functions a p and b p which are these 2 integrals. So, the only difference here is that this is an integral rather than a sum of terms for large number of ns and the coefficients are defined not for different values of n, but for continuous variables frequency variable which is p.

Now, using this itself you can also represent this large double integral in a somewhat different manner in which you can push cos p x inside the this integral and sin p x inside this integral and then you get this form of the Fourier integral expression which is called the phase angle form. In which you find that cos p x cos p v plus sin p x sin p v is replaced with this and this turns out to become a straight forward double integral.

Now, from this double integral with a little more work we can define an integral transform and how it is done is here.

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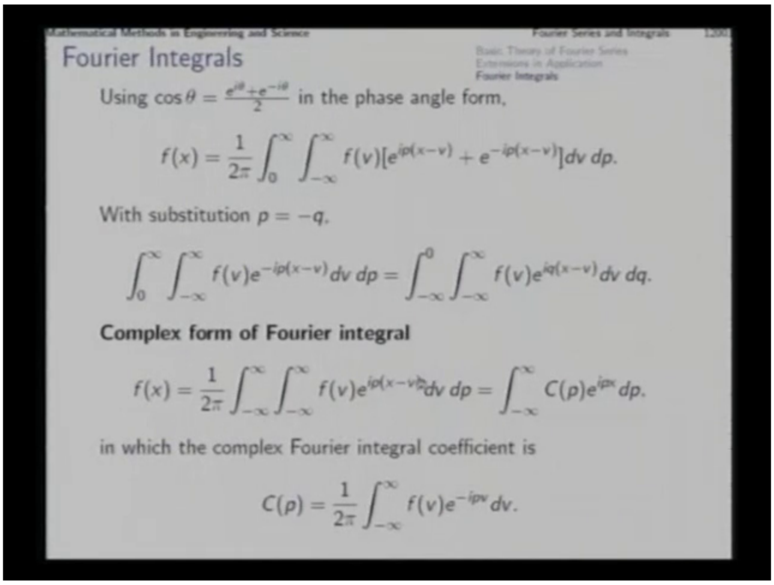

So, in this cosine we can replace in this manner cos theta is e 2 the power i theta plus e to the power minus i theta by 2. So, if we expect if we substitute this expression for the cosine term there then x by 2 comes here and inside, we have got e to the power i p x minus v plus e to the power minus i p x minus v and then we find here that in this double integral d v for that differential the corresponding variable v varies minus infinity to the plus infinity and this frequency variable p varies from 0 to infinity.

Now, this term in the integrant can be omitted if we consider this limit also from minus infinity to infinity that gives us a more symmetric representation to see that what we do is that we can substitute p equal to minus q so; that means, that for this integral if we try to separately evaluate this component then in place of minus p. We will have q here in place of d p we will have minus d q and therefore, the limit of this integral over p will become from 0 to infinity because of the sign change it will become minus infinity to 0 that is first of all it will become 0 to minus infinity and then because of the sign change we will have the limit reversed.

So, now if we recognise this that the integral of this term f v into this turns out to be actually the integral of this same thing over minus infinity to 0 then the addition of the these 2 terms is the addition of an integral from minus infinity to 0 and then from 0 to infinity. That means, this integral and this integral together will give the complete integral from minus infinity to plus infinity and that gives us the complex form of the Fourier integral in which it is a single exponential function here with f v and we have this double integral and then the in the complex form of Fourier integral. This term also rather than cosine and sin has become this exponential function with imaginary index and this c p the amplitude function has become the integral that you get when you take only f v into e to the power minus i p v 1 and plus i p x remains here, right. So, from the original Fourier integral also we got similar a p and b p. Now, in a complex form the 2 coefficients combine together to give us the complex coefficient function.

Now, this actually holds the root of the definition of the Fourier transform let us see that directly here.

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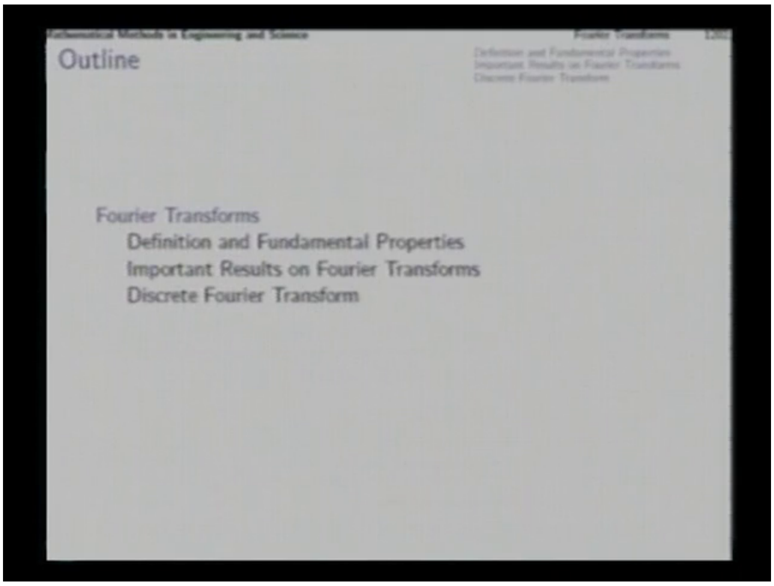

In this chapter, we will not go into great detail, but we will just see how the integral transform called Fourier transform gets defined from the complex form of the Fourier integral.

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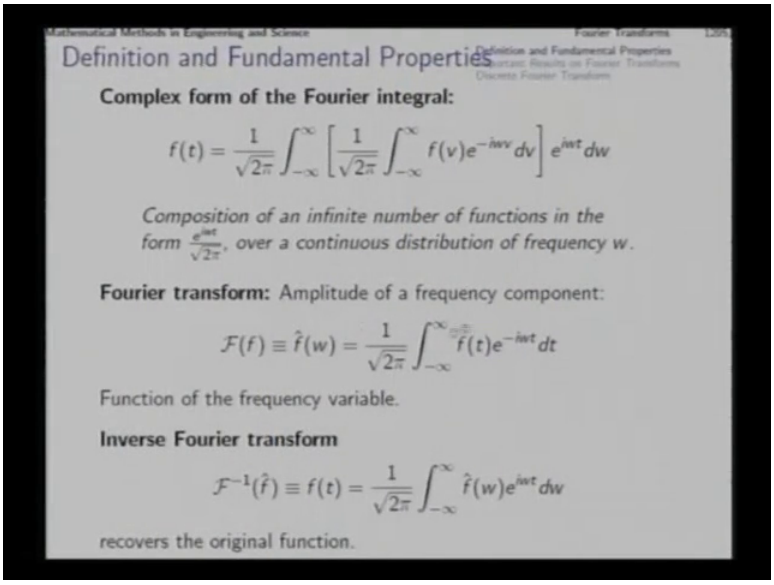

So, with a different notation here in place of x, we have got t here in place of p the frequency variable we have got w here and. So, on otherwise it is the same expression and the 1 by 2 pi that factor has been split into 2 parts; one is here; 1 by root over 2 pi and the other is here.

Now, here this is the same earlier complex coefficient function and, but with this factor here rather than the earlier factor that we had now this can be considered as the function representation in which we are talking about a composition that is addition the sum of an infinite number of functions which are these in the form e to the power minus w t by root over 2 pi that is e to the power minus w t divided by root over 2 pi 2 pi and the coefficients of these super position coefficient of these super position turns of to be this that is the amplitude function.

And amplitude function varies continuously with the frequency. Now this function representation is in a way the linear combination the composition of a large number of frequency components. The frequency components are here and the corresponding coefficients are here corresponding amplitudes are here and this amplitude terms out to be a function of the frequency that you get from here. So, this function of frequency which turns out to be the amplitude is the Fourier transform.

So, amplitude of a frequency component that is this bracketed term here and that is the Fourier transform and this is function of the frequency variable so; that means, a wave form has large number of components of continuous frequencies. And now for every frequency, this gives us the amplitude value and when with this amplitude value and this frequency we construct the term that component completely we have got this and such terms infinite number distributed continuously about the frequency variable is the complete function.

Now, if we rather have this Fourier transform in our hand then that Fourier transform sitting here with e to the power i w t multiplied and d w put here and integrated like this that is in this entire place if we put this Fourier transform, then we have got this. And this integral will recover the original function that is the inverse Fourier transform the way we defined earlier Laplace transform an inverse Laplace transform in a similar manner we have got this Fourier transform and inverse Fourier transform Fourier transform and inverse Fourier transform these are a quite symmetric in their look here you have got e to the power minus i w t to be multiplied and then integrated here you have e to the power i w t multiplied then integrated.

So, this gives you very similar expressions for evaluating the Fourier transform and the inverse Fourier transform. Now, these transforms are extremely in the useful in the solution of differential equations par partial differential equations and also in the field of signal processing we will not go into the detail of that because the main agenda of this lecture is another kind of function approximation in which the objective is different than all the function approximation techniques that we have discussed till.

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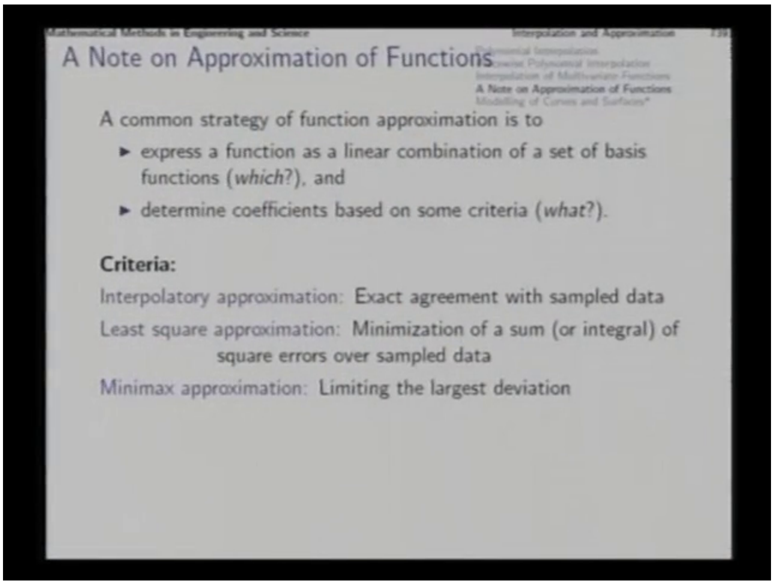

Now, earlier when we were discussing the interpolatory approximation of functions in that context we studied we discussed 3 criteria of approximation of functions one was interpolatory approximation which we studied in that particular lesson. And the other was least square approximation which we have studied in several sections in this course first the linear a least square problem which we studied in the module of linear algebra and then we studied the non-linear least square problem which we studied in the context of non-linear optimisation and finally, the same least square of approximation. We have been discussing till recently which are least square approximation based on the error value developed through integrals rather than sums of finite number of components finite number of error values. So, that least square approximation we have studied already.

Now, the third criteria of approximation of function is a minimax approximation in which the objective is not the sum of squares all over the domain sum of squares of errors all over the domain, but the objective function to be minimised is the maximum error. So, therefore, this kind of an approximation is called minimax approximation and one particular family of Eigenfunctions that is the solutions of Chebyshev equation and called Chebyshev polynomials give us specific interesting properties which are directly useful in the case of minimax approximation.

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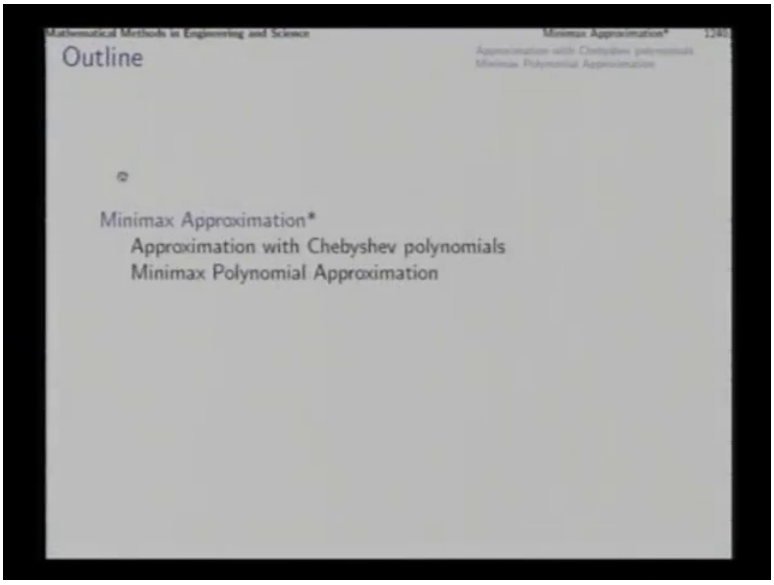

And therefore, this topic of minimax approximation we start with a discussion of Chebyshev polynomials.

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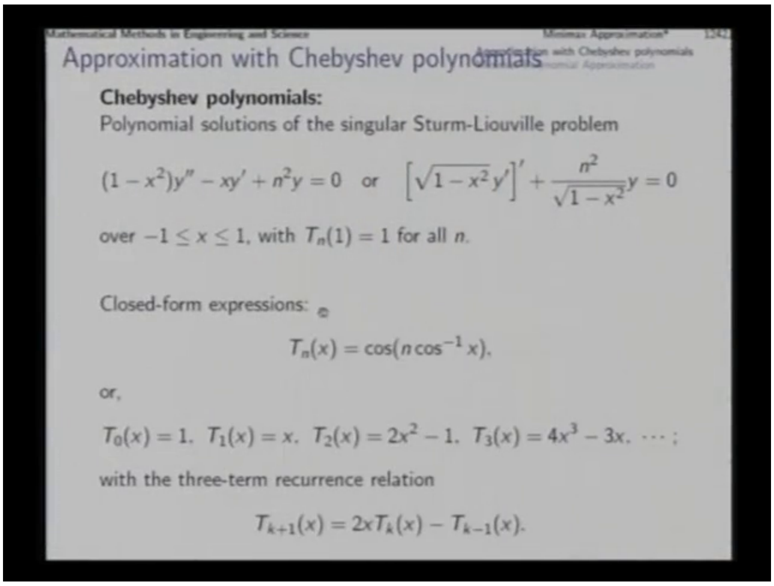

This differential equation resemble the Legendre equation to a great extent in the case of Legendre equation we had a 2 here.

Minus 2 here we got minus 1 now in place of this minus 1 if we had minus 2 then we found that this minus 2 x turned out to be exact derivative of this and therefore, these 2 terms together in the case of Legendre polynomial Legendre equation turned out to be the perfect differential of perfect differential coefficient of 1 minus x square into y prime now in this case that 2 is missing. So, it will this much right now is not in the self adjoint form right now not in the Sturm-Liouville form.

So, if you can; if you divide the entire equation with this square root of 1 minus x square then you will find that here then you will have square root of 1 minus x square and here you will have minus x by square root of 1 minus x square and in that case the terms together will be in the self adjoint form and together they will give us that derivative of this combined term and through the division here you will get this.

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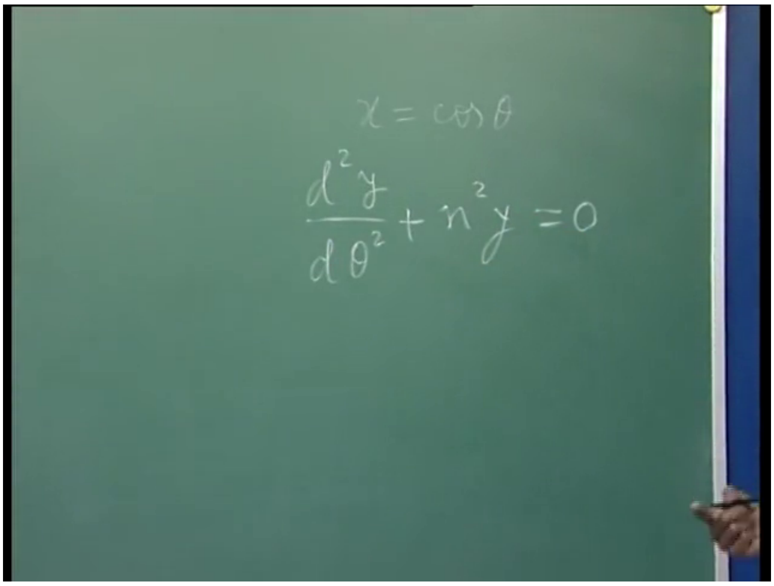

Now, there is one interesting point here that if we change the independent variable here and make this substitution x equal to cos theta, then this differential equation quite easily boils down into this differential equation is the same differential equation which we saw in the case of Fourier series. Now from here, we can very easily see that the solutions of this will be cosine n x and sin n x, they will be the linearly independent solutions of this differential equation.

Now in particular the solution sin n x will be very complicated function, but the other family of solution cosine n x that is part different values of n cos 0 that is one then cos x sorry not x, but theta. So, the solution of this will be sin cosine theta and sin theta. So, the term sin theta when finally, expressed in terms of x back by putting theta equal to cos inverse x will turn out to be quite complicated functions on the other hand the other solution that is y equal to cos theta will turn out to be polynomial sin x. So, you can see that.

So, that is why we are particularly interested in the polynomial solution of this differential equation and note that this differential equation also will define a singular Sturm-Liouville problem just like the Legendre equation because here the function r x which appears here is 0 at x equal to minus 1 and at x equal to 1; that means, you do not need any boundary conditions to define the Sturm-Liouville problem. So, this differential equation will define a singular Sturm-Liouville problem over this interval Legendre equation also did the same thing.

Now, we are particularly interested in the polynomial solutions of this particular Sturm-Liouville problem and one family of solution of that will turn out to be polynomials solutions now the way we discovered this fact in the case of Legendre polynomials it is possible to discover the same fact for Chebyshev polynomials also in the same manner by consideration of the series solution and noting that for integer values of n the series terminates and gets you polynomials equations polynomials functions the same conclusion will be reached through this arrangement in which we replace x with cos theta and solution of this turn out to be cosine n theta sin n theta we discard that sin n theta because that is not that not going to give us polynomials solutions, but cosine n theta will give us polynomials solutions and for all values of n cosine n theta will have value one when we take x as 1.

So, the closed form expression for that particular family of solution which are going to turn out to be polynomials solutions is this that is t and x is cosine n theta theta is cos inverse x. Now, if we put n equal to 0 then we get T 0 which is cos 0 that is one if we put n equal to 1 you will get cos cos inverse x which is x if we put n equal to 2, then you have get cos 2 theta we know that cos 2 theta is twice cos square theta minus 1. Similarly cos 3 theta will be 4 cos cube theta minus 3 cos theta and so on.

So, we can also very easily prove that the family this family of functions T 0, T 1, T 2, T 3, etcetera will satisfy this 3 term recurrence relation that is T k plus 1 will be twice x T k minus t k minus 1 which means that after evaluating the 2 of them, we can evaluate others in terms of polynomials directly rather than writing multiple angle formulas of cosines from this recurrence relation; recurrence relation what other properties these polynomials have these solutions of the Sturm-Liouville problem have

Now, that with a little exercise we can establish a lot of properties of these polynomials in the exercises of chapter 30 nine and chapter forty of the text book we have the steps to establish many of these properties. So, it is expected that the student will go through those exercises and execute those steps to learn this; learn the derivation of these properties first hand here.

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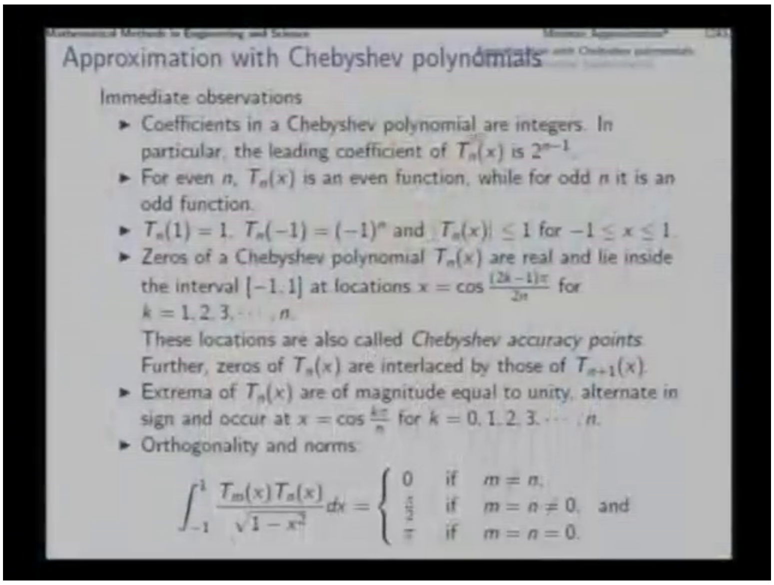

We will just summarise the immediate observations immediate properties for proceeding further coefficients in a Chebyshev polynomials are all integers which is different from the Legendre polynomials in particular the leading coefficient of T n will be the 2 to the power n minus 1 for even n that is for T 2, T 4, T 6, T n x even function because they will have only the even powers while for odd n it is an odd function this is the same as the Legendre polynomials in the case of Legendre polynomials also we noticed this property all the Chebyshev polynomials will have the value 1 at x equal to 1 that is obvious because at x equal to 1 cos inverse x will be 0.

So, for all values of n n into 0 is 0. So, cos of 0 will be 1. So, value of this all these polynomials for x equal to 1 will turn out to be 1; that means, all of them are constant one then we find that at minus 1 the value will turn out to be this minus 1 or 1 depending upon the whether n is odd or even this also is famous Legendre polynomials Legendre polynomials that family also had this property and this is a very interesting property that is the value the absolute value of T n x over this interval is always less than equal to 1. This also is there in the case of Legendre polynomials again 0s also the distribution of 0s of a Chebyshev polynomial also shares a similarity with Legendre polynomials the 0s of the Chebyshev polynomials are all real and lie within this interval.

But the locations of the 0s of Chebyshev polynomial polynomials turn out to be to the turn out to have very particular significance they are at these locations and these locations are called Chebyshev accuracy points why that we will explore further. Further another interesting thing is that the 0s are interlaced by the 0s of T n the n th Chebyshev polynomial are interlaced by those of n plus T n plus 1; that means, the n th degree Chebyshev polynomial will have n 0 and every 0 of the T n function will turn out to be enclosed by 2 consecutive 0s of the next order Chebyshev polynomial this interlacing property is also interesting.

Now, we find that these 3 properties except this part the Chebyshev polynomials share with the Legendre polynomials and this next property extrema of Chebyshev polynomials have a very unique property with no other family of function posseses and that is all extrema of the Chebyshev polynomial T n are of equal magnitude magnitude is unity and they alternate in sign and occur at these values of x 1 maximum. Next minimum next maximum next minimum all these maxima and minima of the same absolute value 1 that is maximum value 1, then minimum value minus 1, then maximum value 1 minimum value minus 1 maximum value 1 minimum value minus 1 and so on.

So, this is a very unique property and this property is called the minimax property, we will see the significance of this property orthogonality and norms. So, this is the integral here is equal to 0 for m not equal to n that is the outcome of the result on the Sturm-Liouville problem here you see that the function p x is 1 by root over 1 minus x square. So, that is why 2 distinct solutions that is 2 solutions of this Sturm-Liouville problem corresponding to different values of n that is different Eigenvalues distinct Eigen values will turn out to orthogonal with respect to the wet function 1 by root over 1 minus x square.

So, the statement of orthogonality of Chebyshev polynomials turns out to be like this. So, this is the statement of orthogonality and the definition of norms will come when you put m equal to n and for m equal to n which is not 0 we get it pi by 2 and for m equal to n which is 0, we get pi this is similar to what we found in the case of the Fourier coefficient.

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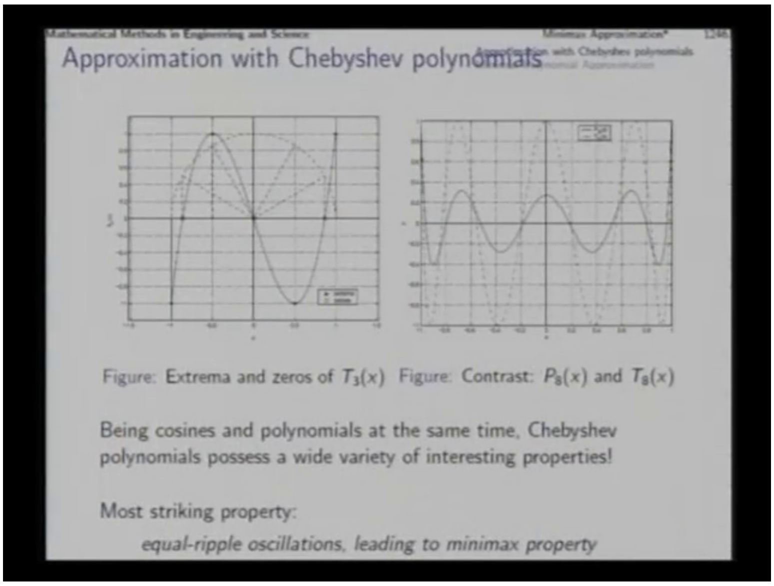

Now, with these properties we can have a closed look at one such sample Chebyshev polynomial between minus 1 and 1. So, between minus 1 and 1 the Chebyshev polynomials varies like this you see the 0s on extreme of this turn out to have a uniform distribution over the theta variable say for theta equal to 0 x is 1. So, the value of Chebyshev polynomial is one then for theta equal to 30 degree pi by 6 x is here cos of cos 30 degree that is root 3 by 2.

So, at that we have got a 0 are the Chebyshev polynomial. So, then 0; 30 degree than 60 degree at theta equal to 60 degree, we have got this negative value minus 1. So, here was 1 extrema value maximum next minimum. So, at 30 degree at 0, we have maximum value at 30 degree, we had 0 at 60 degree, we have the minimum value minus 1, then at 90 degree x equal to 0, we have got T 3 x also 0. So, that is the second 0 from this side, then again at 120 degree, we have got the maximum value at 150 degree, we have got a 0 at 180 degree, we have got this extrema value.

So, the 0s and extrema are distributed like this the extrema values have the same absolute value one this maxima at value one this minima at value minus 1 this is a distinct property which is not shared by any other family of functions and as you compare it with say Legendre polynomials we find that this is the graph of the 8 order Legendre polynomial and 8 order Chebyshev polynomials superposed super imposed. So, this is the graph of p 8 Legendre polynomial and this is the graph of Chebyshev polynomial you can see that the Chebyshev polynomials has this equal ripple property that is equal amplitude oscillations.

So, from here it goes all the way down the a case of Legendre polynomials as you proceed towards the centre of the domain near 0 the amplitude tends to come down compared to here it comes down towards the centre in the case of Chebyshev polynomials every extremum is at the extreme value of minus 1 or 1. So, 8 order polynomial it has got go going down going up going down like 8 such trends you have and all the 0s you can see here 1, 2, 3, 4, 5, 6, 7, 8 and they are distributed uniformly over the theta variable cos inverse x.

So, being cosines and polynomials at the same time Chebyshev polynomials possess wide set of interesting properties which have this unique application in minimax approximation the most striking property is this equal ripple oscillations leading to the minimax property.

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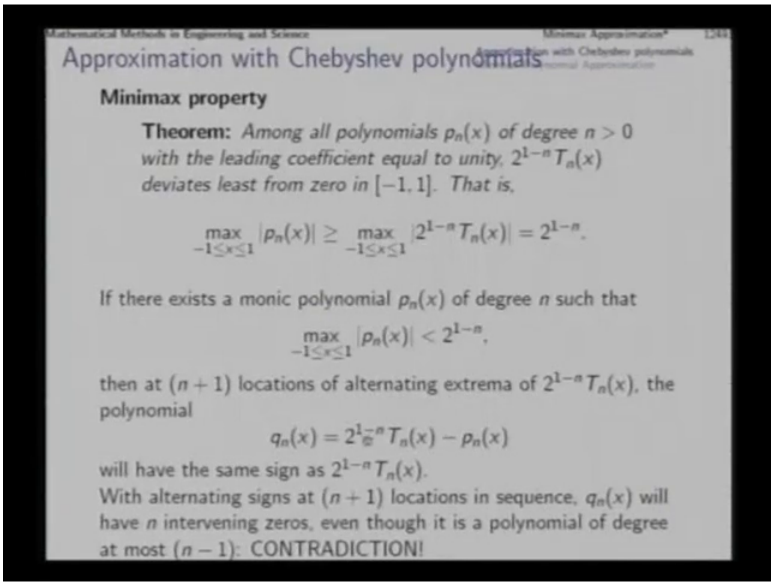

We will see this in 2 stages first what is this minimax property of Chebyshev polynomials the important result is that among all polynomials p n x of degree n with the leading coefficient equal to unity why this is needed because if we want to compare 2 polynomials then we can make no comparison if one of the polynomials can be multiplied by an arbitrary number.

So, we need to fix something for both the polynomials to make a worthwhile comparison. So, we say that we will fix the coefficient of the largest degree term there is we are talking about monic polynomials. So, the leading coefficient will have unity; that means, x to the power n plus a 1 x to the power n minus 1 plus a 2 into x p minus. So, and so on. So, the leading coefficient is one. So, among all such polynomials of degree n with the leading co coefficient unity that is among all monic polynomials of degree n this is the one which has the least deviation from 0.

Now, why this factor because we already know that T n x has the leading coefficient which is 2 to the power n minus 1; so, to make it monic, we must multiply it with 2 to the power 1 minus n. So, or divide with 2 to the power n minus 1. So, therefore, the monic version of the nth order Chebyshev polynomials is this and among all monic polynomials of nth degree the claim is that this one will have least deviation from 0 over this interval.

So, mathematically we can say that maximum over this is max this domain this interval of absolute value of p n x is always greater than equal to the corresponding value for this particular function. So, all other polynomials we will deviate more from 0 and this will deviate least and at least deviation turns out to be this now this equality part is easy to establish because we already know that T n x will deviate between minus 1 and 1.

That means the absolute value of the deviation of T n x is 1. So, absolute value of the deviation of this factor into T n x will be this factor itself this unity equality part is quite obvious for the inequality that is all other polynomials will deviate more we need to do something. So, for that we first assume let suppose there is some other polynomial other than this which has even less deviation. So, if there exist some monic polynomial p n x of degree n; which deviates less which deviate less than this if you assume that like that; then what we can do is that we considered the values of this candidate polynomial at the extrema of the Chebyshev polynomials.

Now, already we know that the nth degree Chebyshev polynomial has n plus 1 alternative extrema positive negative positive negative and so on; so, each with this absolute value that we already know; so, at those extremum points of the Chebyshev polynomial we consider the value of this. Now, these points are giving us the maximum deviation of this which is 2 to the power 1 minus n and we know we have assumed that this candidate function deviates less. So, if it deviate less, then it will never be able to cross the extremum value of this.

So; that means, that if you consider this difference then this part in the difference will always dictate the sign of the difference polynomials q n x right. So, if this is positive then whatever be the value of this the difference will be positive if this is negative then whatever be the value of p n the difference will be negative because this fellows value has been assumed to be less than the extrema value of this and we are evaluating at the extremum point,. Then this difference polynomial will have the same sign as the first part now we already know the nature of the signs of this in the sequence that is n plus 1 locations n plus 1 alternating extrema this function has. So, its sign will be alternating at that these n plus 1 locations positive negative positive negative and like that.

So, with alternating signs at n plus 1 locations in sequence T n x and therefore, q n x also will have n intervening 0s; so, because this q n x has the same alternation of sign as this because this one is going to dictate that the sign of q n. So, q n with alternating signs at n plus 1 locations will have n intervening 0s positive here negative here. So, in between q n x must have a 0. So, between n plus 1 extrema of alternating sign, it will not extrema of this difference function, but with alternating signs at 1 plus locations in sequence it will have n intervening 0s. But then that is something very awkward because both of these were monic polynomials which means that in both of these cases the coefficient of x n was unity and in that case in the difference the x n term will get cancelled and this polynomial is actually an n minus 1 degree polynomials, but then we have just now, found that it has n intervening 0s how can an n minus 1 degree polynomial have n 0s.

So, this leads to a contradiction which means that this assumption of a monic polynomial of less variation less deviation is certainly; so, through contradiction we get this result that among all polynomials of degree n it is the Chebyshev polynomial the scaled version of it to make a monic polynomial deviates least and this turns out to be the deviation now how do we utilise this property.

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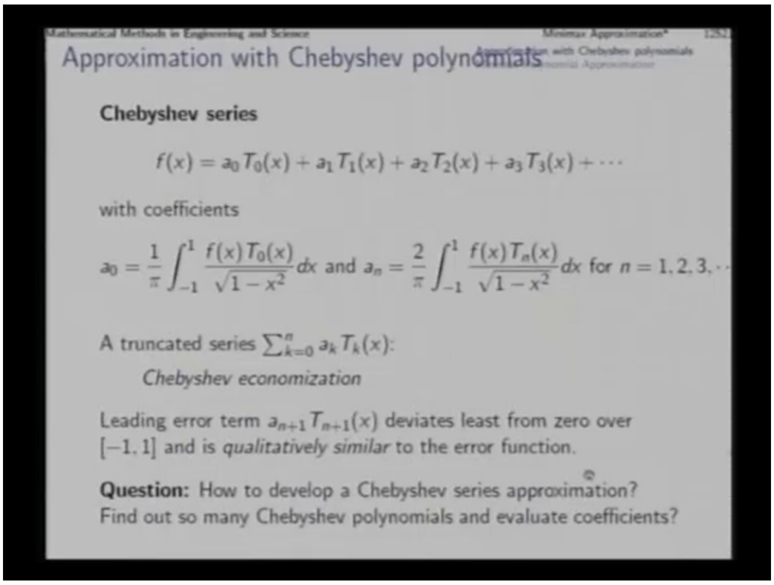

Let us explore with a linear combination of the Chebyshev polynomials with have this series which is called the Chebyshev series for which we can construct the coefficient a 0, a 1, a 2, a 3, etcetera as we have earlier studied that is how to construct the generalized Fourier coefficient for this Chebyshev series generate the Chebyshev coefficient for which we have these expressions; now typically we would not be able to construct this infinite series. So, we typically try to represent a function with a truncated series only up to a finite term and this approximation is called the Chebyshev economization we are representing the function in an economical manner.

Now, the leading error term in that case will be the next term that is up to the term T n we are already incorporating in this truncated series the next term which will be the leading error term will be this and we know from the foregoing discussion that among other among all the candidates for the polynomials this will deviate least from 0 over this interval and therefore, it is qualitatively similar to the error function. Then the question arises that with this how to develop a Chebyshev series approximation do we try to find out.

So, many Chebyshev polynomials and then evaluate the coefficients and then construct the series the truncated series while that could be an option the good news is that even that much computation is not really needed to be done there is a little shortcut to do that. So, first of all this is a little bit of pre processing involved.

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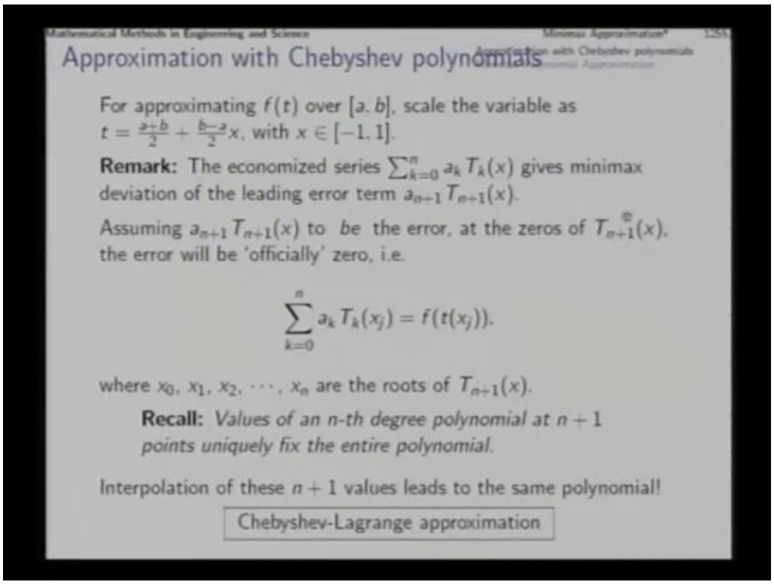

If the required domain of the function description function representation is not minus 1 to 1, but it is a to b then we always can scale it till the variable t to x with this in which x belongs to this, then onwards we work with this interval of x the economised series this one which we are constructing that will give the minimax deviation of the leading error term which will deviate least and that deviation will be the will have the equal ripple property which every Chebyshev polynomials has.

Now, since we are discussing with reference to the leading error term now for a moment if we concentrate on this leading error term only and pretend that this happens to be the actual error and assuming this to be the actual error we know that there will be n plus 1 values of x were this actual error officially turns out to be 0 why because T n plus 1 has n 0s. So, there are n values of x from minus 1 to 1 in that interval of interest in which this function is 0.

So, these are at the 0s of the n plus 1 th Chebyshev polynomials. So, at these values of x we find that when if we take the exact actual function value at these points then there will be the correct values through which we can interpolate a function. That means, that we can take these points these values of x and at these values of x which turn out to be the 0s of this we evaluate the function we have got the values of the function at n plus 1 values of x and we know that if this happens to be the actual error then the polynomial that will be interpolated through these n plus 1 points will deviate at least; that means, that after we have got the 0s of this function this Chebyshev polynomials n plus 1 degree and got the Chebyshev accuracy points the so called Chebyshev accuracy points, then the polynomial interpolated through these values at the Chebyshev accuracy points through any other means possibly with the Lagrange interpolation itself will give us the Chebyshev series approximation in terms of the actual function that we develop.

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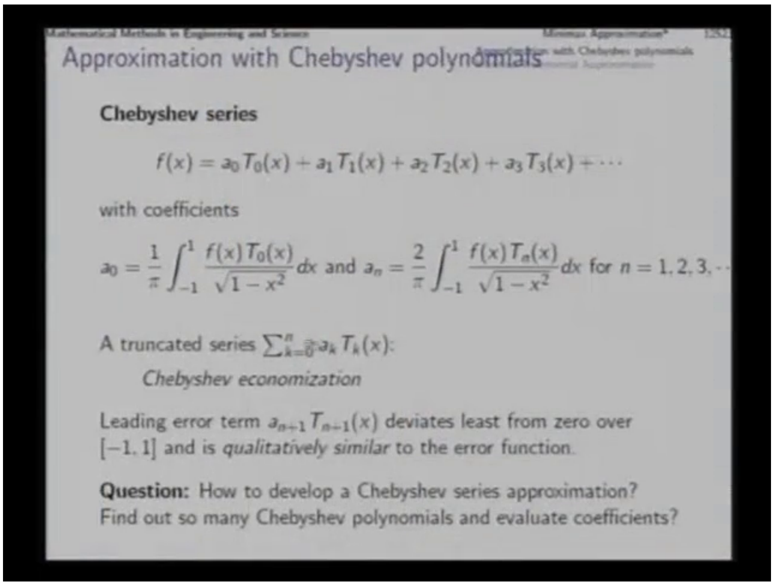

Because here we were seeing that this series will have 0 error at those values which are those values of x which are roots of this 0s of this.

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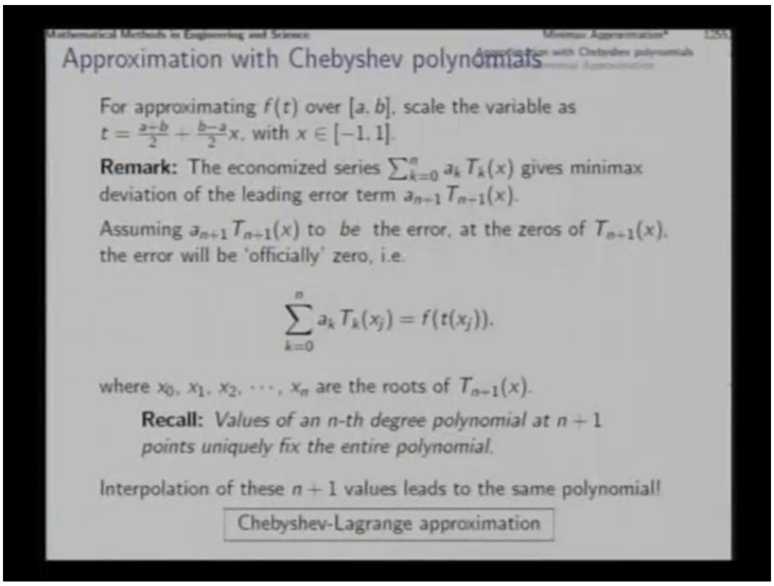

And now we are telling that through these same points with we can construct a polynomial which is through Lagrange interpolation.

Now, n plus 1 values of a an n th degree polynomial fixes the polynomial uniquely. So, therefore, whether we construct the series by finding out the Chebyshev coefficients a 0, a 1, a 2, etcetera or whether we construct that Lagrange interpolation over those points where we know the correct value of the function it will lead to the same polynomial. And therefore, from the foregoing theory Chebyshev polynomials we can just take at which points we give the correct values of the function and then from the values at those n plus 1 points we simply construct the polynomial through Lagrange interpolation this is called Chebyshev Lagrange approximation.

Now, this much on approximation with Chebyshev polynomials themselves now the minimax property goes beyond because here this assumption has worked that the leading error is the important component of the error, but that will not be always the case because in the actual complete series there could be components there could be contributions from T n plus 2 T n plus 3 and so on. So, if those components those contributions are significant then even though the approximation resulting from Chebyshev Lagrange consideration gives us qualitatively similar error values, but still it will not be the perfect minimax approximation even the perfect minimax approximation perfect minimax approximation you can construct.

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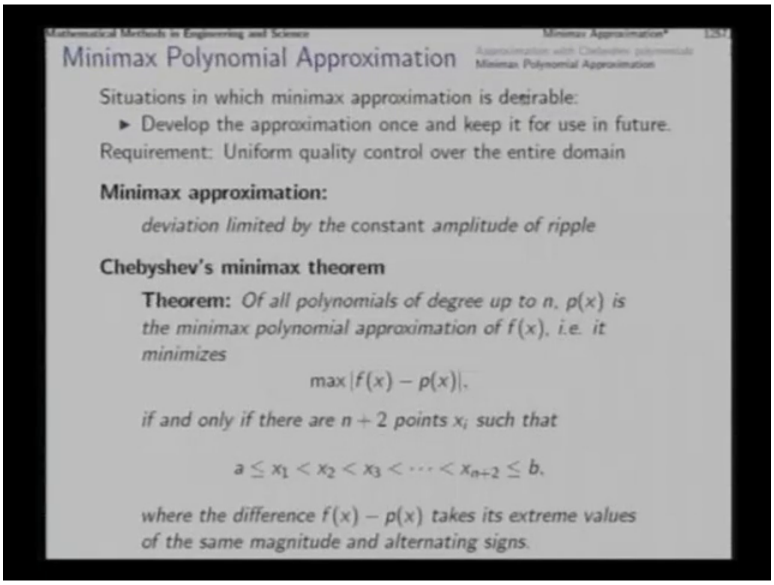

And that is based on this very important property very important theorem of by Chebyshev that is now before that make note that the situations in which the minimax approximation is desirable are those in which we want to develop the approximation once and keep it for use in future a priori.

We do not know at which point the function value will be needed and therefore, it will be a good idea if we can say something regarding the uniform quality control that the function evaluated by this method will never be too wrong too much mistaken that is there is a limit on the maximum deviation that it may have. So, minimax approximation gives us that guarantee that the maximum deviation is limited the maximum deviation is minimised and the interest thing is that that approximation is going to have this minimax property which has constant amplitude of the variation and this is the Chebyshev minimax theorem. That is how do you recognise that a particular represent has this minimax property the criterion for recognising that minimax approximation is the same as the maximum deviation going to be same in a absolute value at all the extrema that is equal ripple oscillation is directly related to the minimax property.

So, the Chebyshev minimax theorem says that of all polynomials of degree up to n a polynomial p x is the minimax polynomial approximation of a function that is it minimises this; the maximum deviation if and only if there are n plus 2 values of x where this difference assumes alternate maxima and minima. So, where that this difference takes its extreme values of the same magnitude and alternate signs; that means, the exact properties that we saw in Chebyshev polynomials that is it has extreme maxima and minima values of the same absolute value 1; 1 minus 1; 1 minus 1, they are; they were the extrema and that actually confers on the Chebyshev polynomials its minimax property similarly for any approximation the approximation being minimax is equivalent to its equal ripple characteristic in the following.

We will not really have a formal proof of the theorem, but we will in a way in a schematic manner we will see the line of proof which also gives us the seed of an algorithm to construct that minimum minimax polynomial approximation suppose you have got a function f x and for which you have got a polynomial approximation of the fourth degree which is the polynomial p x and the error function is f x minus p x.

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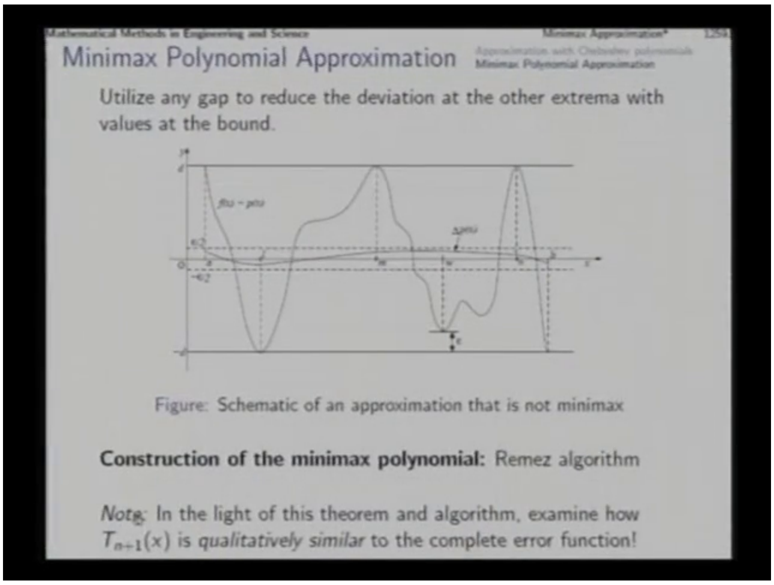

Now, the theorem says that p x is a minimax polynomial approximation of f x if the error function f x minus p x has n plus 2 that is 6 extrema of equal magnitude and alternating sign. So, d minus d, d minus d, d minus d 6 extrema of same magnitude and opposite signs if f x minus p x has that characteristic then p x is the minimax polynomial approximation of f x this is the statement of the Chebyshevs theorem.

Now, suppose the polynomial that we have constructed does not have this property that is it lacks that property only by a little that is five extrema are of the same value d minus d, d minus d; this particular extremum is not at minus d, but at a little above that is this is not really the farthest extremum value that we were looking for; that means, here in this case there is a gap. So, in that case this is not the minimax polynomial approximation.

Now, if there is any such gap now this idea of utilising any gap to reduce the deviation at the other extrema gives us the conviction that if not a formal proof of the minimax theorem and also in a way a very fruitful algorithm yes suppose for being the minimax polynomial approximation this extremum had to be here at value minus d now there is a gap it does not reach here. So, we say that we might increase this deviation that is rather than this value we can increase the deviation that is take this minimum even lower and at the cost of this deviation we try to reduce the deviation at other extrema, then it will reduce the maximum value of the error. So, for that what we do if this gap is a man is of magnitude epsilon then we construct a polynomial with largest possible value epsilon by 2.

So; that means, it is bounded within y plus minus epsilon by 2 we consider a polynomial of that type which is bounded in its value within plus minus epsilon by 2 that kind of a polynomial we call as delta p x that is the polynomial by which p x has to be changed and we imposed the condition that this delta p x. Other than having no value beyond this plus minus epsilon by 2 band is positive here at this extremum negative here at this extremum is then positive here at this extrema positive here at extrema and again negative here at this extremum.

So, at all the five extrema mini maximum maximum maximum at these maxima we are asking for positive; positive and positive values of delta p and at the 2 minima here and here we are asking for negative and negative values for delta p. Now note that for satisfying these requirements what delta p has to do delta p has to cross how many times from positive to negative to positive and then finally, negative here since the largest magnitude extrema. Here there is a pair of continuous maxima without there being a minimum in between this and this point and this point m and n the function the polynomial delta p did not have to change its sign. So, it had to change its sign once here second time here and third time here. So, size changing sign is perfectly fine fourth degree polynomial. So, we can certainly construct a for a fourth degree polynomial which is positive here negative here positive; positive and negative here.

So, that kind of a delta p, we can construct and we can have a hold over its maximum value also because if its maximum value turns out to be larger than epsilon by 2. Then we can multiply the polynomial throughout by a small number to bring it within epsilon by 2 and this limiting of its absolute value within epsilon by 2 will ensure that whatever damage the polynomial p x is caused through the addition of delta p will not damage this point too much beyond as other values come closer as other values come closer to 0 at other extremum points this one will not go beyond epsilon by 2. So, the best improvement at other points that is possibly is epsilon by 2 and even after the worst damage at this point it will not go beyond epsilon by 2.

So; that means, through this exercise we can develop a suitable delta p which when added to current p will give us a new polynomial in which all these extrema will be reduced in their magnitude all these extrema will be reduced in magnitude and perhaps this extremum will be in peace in is magnitude, but never beyond what will be the resulting value of the other extremum; that means, that any gap like this can be utilised to reduce the deviation at the other critical extrema.

Now, consider the same situation if this point were also here in that case in order to reduce all the extrema, we would need the values to come down at these 3 maxima and to come up at the these 3 minima if there were no gap, but that would not be possible because in that case delta p would be required to be positive negative positive negative positive negative a fourth degree polynomial can never show never have so much; so, many of oscillations.

So, therefore, if there is no gap like this then that will mean that we cannot improve the approximation further on the other hand any gap that exist like this we can always utilise that gap to reduce the variation at the other points. So, this is the you know schematic demonstration of the truth of the minimax polynomial in other standard books on approximation theory you can have a look at the formal proof incidentally this same reasoning turns out to be the basis of the construction of the minimum minimax polynomial approximation and that algorithm is called Remez algorithm the general remez algorithm is actually for rational approximation rational function approximation. But the same theme can be utilised for approximation with polynomials also and for this minimax polynomial approximation we will find that the Chebyshev polynomial turns out to be a very good starting point with the help of Chebyshev Lagrange approximation as we have seen some time back.

So, you will find that in the light of this general minimax theorem you will find that T n plus 1 x the n plus 1 th degree Chebyshev polynomial which we considered earlier was actually qualitatively similar to this complete error function with its equal ripple properties. Now, this completes our module on approximation theory next couple of lectures, we will devote on a quick recapitulation of the very important partial differential equations and after that couple of lectures on partial differential equations we will consider complex analysis.

Thank you.