Mathematical Methods in Engineering and Science
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Module – VII
Application of ODE's in Approximation Theory
Lecture - 03
Approximation Theory and Fourier Series

Good morning, this lecture we start with a few comments on what we discussed in the previous lecture.

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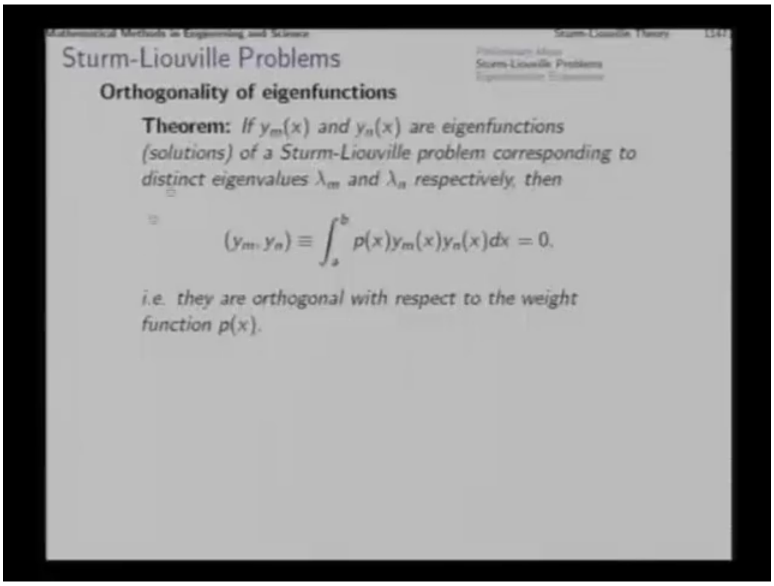

In the previous lecture, we studied Sturm-Liouville theory and the center piece of that theory is this theory. This says that if Y m and Y n are 2 eigenfunctions of a Sturm-Liouville problem corresponding to distinct eigenvalues, then these 2 are orthogonal to each other with respect to the weight function p x appearing in the Sturm-Liouville problem.

Now, we say that this particular property of the eigenfunctions of a Sturm-Liouville problem, enables us to use the eigenfunctions of a particular family as basis members for representation of continuous functions, because these will constitute a complete set of basis function for continuous functions with piecewise continuous derivatives. Now the question is that why do we really required orthogonality. Linear independence should be enough for representation of functions as we did in the case of vectors in ordinary vector spaces, to appreciate the importance of orthogonality let us see a situation in the case of ordinary vector space and to keep the discussion simple.

Let us take the vector space to be of dimension 2, which we can represent on the board. So, suppose we have got a vector space of dimension 2 and to represent all arbitrary vectors in that vector space we want to form a basis.

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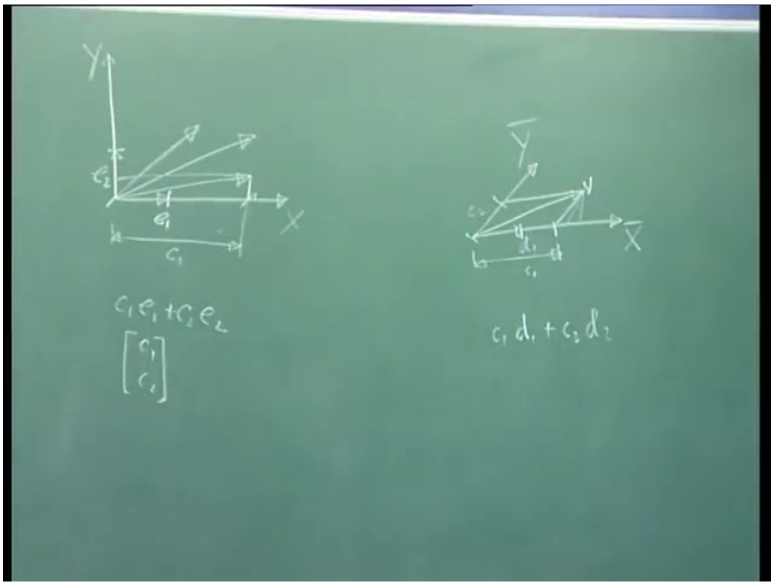

Usually in most of the scientific applications, we do take a pair of orthogonal basis members. In Cartesian geometry we take this as x axis this as y axis, the unit vector in this direction is taken as 1 basis member basis and the unit vector in this direction is taken as the other basis member right and in terms of these 2 basis members let us call them e 1 and e 2, and for that to represent this function this vector we draw a line parallel to the y axis and that cut here at right angle and whatever is the length here that is say c 1. Similarly for this this this length is a c 2 and then this vector get represented as c 1 e 1 plus c 2 each and since e 1 is 1 0 and e 2 is 0 1.

So, the representation of this turns out to be c 1 c 2 right. So, this we know, but this is orthonormal basis at least orthogonal will be looking for if the axes are at right angle; however, you will say that this was not necessary; as long as the 2 basis members are linearly independent which will make them the basis members, the representation should be all right and that you can see if you say that i will take a oblique coordinates and say that this is our X bar axis and this is our Y bar axis. Even in this pair of axis we could represent a vector say we want represent this vector for this purpose what we do? We draw a line from this point parallel to this and another line parallel to this and then say that this turns out to be this length turns out to be c 1 or more precisely you could say that this vectors length divided by the basis members length turns out to be the first coordinate and similarly in this case you will have the second basis member and here also you could say.

That suppose this unit vector is taken as d 1 and similarly this basis this here whatever is the basis member that is taken as d 2, and then you could still say c 1 scalar in to d 1 basis vector plus c 2 scalar in to d 2 basis vector in this direction will still give the representation like this it should be enough. So, in this vector space we find that as long as the 2 selected vectors are linearly independent, they will be able to form a basis and we can represent arbitrary vectors as linear combinations of the basis vectors why that is not enough for representation of functions.

In the case of functions why are we so, eager to ensure orthogonality of the basis members. We will see the reason if we consider the case that this vector in this vector space, if we want to represent the vectors with less number of basis vectors for example, in this case suppose we say that the vector could be in any direction, but we want to represent the vector only with the help of the first basis member and the second basis member we will not keep; in that case here we will still say that.

The best representation that this vector can get if we are going to use only the first basis member that will be c 1 e 1 because we have got this component, which is orthogonal to this basis member perpendicular, without any regard to this. Now in this case we cannot say that, in this case we cannot figure out in which direction to draw this line from here in this case you see we dropped a perpendicular in the case in which we know that the supposed complete set of basis members are orthogonal to one another in that type of situation we know that if we want to represent the vector with a linear multiple of even only then, we know that the length c 1 will be found if we drop of a perpendicular here.

And that will give us this. In this case if we say that this basis vector d 2 this basis vector d 2 is not in our hand we want to construct whatever best possible representation of this vector is possible only with the help of d 1, then you will not know in which direction to draw this line because this vector is not there at all in our hand. So, whether to draw this line here or here or here we do not know then we will say that we are confused because we do not know the complete set of basis members of which this partial recommendation is one part. Similarly say if you want to represent the vector from this point on earth to an aircraft there.

Now, if you are allowed see vectors then you will say that this much east this much north and this much upward. On the other hand if somebody says that no no no we do not want upward we want to find its position only in terms of east and north then what you do? You just take this much east this much north and that is the ground position exactly above which the aircraft is currently flying. This you could do because the 3 basis members are orthogonal if the 3 basis member of oblique of this shape, then you will not know how to drop this kind of a line.

So, which parallel pipette you need to make you do not know the angles, in the case in which you are already talking about an orthogonal set of basis members, then you know that the component along each axis component along each direction will be found in which you drop a perpendicular as here and therefore, the way we worked out the components along the basis members in the previous lecture will make direct sense.

Now, in the case of finite dimensional vector spaces it does not matter too much as long as you can supply a complete set of basis members. In the case of function space where the vector space is of infinite dimensions, whenever you want represent a function you have to represent with a finite subset of its dimensions and therefore, it becomes important that even without enumerating explicitly, all the basis members which you cannot because they are infinite you should be able to make a respectable representation and then you have this kind of situation where you want to represent a 2 dimensional vector with only one basis member. The other basis member is not taken into consideration or a 3 dimensional vector you want to represent only with its projection on a 2 dimensional sub space.

So, whatever function representation we make we basically try to project an infinite dimensional vector in a finite dimensional sub space, because the infinite series for computational purposes needs to be essentially truncated at some point. So, therefore, then it is a question of representing the functions with, limited number of basis member then it is very important to have the basis members which are mutually orthogonal and therefore, in the case of representation of functions in the function space, the orthogonality property of the basis members turns out to be extremely important. In contrast to the finite dimensional vector spaces in the ordinary linear algebra sense, in which the number of basis members was finite and it was most of the time possible to enumerate all the basis members.

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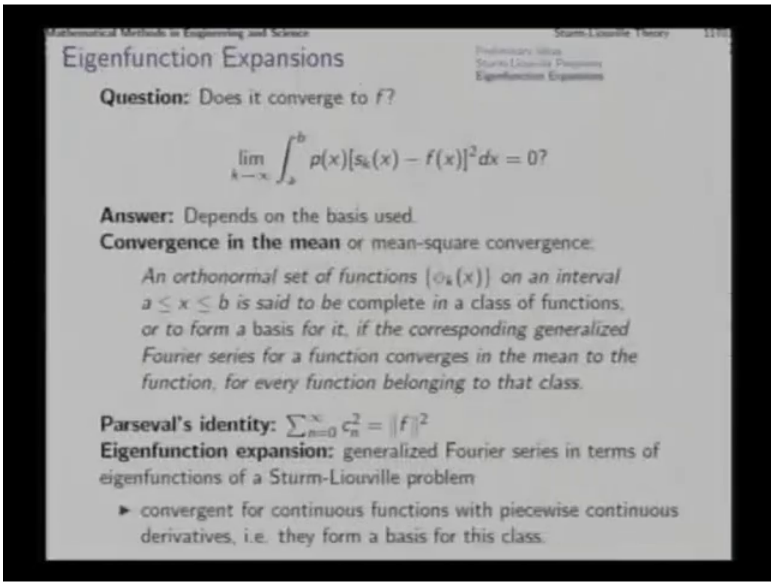

So, this is one important point because of which orthogonality is important. Now another point we made in the previous lecture is that eigenfunction expansions will give us generalized Fourier series its representation in terms of eigenfunctions of a Sturm-Liouville problem, which will be continuous which will convergent for all continuous functions with piecewise continuous derivatives that is for this kind of functions for this class of functions the eigenfunctions of Strum Liouville problem will turn out to give a complete set of basis members.

Now, there are. So, many Sturm-Liouville problems possible and each of them will provide us one family of eigenfunctions, which one which family to take. Now in different kinds of applications different families of such eigenfunctions are found suitable. So, in particular Legendre polynomials turn out to be special even among these special functions for example, all the functions which we develop like this as eigenfunctions of different Sturm-Liouville problems, they are called special functions.

So, Legendre polynomials are give us one family of such special functions; similarly laguerre polynomials will be another family of special functions special function another family of another family of special function. Among all these Legendre polynomials are somewhat more special, in the sense that in the case of any such family of eigenfunctions we find that they have an orthogonality property and that orthogonality property is with respect to the weight function p s which appears in the Sturm-Liouville problem. In the case of Legendre polynomials the corresponding p s is unity and therefore, the orthogonality in the case of Legendre polynomials turns out to be with respect to one.

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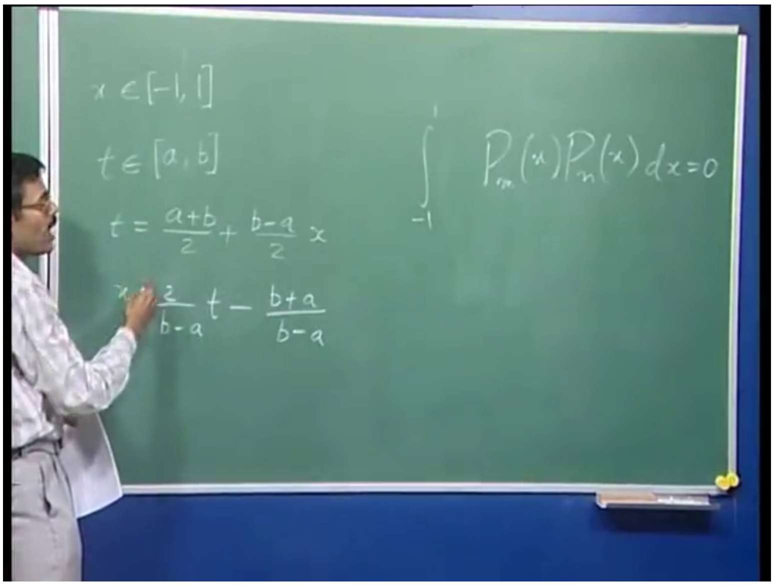

That is this P x is unity there and you have the orthogonality property as simply this that is this orthogonality of Legendre polynomials is defined with respect to unit weight function.

So, that makes orthogonality even more special among the families of special functions; however, as we have earlier seen that they are orthogonal over the interval minus 1 to 1 right and therefore, Legendre polynomials in the form of their linear combinations can represent the continuous functions over this interval.

Another question arises that whenever we want to represent a function, all the time we do not want to represent them over this interval only. Sometimes we may need to represent it represent a function or represent some functions over an interval, which need not be this, but that is a minor issue because suppose we take x in this interval for the purpose of use like this, that is for the reference of the Legendre polynomials.

Now, whatever is our domain of interest, we can say that another variable a varies within this interval and between this variable which will fit the domain of orthogonality of Legendre polynomials and this variable which is the which is in the domain of our interest we can directly establish a scaling and that scaling will be given as.

Now, see if you take x equal to 0 then you get a plus b by 2 the midpoint of this interval, and if you take x equal to 1 then this minus a by 2 and this a by 2 will go up and you will have b by 2 plus b by 2 which is b. So, for 1 you have b here, for minus 1 similarly minus b by 2 and plus b by 2 will go and you will have a by 2 minus minus plus and another a by 2 which will give you a.

So, over this interval if you apply this reparameterization then you get the correct domain for t. So, the inverse also you can establish that is if you need to transform from given values of t to x so; that means, that if you need the value of a function at t equal to something, then from the inverse expression which will be this. So, that value of t you put here, get the x and with that value of x in this interval you use the theory to get the coefficients and then you get a polynomial and that polynomial in terms of x will come to you will appear and then in that in the place of from there you make a conversion from x to t and then you get the expression in terms of t. But you can do this reparameterization this scaling of the variable as long as that interval of your interest is finite.

If you need to represent functions over an infinite interval which can happen in 2 ways one is semi-infinite and the other is infinite.

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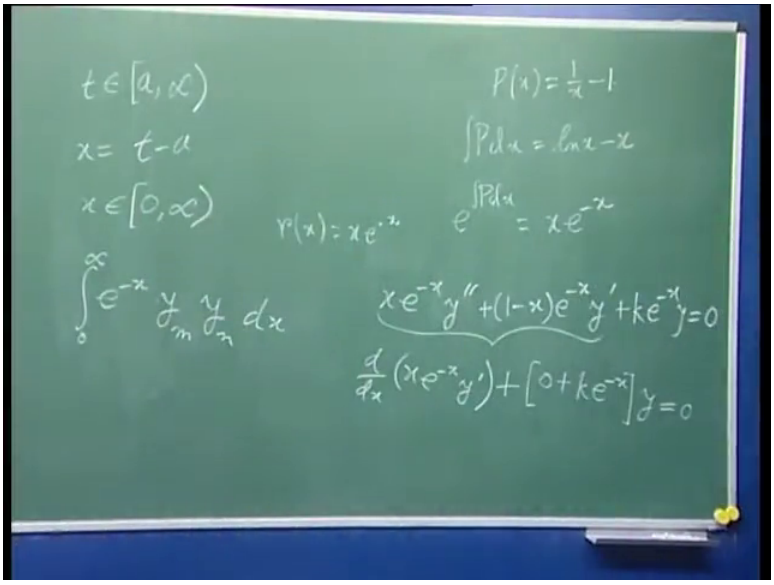

If your domain of interest turns out to be this, then what you do? No amount of rescaling will make this infinite interval shrink into the shrink to the finite interval of minus 1 to 1 and for function representation over this intime infinite interval, you will not be able to use Legendre polynomials as it is. So, then you look for some other family of eigenfunctions.

One issue is very clear that if you use the rescaling which is like this, then over t or say let us say call this variable as t call this variable as t. So, if you use this rescaling then as t varies from a to infinity, x will vary from 0 to infinity. So, which is a semi infinite interval right.

Now, in this you will need a family of eigenfunctions which will be orthogonal over the interval 0 to infinity and if you look for the suitable family for that purpose you see here this equation.

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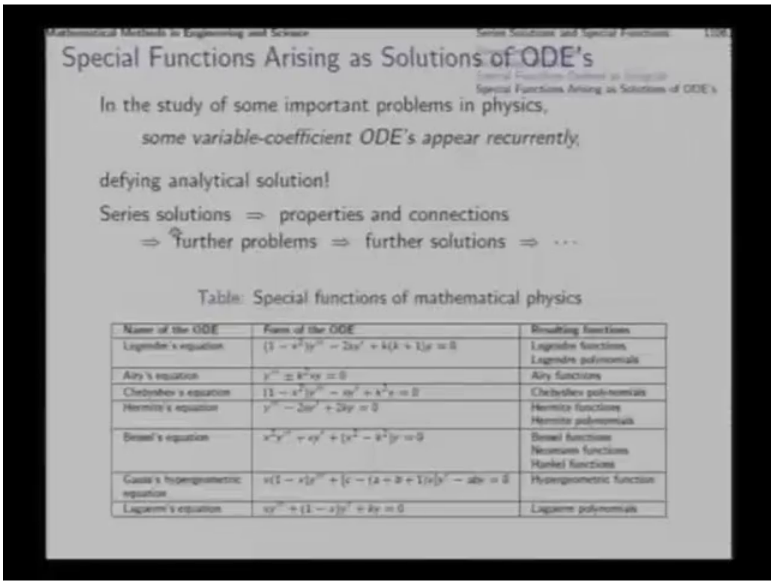

If we try to put this in the self adjoint form or the Sturm-Liouville form in the form of Sturm-Liouville equation standard equation, then what we need to do? We first need to divide this with x getting the coefficient of y double prime as 1, then whatever will appear here 1 minus x by x that is p and then P of x will turn out to be 1 minus x by x which means this and therefore, we will get integral P d x that will turn out to be integral of this which is l n x minus x and then we will get the integrating factor, which will be e to the power this; that means, e to the power minus x into x.

So; that means, this. So, that is normal form. So, the standard form the differential equation needs to be multiplied with this integrating form integrating factor to cast it into the self adjoint form or the standard form of the Sturm-Liouville equation. So, with x the standard form or the normal form is already multiplied when you have this because to get the normal form we had to divide by x. So, further we need to multiply it with e to the power minus x as we do that as we multiply this equation throughout with e to the power of minus x then we get this equation and note that that term from here to here turn out to be exact derivative of x e to the power minus x y prime we can verify it.

Now, x into e to the power minus x into y double prime and then y prime into derivative of this, derivative of this will be 1 into e to the power minus x which is here and then x into e to the power minus x with the negative sign, x into e to the power minus x with a negative sign that is here. So, the terms from here to here turn out to be this and then we have here 0 plus this y is equal to 0.

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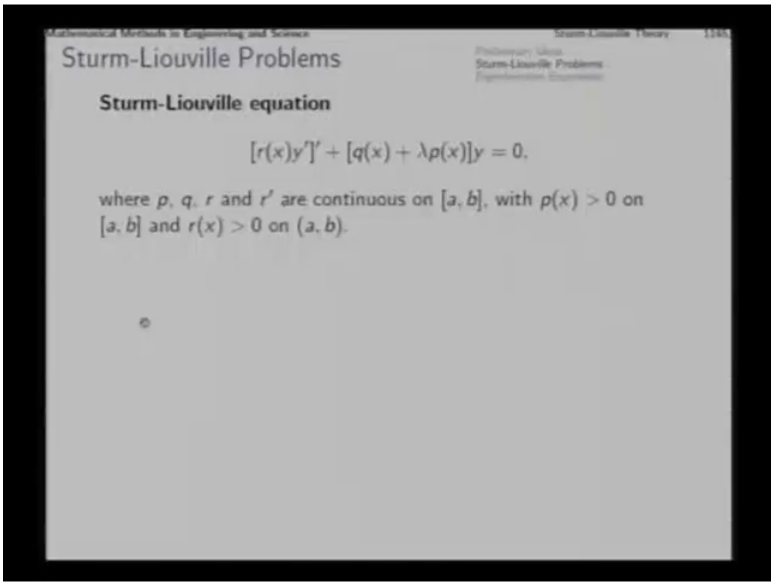

So, for this problem we find that r x in the Sturm-Liouville equation r x in the Sturm-Liouville equation turns out to be x into e to the power minus x ok which is 0 at x equal to 0, and also at x equal to infinity.

At x equal to infinity you will find that you have got infinity into 0 or infinity by infinity form, but you can verify that its limit will be 0. So, its limit is 0. So, this function r x is 0 at x equal to 0 and in the limit it is 0 at x is equal to infinity also. Now since r x is 0 at the 2 endpoints of this interval; that means that this differential equation that we have will define a singular Sturm-Liouville problem, over this semi-infinite interval with no boundary conditions necessary.

And therefore, its solutions laguerre polynomials will be orthogonal mutually orthogonal over this interval with respect to the weight function p x which is e to the power minus x here. So, in the case of laguerre polynomials, the statement of orthogonality will be this the orthogonality will be with respect to this, but with respect to any function any suitable weight function that you get it must be positive definite function which it is.

So, you will find the orthogonality in this manner, over this interval. So, if you want to express if you want to represent and manipulate functions over this semi-infinite intervals, for that purpose you take not Legendre polynomials, but laguerre polynomials. Similarly if you want the function representation over infinite interval infinity on both sides, then you look for a different family of Sturm-Liouville different function of the family of eigenfunctions solutions of a different Sturm-Liouville problem.

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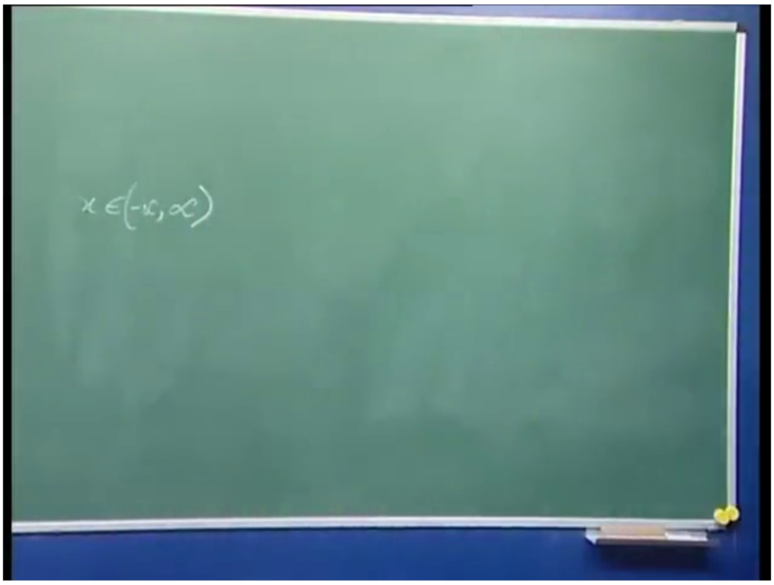

Which will be orthogonal mutually orthogonal over this entire interval and for that also we have one such differential equation here you find Hermite equation.

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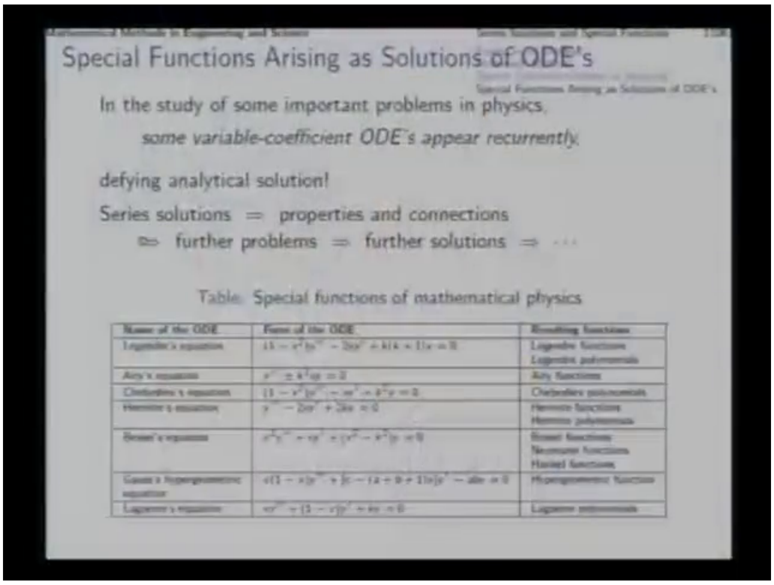

If you multiply this entire equation with e to the power minus x square then see what we get.

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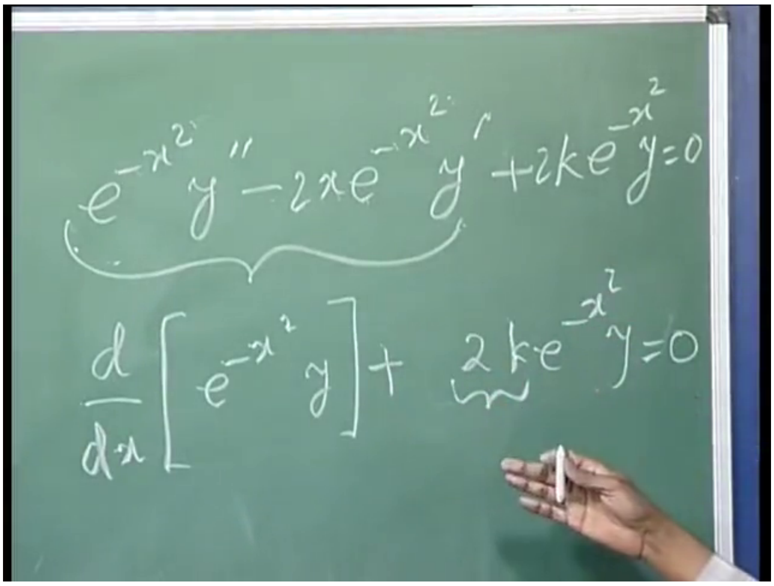

If you multiply this with e to the power minus square this Hermite equation then we will get do you note do you notice that this is the exact differential coefficient of something; because the derivative of e to the power minus x square is e to the power minus x square in to the derivative of this, which is minus 2 x which is sitting here.

So, this entire stuff from here to here will be the derivative of e to the power minus x square into y prime, plus here we have got this and with this eigenvalue e to the power minus x square will be the weight function and therefore, in the case of hermite polynomials, which will be the solutions of the singular Sturm-Liouville problem defined by this over this entire interval, those polynomials hermite polynomials will be orthogonal with respect with each other orthogonal to each other with respect to the weight function which is e to the power minus x square.

So, this way for infinite interval we can use hermite polynomials, for semi-infinite intervals we can use laguerre polynomials the solution of this and for a finite intervals after rescaling we can use Legendre polynomials themselves. Now for a finite polynomials for finite interval there are other proposals also possible and so, it could be for some infinite cases also and there for different kinds of purposes we look for different families eigenfunctions. To such further special cases we will discuss in this lecture and in the coming lecture and that will give you something more than what is Sturm-Liouville itself gives Sturm-Liouville theory gives us first orthogonality and then completeness of the basis and further this least square approximation and in that sense.

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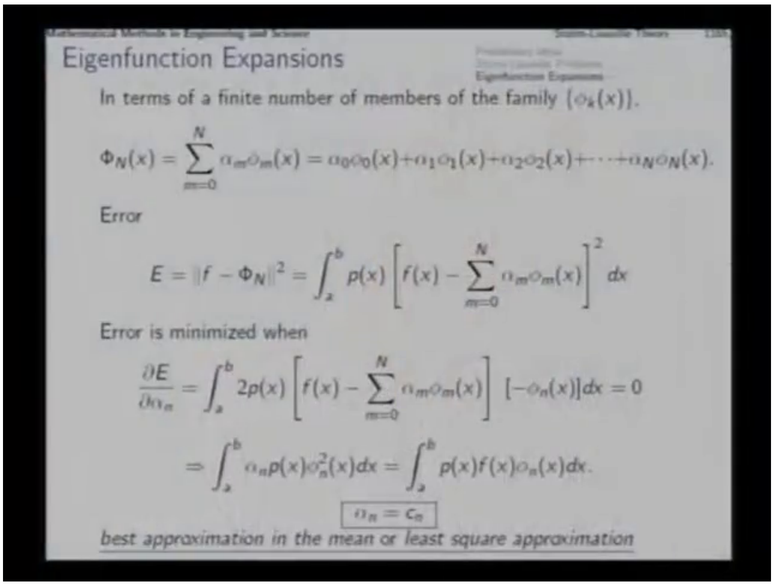

We have got in the Sturm-Liouville theory in the eigenfunctions of the Sturm-Liouville problems, a handle a tool to make least square approximation of functions in the integral sense. Long back when we were studying the interpolation and approximation of functions, in that context we discussed that interpolatory approximation is just one way of function approximation. Another very common way of function approximation is least square approximation.

Now, in the least square approximation when the squares are finite in number and collected over discrete samples, then we have one way of making the least square approximation in terms of the finite terms and in terms of integrals of the error we have got the least square approximation from the Sturm-Liouville theory. Other than least square of approximation of continuous functions with piecewise continuous derivatives other than this, that is after assuring this much what more we can ask for there are to particular themes that we will be exploring fresh further. One is that can we represent functions which have discontinuities and that gives us our next topic, which is Fourier series.

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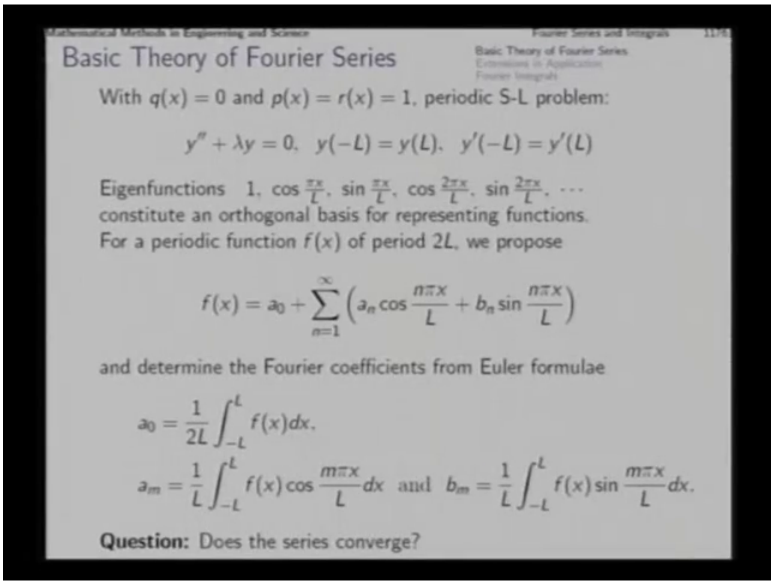

For this purpose for exploring the Fourier series let us take this differential equation. If we consider this as a Sturm-Liouville equation, then we find that u x is 0, p x and r x both are 1 r x is 1 because the coefficient here is 1 and in the it is already in the Sturm-Liouville form and this is simply lambda y rather than q plus lambda p y. So, q is 0 and p is 1. So, this is a Sturm-Liouville equation with eigenvalue lambda and p x q x r x like this.

And this tells us that r x is 1 which is constant. So, whatever interval we define over that r x is going to be constant in particular if our interval is a b then r a and r b are equal. So, with r a and r b equal we can define a periodic Sturm-Liouville problem with this kind of boundary conditions y a is equal to y b and y prime a is equal to y prime these are the periodic boundary conditions which define a periodic Sturm-Liouville problem with this differential equation with this self adjoint ODE. Now here the interval a b is minus L to L of length 2 L now we can find out that eigenfunctions of this turn out to be 1 cos pi x by L sin pi x by L cos 2 pi x by L sin 2 pi x by L and so on.

And this family of functions will constitute an orthogonal basis for representing function so far so good and this family of functions will also give us the least square approximation of functions with limited number of basis members consistence considered which every family of eigenfunctions Sturm-Liouville problem must give.

But this particular set of basis members this particular family of basis members offers something more it offers another facility.

So, if you want to represent a periodic function of period 2 L, because we have apply periodic boundary condition. So, if we want to take a periodic function then whatever is the representation over this particular period minus L to L that same thing will go on continuing from L to 3 L 3 L to 5, L and on this side minus 3 L to minus L minus five L to minus 3 L and so on. So, if you have got a periodic function of this period then we can propose the function in this manner as an infinite sum of as an infinite linear combination of these basis members, and we can determine the Fourier coefficients like this, which we can determine which we can derive this Euler formulae can be derived from the standard Sturm-Liouville, that we discuss in the previous lecture.

And. In fact, these are the precursors of the general Sturm-Liouville and that is why this this Fourier coefficients and Fourier series were developed earlier and therefore, when the more generalized eigenfunction expansion was developed by mathematicians, then the corresponding series was called the generalized Fourier series this is the original Fourier series.

Now, you might make a note that in the case of the coefficients of cosine and sin terms here, we are dividing by L to get this coefficient, but in the case of finding the coefficient rela corresponding to the first eigenfunction one, we are dividing this integral by 2 L. The reason is that the norm of this member in the family is 2 L square root square root of 2 L in these cases the norm is square root of L.

Now, with these coefficients defined according to the rooting (Refer Time: 34:00) of Sturm-Liouville theory, you will get the Fourier series of a function which is periodic with period 2 L. Now the question is, till now we have been discussing all those facilities that Fourier series gives a which a which any suitable family of eigenfunctions of a suitable Sturm-Liouville problem would give anyway, but Fourier series offer something more. Fourier series will give us the convergent series representation for even certain discontinuous functions, within which in general eigenfunctions of a Sturm-Liouville problem need not give or may not give is not guaranteed to give.

So, that is something which Fourier series gives in addition to which were which it must give as a family of eigenfunctions of a Sturm-Liouville problem. The additional facility that Fourier series gives is ensured by this particular result which you get if the function satisfies Dirichlets conditions.

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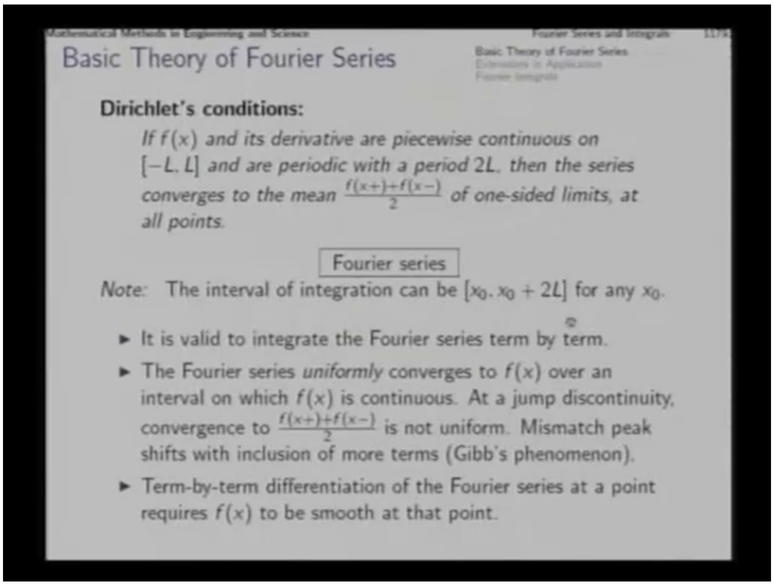

If f x and its derivatives is if and its derivative are piecewise continuous on the interval and are periodic with a period 2 L then the series given earlier converges to the mean of one sided limits at all points; that means, that derivative is piecewise continuous that is needed for any Sturm-Liouville problem, in the general case of the Sturm-Liouville problems for the function that we want to represent was needed to be continuous itself the function itself was needed to be continuous.

Here we are saying that even if the function itself is also just piecewise continuous that will be good enough and in that case at the points of discontinuity, the Fourier series estimate will converge to this as more and more points are taken and this is a very sensible estimate because if there is a discontinuity at a point which is of this nature then at this point the Fourier series representation converges to the average of these.

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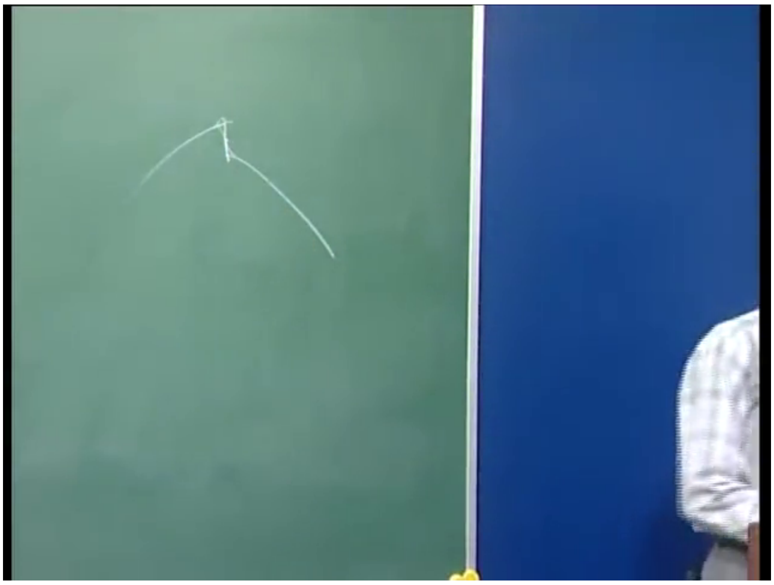

This is the convergence in the mean at those points, where it is continuous the average of the 2 limits turns out to be the function value itself. In this case where one limit is here and other limit is here the Fourier series will converge to this point and with this sense the Fourier series is able to give a representation to even discontinuous functions and this is the additional facility additional ability that Fourier series provides compared to other families of eigenfunctions.

Now, rather than minus L to L the interval could be any x 0 to x 0 plus 2 L now; that means, that since L has been taken as a symbol variable so; that means, that whatever interval finite interval you want you can put in this. Now a few important properties it is valid to integrate the Fourier series term by term even if it has a discontinuity of this kind and this makes very good sense because integration is actually smoothening process. So, if the actual function has discontinued of this kind, the integral actually will remove the discontinuity that is in the sense that in the integral you will not find this discontinuity.

Now, this is regarding the integral the function value itself at that point where you have this this kind of a discontinuity that is at a jump discontinuity, convergence to this is not uniform. So, at that point the function value is not reliable and it should not be for that matter because the function value is not this. So, at this point function value is not reliable. So, around this point the convergence could be anything like this or like this and for that matter it is also noticed that there may be a little rise just near the discontinuity.

So, this is very interesting because the more and more terms you include in the series the mismatch peak shifts a little bit this is called the Gibbs phenomenon. So, at the location of the all in the immediate vicinity of the jump discontinuity, the value of the Fourier series estimate may be unreliable what about differentiation as you see that integration is smoothing process, differentiation is a process in which the discontinuity actually increase. So, therefore, term by term differentiation of a Fourier series is valid at only those points where the function is smooth at these points. So, the derivative at this point will not be valid.

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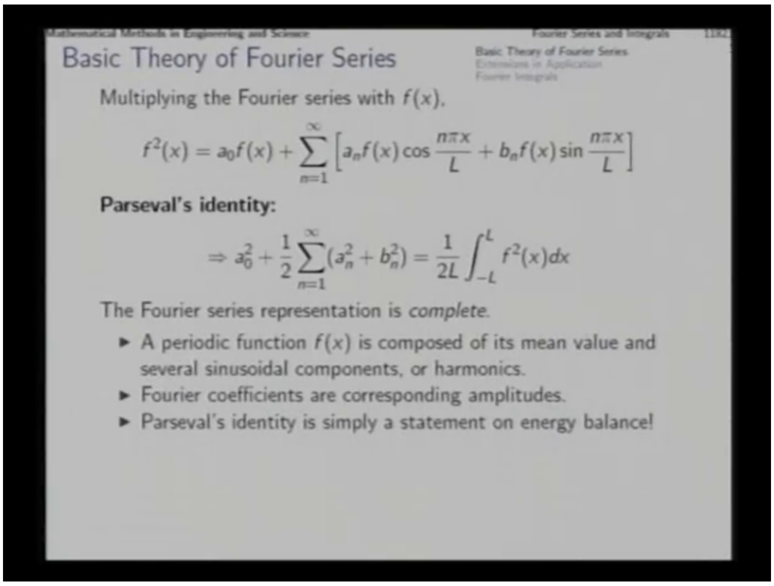

Now, you can find out the statements of the standard results like basis inequality or parsevals identity in the context of the Fourier series, in the usual manner in which we did it for the general Sturm-Liouville problems. And in many scientific applications you will note that a periodic function f x is composed of its mean value and several sinusoidal components known as harmonics and they this mean value is given with the average part, which is here this is the average value mean value and then this these 2 terms with n equal to 1 is for the first harmonic and then second harmonic and so on

So, in any periodic function you can have the separate terms which is 1 is the mean value average value, and then several sinusoidal components with higher and higher frequencies. So, those frequencies will appear here in this manner n pi by L will be the frequency and in that context parsevals identity which is this is simply a statement of energy balance the total energy of a wave is equal to the energy of the is equal to the total of the sum total of the energies of all the harmonics taken together along with the average.

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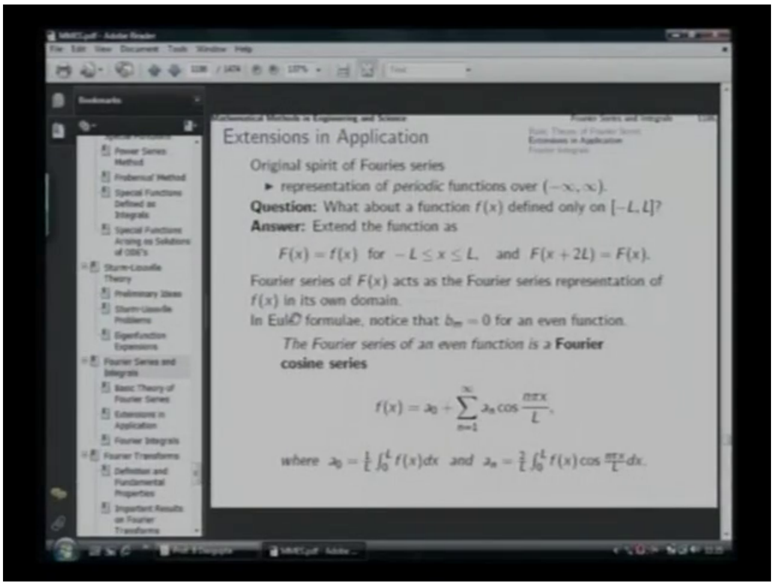

Now, there are a few extensions which we apply when we need to have series Fourier series in special situations for example, the original spirit of Fourier series is the representation of periodic functions over this infinite interval. So, that is minus infinity to plus infinity that periodic function is represented completely. Now what about a function which is defined only on a finite interval and outside that there is no definition what we do for that function. So, what we do, we make an extension of the function which is periodic. So, small f is a function which is defined only over this outside this it is not even defined. So, we say that what about capital F, which we define as small f over this interval and outside that interval we say that we make a periodic extension of it.

So, whatever is f x over minus L to L the same thing we go on repeating for capital F beyond this interval now according to the original spirit of Fourier series then we can develop a Fourier series for capital F which will exactly match with small f over that interval of interest now further function small f for which we are looking for the function representation the values outside. This interval will have no meaning, but whatever is the value whatever is the series representation over this interval will be the same as that for f x capital F x. So, this is the periodic extension of a function which is not periodic. So, this is a non-periodic function defined over a finite interval we make a periodic extension of it.

Now, you will make another note that in the Euler formulae when we try to find out the Fourier coefficients then for an even function we found that the coefficients corresponding to the sin term is 0 and that shows that the Fourier series of an even function turns out to be a Fourier cosine series in this manner the sin terms are absent. So, there we can find out the coefficients like this for even function we need not integrate from minus L to L by twice the integral from 0 to L we can find this.

Similarly, for an odd function the cosine terms will be missing and a sin terms will be there and of course, the average also will be missing because for odd function from minus L to L average value will be 0.

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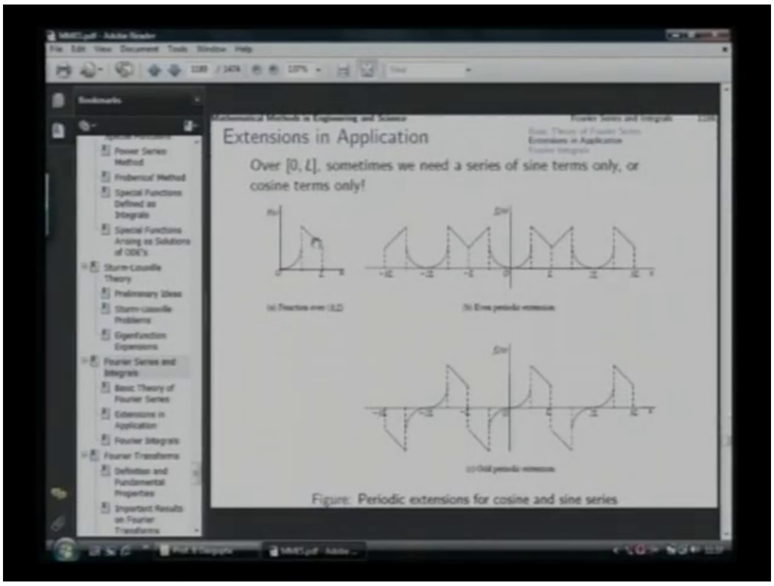

So, similarly we can get a Fourier sin series for an odd function and that gives us another important tool in our hand another important weapon in our hand sometimes we need a series of only sin terms or only cosine terms this kind of a requirement we will face when a few vectors down the line you will be studying partial differential equations. So, there in order to satisfy certain boundary conditions, we will need only sin terms or only cosine terms.

So, in that case what we can say that over 0 to L, if we need only sin terms or only cosine terms then we can make first a and odd extension over minus L to 0 or an even extension over minus L to 0 and then for that entire function from minus L to L we can go on repeating that is a periodic extension. So, suppose this is our function over 0 to L like this and then like this it has a jump discontinuity at this point and for that matter another at this point. So, for representation of this function in the form of a cosine a series what we do is that whatever is this function from 0 to L we make a symmetric reflection of it from minus L to 0 like this.

Now, this becomes from minus L to L this becomes an even function and for this even function we make a periodic extension which will look like this the same thing gets repeated from minus L to minus 3 L to minus L here and again L to 3 L here 3 L to 5 L and so, on. So, now, the Fourier series of this will turn out to be a cosine series similarly if we want a sin series then here we make a make an antisymmetric reflection. So, 0 to L the function is defined like this over minus L to 0, we defined the extension in this manner and then repeat that sequence minus L to L and this will give us a sin series.

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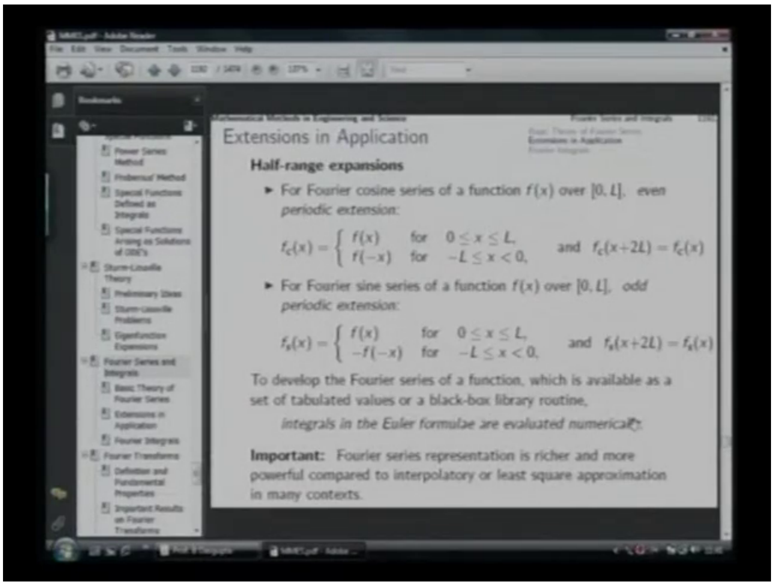

So, these were the corresponding series that we get out of it are called half range expansion which are valid only for the half range from 0 to L, beyond that it has no meaning beyond that its values have no sense. So, for Fourier cosine series this is the even extension and then this is the periodic extension. Similarly for Fourier sin series this is the odd extension that we make which pictorially we saw just now and this is the periodic extension.

So, these processes give us ways to get sin series or cosine series for non-periodic function with the definition only over limited finite intervals. In a special situation where we have the function values available only in terms of a table that is at different values of x we have got the values of f x, how to develop the Fourier series for such a function which is available only as a set of tabulated values or a black box library routine that is whenever we call the library routine, we get the value for that we can still develop, the integral which is needed for the evaluation of the Fourier coefficients through numerical integration process.

Sometimes it may be it may happen that we have got values of the function from some experiment, which can be conducted over only limited values of x there and that also not at constant intervals. In such situations also we can use numerical integrations in order to develop the Fourier coefficients and in any form that we have the data regarding the function values, then from that we can work out the Fourier coefficients with the help of other formulas integral may be needed to be evaluated numerically.

Now, from the foregoing discussion that is that the Fourier series can give you infinite series representation of functions, even those functions which have jump discontinuities like this we find that apart from giving least square approximation Fourier series representation is even richer and it is more powerful compared to other kinds of representations. One problem however, is still unaddressed we considered the original Fourier series for infinite interval, if the series itself is if the function itself is periodic.

Now, for non-periodic functions which have definitions on the a word a finite interval that is in which that finite interval is of our interest and nothing beyond it then we could make a periodic extension of that finite interval itself and we could still represent the function in the form of a Fourier series a for the entire infinite interval and out of which that particular interval will make the correct sense the rest of it we can ignore. What about a function which is defined over the infinite interval, but which is not periodic. So, for that what we can do? That takes us to the concept of Fourier integral that is how to apply the idea of a Fourier series to a non-periodic function over on infinite domain.

Now, this is a single period of infinite size. So, for that what we do? We take a single period and magnify that to infinite size for that purpose.

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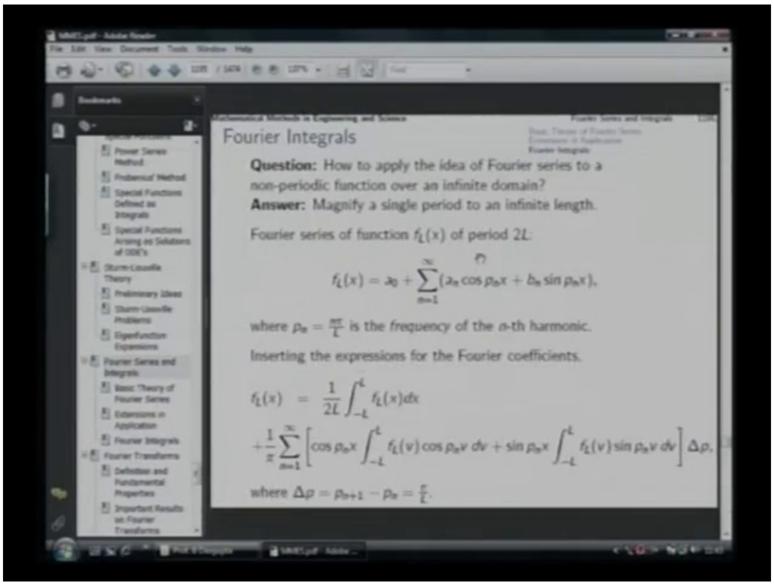

Let us consider this Fourier series of function f L of period 2 L, which will look like this an infinite sum a series n h p n is n pi by L and that is the frequency of the n th harmonic in the ordinary Fourier series we had n pi by L sitting here right. Now what we do we insert the expression for the Fourier coefficients a n and b n and a 0 then in place of a 0 we have got this for a n we have got this f L in place of f L x we are writing f L v because this variable then actually is the dummy variables for this integral and nothing else. Now we cannot use x because x has remaining outside this integral. So, we have f L v cosine n pi v by L into d v. So, this is the Fourier coefficient.

Now, here this Fourier coefficient in that here we have removed the 1 by L and put 1 by pi here with this delta p here, because delta p is pi by L why delta p is pi by L because with a large number of terms we have 1 term which is n pi by L the next 1 will be n plus 1 pi by L what is the difference of the 2 values of p p n and p n plus 1 that is pi by l. So, that is delta p. So, that delta p pi by L is sitting here. So, pi by L into 1 by pi that is giving us the 1 by L which we needed in the ordinary Fourier series.

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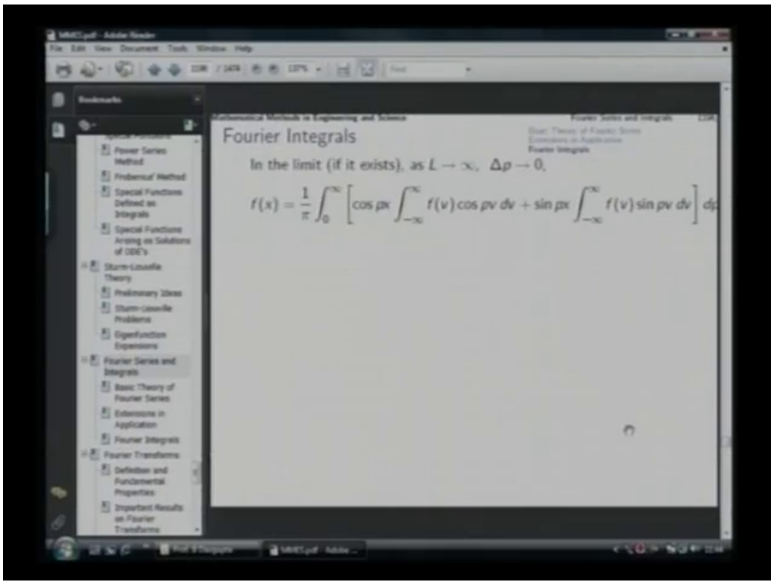

Now, if we can find the limit, then in the limit if we take this as L tends to infinity we are magnifying a single interval of size 2 L minus L to L. Now we are saying that as L tends to infinity we are actually stretching this single interval to infinite size minus infinity to plus infinity. So, as we stretch that single interval from minus infinity to plus infinity, these will turn out to be integrals from minus infinity to plus infinity and pi by L will turn out to be extremely small; that means, delta p will tend to 0, then we will be calling it d p and then this sum will be corresponding to p 1, p 2, p 3, p 4 each varying from the neighbours by extremely small distances which is d p; that means, this sum of discrete items discrete terms will get replaced with the sum of infinite terms which are continuous and that is an integral.

So, this sum from n equal to 1 to infinity will become the integral from 0 to infinity and in between this minus L to L integrals for the coefficients will turn out to be simply integrals from minus infinity to infinity and delta p can be now replaced with d p. So, this turns out to be not a sum of large number of terms not an infinite series, but an integral. So, this in the limiting case in the limit the Fourier series goes to Fourier integrals and that is the way to represent a periodic functions, non periodic functions which are defined over the entire interval and entire infinite interval minus infinity to plus infinity entire real line.

So, in the next lecture we will start from this point and study a few interesting forms of the Fourier integral and out of one such particular form, we will also make a quick definition for what is known as Fourier transform. And after that we will consider in the next lecture another special facility that a particular Sturm-Liouville problem will give us in the eigenfunctions of the Chebyshev problem and that is the family of Chebyshev polynomials. So, these are the issues which will be discussing in the next lecture.

Thank you.