Mathematical Methods in Engineering and Science
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Module - VII
Application of ODE's in Approximation Theory
Lecture - 02
Sturm-Liouville Theory

Good morning. In today's lecture we will be studying Sturm Liouville Theory, which will take us from the domain of ordinary differential equations to the domain of approximation theory.

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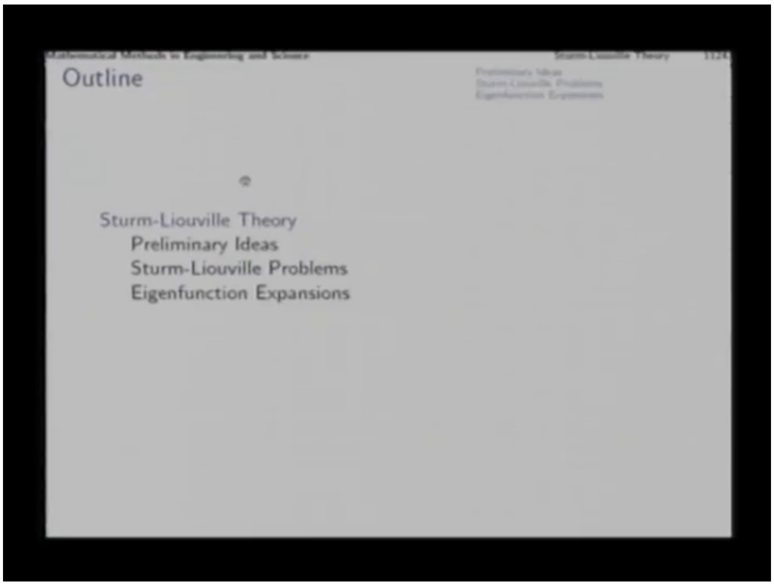

So, in between the two theories this Sturm Liouville theory forms the bridge. Now recall that till now a lot of our discussion has been in the context of initial value problems, with boundary value problems getting very small share of our focus. In the context of Sturm Liouville theory we concentrate on boundary value problems, in a particular manner.

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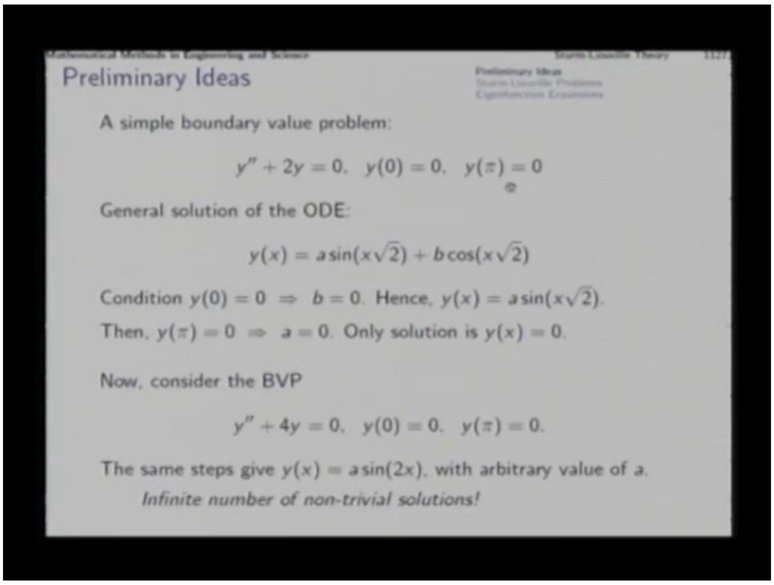

Consider this simple boundary value problem, y double prime plus twice y is equal to 0 with both the boundary values of y at x equal to 0 and x equal to phi given to be 0 everything homogeneous the differential equation as well as the boundary conditions.

We try to solve this boundary value problem first the ODE. As we try to solve this ODE we know that we will construct the auxiliary equation which will be m square plus 2 equal to 0. So, m will be found to be plus minus root over 2 and based on that we will form we will form this solution general solution of this differential equation, up to this point it is fine. Then we will try to supply this boundary conditions to determine the values of a and b. Now as we force this condition at x equal to 0 y is 0 then at x equal to 0 this term will vanish.

So, we will find that b cos 0 is 0 now cos 0 is 1. So, it will be b equal to 0. So, from the first condition we find b equal to 0. So, putting that then the solution reduces to only this much, this y x is a sin x root 2 then we apply the second boundary condition. At x equal to phi also y is 0. So, as we put x equal to phi, we find that a into sin phi root 2 is 0 now sin of phi root 2 is not 0. So, a has to be 0 now if we put a equal to 0 here then we find that y is equal to 0 this is the only solution possible which you we knew before we started the entire process. So, that is in this particular situation with these boundary conditions this differential equation has only the trivial solution and no non trivial solution.

In a similar manner we try to solve another boundary value problem in which everything remains the same except that this 2 changes to 4 and we consider this boundary value problem. Exactly the same steps will give us in this case this solution, a sin 2 x equal to y equal to a sin 2 x in which a can be arbitrary. Now you find that in this case we could find only the trivial solution there was no non trivial solution at all. In this case we have got infinite number of non trivial solutions with every for every value of a we have got a non trivial solution. So, what makes the difference between this case and this case?

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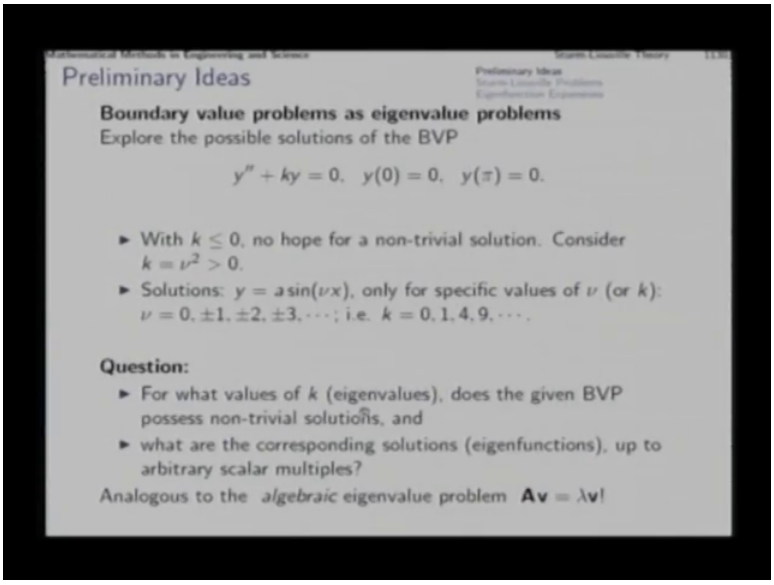

In a more general setup we can consider this boundary problem with k here, where earlier we had 2 in one case and 4 in the other case. Now you say that for which values of k we will be able to find non trivial solutions of this boundary value problem. We repeat the same steps and find that with k less than equal to 0, there is no hope of finding a non trivial solution and why so? Because with k negative, we find that this will give rise to an initial equation an auxiliary equation which is m square minus 1 equal to 0, m square minus something equal to 0 not necessarily 1.

So, m square minus something equal to 0 and that will mean that m will be will have 2 real solutions, m will have 2 real values and there will be exponential solutions. Now if you find an exponential solution which will rise from the 0 value at some point then it will never return to 0 value again. Therefore, negative k we cannot hope for a solution, with 0 k also there will be no non trivial solution because in that case y double prime will be 0 which means that y prime will be constant which will mean that y will be a linear function and a linear function also if it rises at some point from 0, then it will never come back.

So, with k equal to 0 also we do not find any solution for this except the trivial solution which is y equal to 0 which will always remain. Now we consider positive values of k now if k is positive then we can call it nu square. Now we find that with nu equal to plus minus 1, plus minus 2, plus minus 3 etcetera we can find solutions of this BVP and they those solutions will be a sin nu x. So, in the earlier case with k equal to 2 we could not find the solution, with k equal to 4 we could find the solution because this was the case which fit fits there.

So, we find that for quantized values of k that is 1 4 9 16 and so on we get non trivial solutions of this BVP and corresponding values of nu are 1 2 3 4 etcetera right. Now for the purposes of recording these solutions we include 0, to include the trivial solution also in this factory. So, nu equal to 0 will give the trivial solution, nu equal to 1 will give us a sin x nu equal to 2 will give us a sin twice x and then similarly a sin trice x and so on. Will be the non trivial solutions all of these solutions are bound at the 2 values 0 and phi. This is like a string which can stay tight or which can have this kind of a shape or this kind of a shape and so on.

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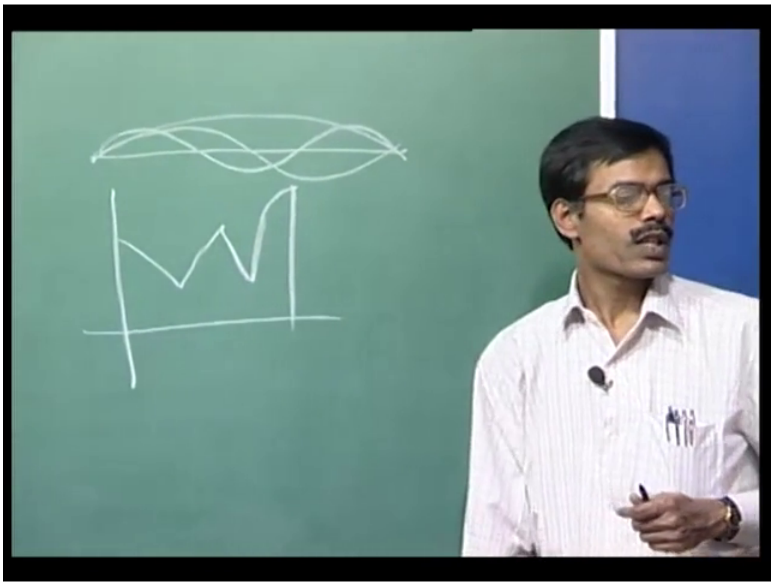

So, the different shapes of the string will be one is this, another is this, is another is this, another is this, and so on for values of nu as 0 1 2 3 and so on. Now and now the size the amplitude of the solution is here a; as that can be varies continuously you can have the same solution copied several times with different scale factors. So, that is simply the scale factor and that scale factor tells us that for the single that amplitude that shows us that for the same value of nu among these a positive integer values, you can have infinite solutions every solution for every value of a.

Now, we ask these questions that for what values of k or Eigen values does the given BVP possess non trivial solutions and the answer to that in this question is k equal to 0 1 4 9 etcetera and the second question is what are the corresponding solutions we call them Eigen functions. These particular values of k are called Eigen values and these particular corresponding solutions are called Eigen functions up to arbitrary scalar multiples, which is here in the amplitude. Now these 2 questions look very similar to the algebraic Eigen value problem and which is A v equal to lambda v for which we ask the question that for which values for lambda this system possesses solutions non trivial solutions and what are the corresponding solutions v.

So, this is the reason that we get a problem of this kind which we will be calling as the Eigen value problem, that in the case of this problem we called it the algebraic Eigen value problem. This qualifier this adjective algebraic was used to basically qualified this particular problem from the Eigen value problem of the differential equations, which we are studying now. And as this particular boundary value problem framed in the form of an Eigen value problem in this manner, has a direct resemblance with the algebraic Eigen value problem which is this similarly we also have a resemblance of the Hermitian matrix or in the case of real numbers the symmetric matrix with what we call in the literature of differential equations as the self adjoint differential operator. And that is the next preliminary idea that we try to develop before going into the actual Sturm Liouville theory.

So, for considering for figuring out what is this self adjoincy of a differential operator, we again start from some simple notions.

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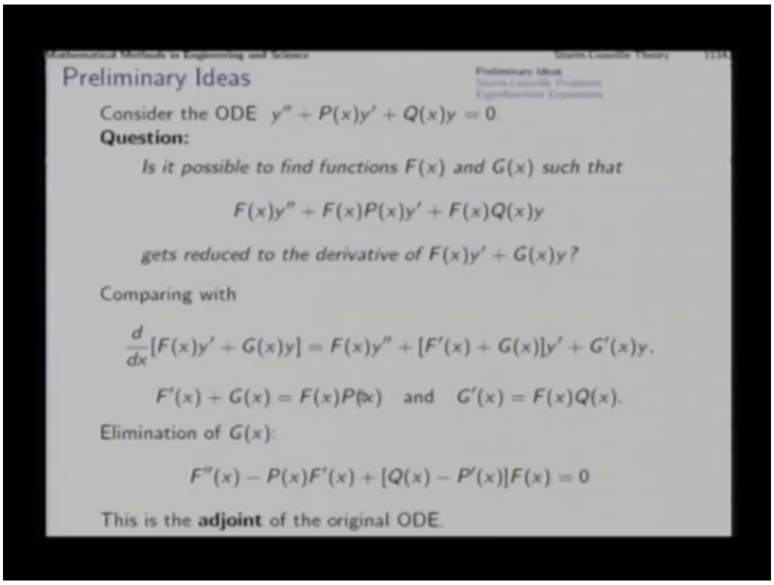

Consider this second order homogeneous ordinary differential equation, linear homogeneous equation and in the corresponding first order equation the way we try to solve the equation Labseque equation in that we had y prime plus P x y equal to something. Now homogeneous version will be y prime plus P x y equal to 0 and there the way we try to find the solution was to find out an integrating factor F with which we can multiply the left hand side to make the entire left hand side the derivative of something. A similar exercise if we try here then let us say let us see where do we reach.

We say that is it possible to multiply this entire differential equation with something some integrating factor F, such that this entire stuff F into this entire thing turns out to be the exact derivative of a an expression which is which involves up to first order derivatives in y first order derivative in y. That is can we find out a function F and an (Refer Time: 11:42) function G such that F into y double prime plus F P y prime plus F Q y turns out to be the exact derivative of a first order differential expression of y, in which y prime and y will be involved and y double prime will not be there exact differential of this.

So, we know that if we can do that, then the coefficient of y prime in that will be F because if derivative being this will mean that there must be a term in which we will have F y double prime. So, that is here. So, that is why this term must behave, but what about the rest of the terms will they oblige to match these terms correctly. So, for that what we do we say that let us propose some F and (Refer Time: 12:34) F G such that F times this entire left hand side turns out to be the exact derivative of this involving F and G. So, we try to differentiate it exactly and find F y double prime matching this first term plus F prime y prime which is here plus from here G y prime which is here plus G prime y which is here.

Now, as we try to match this first term with this you know as we try to match this entire expression with this entire expression the first term matches what about the others. Coefficient of y prime here it is x P here it is this. So, we say equality will require that this is equal to F P and then here coefficient of y here it is G prime, here it is x p. So, equality will require this now what we have got in this line? In this line we have got 2 differential equations in the 2 functions that we were asking for that makes sense 2 unknown functions which we would like to determine and for that 2 differential equations coupled yes this and this are coupled.

So, we say that for solving 2 unknown functions from 2 differential equations, we can try elimination that is we try to our focus is F the function which is supposed to turn out to be the integrating factor. So, we try to eliminate the other function G x. So, how to eliminate that? We can differentiate this and their G prime will appear, there we will insert this value of G prime. So, as we differentiate this we get F double prime plus G prime. So, F double prime plus G prime. So, for G prime we put F q. So, that is F Q is equal to derivative of this F prime P plus F P prime.

So, those 2 things we bring to this side this is minus P F prime this is minus P prime f. So, the 2 parts 2 components of the derivative of F P with negative sign as its come to this side. Now see what does this mean? This means that it will be possible to find F x and G x such that F x into this will turn out to be the exact derivative of an expression like this, if the function F x satisfies this. Now what is this? This is again a second order differential equation in F, which is of the same type as the original differential equation. This differential equation is called the adjoint of this differential equation so; that means, the adjoint of this differential equation is that other differential equation which an integrating factor has to satisfy such that that integrating factor multiplied with this will reduce it to the exact derivative of an expression like this.

Now, in that in terms of helping in the solution process of this differential equation, did this analysis make any contribution really know because this was a second order differential equation in which coefficient of y prime was unity here also this is also differential equation in which the coefficient of F prime F double prime is unity, here the coefficient of y prime was P here the coefficient of F prime is minus P something similar, here the coefficient of y was Q here the coefficient of F is Q minus P prime, there is no reason to believes that this will be simpler compared to this in general right.

So, in terms of procedure to solve the differential equation this discussion did not help, but the concept that we develop further in this line of argument will help in some other place.

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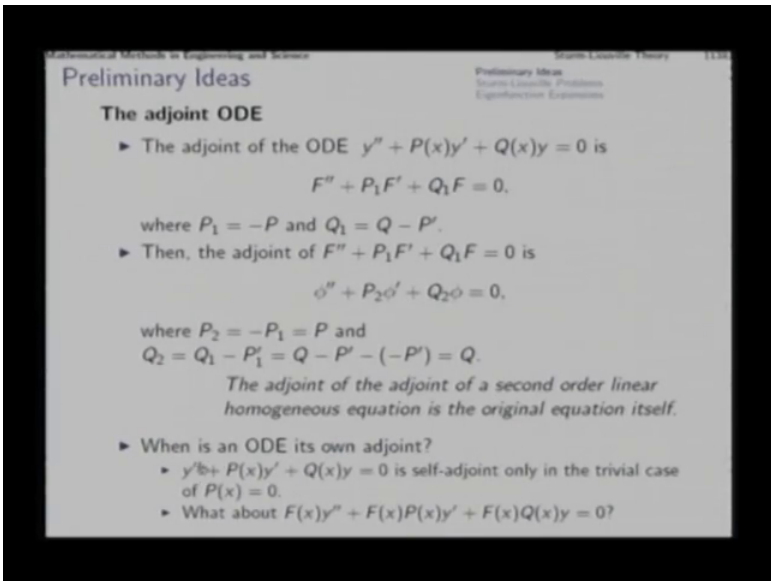

So, the next question that we ask is that what are the properties of this adjoint? For example, till now we have seen that the adjoint of this differential equation turns out to be this, where the coefficients of y double prime and F double prime are one in each case in both the cases, if the original differential equation had a coefficient P x here then the adjoint will have P 1 of x which is just the negative of P. Similarly if we had Q here then we have got a Q 1 here which is Q minus P prime, this we have seen just now here Q minus P prime. So, this much we have got.

So, this is the adjoint of this differential equation where P 1 is minus P and Q 1 is Q minus P prime. Can we find the adjoint of this differential equation? Adjoint of this differential equation will be another such differential equation phi double prime plus P 2 plus P 2 phi prime plus Q 2 phi equal to 0 in which P 2 will be minus P 1 and Q 2 will be Q 1 minus P 1 prime exactly like this. So, P 2 is minus P 1 which is P and Q 2 will be Q 1 minus P 1 prime Q 1 minus P 1 prime, but Q 1 itself is Q minus P prime. So, in place of this you write Q minus P prime and P 1 itself is minus P.

So, in place of P 1 prime you write you write minus P prime now you find that this 2 cancel this 2 cancel and you get back Q. So, what we have got? We have got that the adjoint of this turns out to be phi double prime plus P phi prime plus Q phi which is exactly the same as this differential equation. In this the name of the unknown function was y, here the name of the unknown function is phi, but they are the same differential equation. So, what we establish out of this? We say that the adjoint of the adjoint of a second order linear homogeneous equation is the original equation itself. The way you used to say that the transpose of a matrix you can find and then you can say that transpose of the transpose of a matrix is the original matrix. So, that transposition or the conjugate transposition have has the same relationship as the adjoint ODE s here.

So, adjoint of the adjoint of a second order second order linear homogeneous equation is the original equation itself. Now transpose of the transpose of the matrix is the original matrix then you ask that which matrix is its own transpose and as answer you found that it is a symmetric matrix or which matrix is the conjugate transpose of itself, as answer you found that it is the Hermitian matrix. A similar question you can ask here, when is an ODE its own adjoint. So, for asking this question you find that you have this differential equation and you say that if this differential equation is to be its own adjoint, then its adjoint which is this this equation with P 1 and Q 1 like this, then you will find that P n P 1 has to be have to be equal, but P n P 1 are negatives of each other.

So, for being equal as well as negative it has to be 0 so; that means that this kind of a differential equation can be its own adjoint only in the trivial case, when P is 0. So, if P is 0 then you will find P prime is also 0 and Q 1 is equal to Q in that case it will be at self adjoint that is the ODE will be its own adjoint.

So, with P x equal to 0 in this case every differential equation every second order differential equation like this with P x equal to 0 will be self adjoint this is trivial, but what about something into this that is if you have this as the differential equation? F y double prime plus F P y prime plus F Q y equal to 0 that is if this differential equation is multiplied with a factor F F of x then you say that under what condition this will turn out to be self adjoint. That means, after multiplying this standard form of the ODE with the integrating factor, when we can say that now it is self adjoint even if P is not 0 and for that we ask the question in a slightly different manner.

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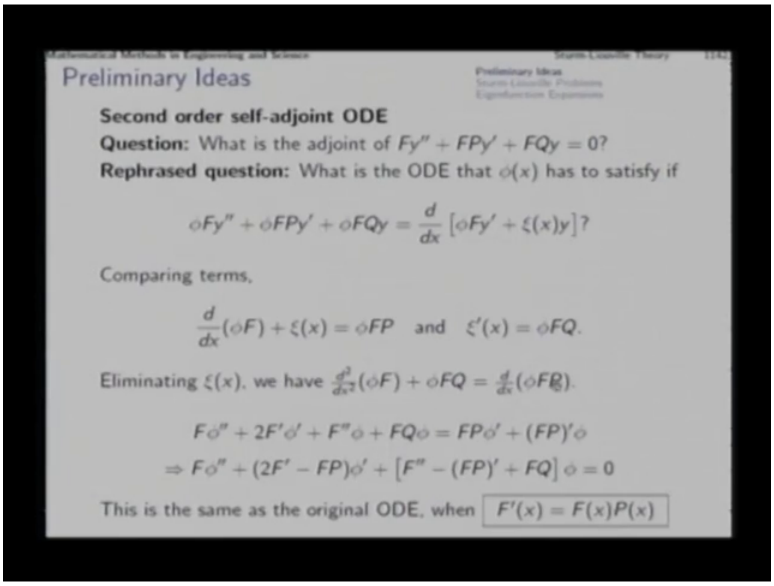

First we try to ask; what is the adjoint of this equation, and then after finding the adjoint we compare that adjoint equation with this and then ask under what condition these 2 are actually the same differential equation. So, we ask this rephrased question of this, what is the say when we want to find out what is the adjoint of this. So, we if we can look for the differential equation with the adjoint has to satisfy, then we have reached some level of answering this question. So, we rephrase the question and say what is the ODE that phi has to satisfy if phi multiplied with is makes it an exact derivative like this. Phi multiplied with this whole thing phi F y double prime plus phi F P y prime plus phi F Q y equal to exact derivative of a first order derivative expression like this first order expression.

So, the first term of that must be phi F y prime so that its derivative can account for this term plus something into y. So, we compare terms the derivative of this is a graphly derivative of this, derivative of this will involve 2 terms; one is psi y prime and the other is psi prime y. So, psi y prime must be equal to this. So, phi F P must be equal to the this,

this will produce another term for the matter the derivative of this is phi x y double prime plus derivative of phi F into y prime. So, derivative of y prime phi F into y prime. So, that will have with itself this part ok.

So, phi F derivative into y prime plus psi into y prime. So, phi F derivatives plus psi should equal the coefficient of y prime here. So, that is this and the coefficient of y from here will be psi prime and that must be equal to phi F Q that is this. So, these 2 conditions must be satisfied. Now again our question was what is the ODE that phi has to satisfy if it is going to work as the integrating factor for this. So, psi is an extraneous function which we had to accommodate to satisfy the form. So, here in this system of 2 differential equations in 2 unknown functions phi and psi, we want to get rid of psi. So, we differentiate this and in that way psi prime appears we insert this, ok.

So, eliminating psi we differentiate this we get second derivative of phi F plus psi prime in which space we insert this is equal to derivative of this which is this. So, this is the differential equation which phi must satisfy, if it is going to make this an exact derivative through multiplication that is the adjoint of this differential equation will be this differential equation which phi must satisfy. So, let us expand it and write it in proper form. So, second derivative of phi F that will be F phi double prime plus twice F prime phi prime plus F double prime phi this is the second derivative of phi F plus this which is here plus equal to sorry equal to the derivative of this F P into phi prime plus F P prime into phi.

Now, we collect the terms together second derivative terms here only one,first derivative term one from here and the other from here that is twice F prime minus F P. Phi term without any derivative. So, F double prime will be here and then minus F P whole prime plus F Q that is here. Now you say that this is the differential equation which phi must satisfy to play this role. That means, this is the adjoint of this equation, now we ask this question that is this self adjoint it will be self adjoint when its adjoint which is here turns out to be the same differential equation could be because here the coefficient of y double prime was F here the coefficient of y double prime is F same.

Now, if the other coefficients are also equal then this differential equation and this differential equation will be exactly same, when will that happen? When F P is equal to this which will mean twice F prime equal to twice F p; that means, F prime equal to F P that is F prime equal to F P and if F prime is equal to F P then here you see this is F P. So, if it is F prime then F prime prime F double prime then this F double prime and this F double prime will cancel each other and what will remain is F Q, which is the same as here; that means, this satisfaction of this simple condition single condition will ensure that this differential equation turns out to be exactly the same as this differential equation, but the way we derived this we know that this differential equation is actually the adjoint of this.

So; that means, that this differential equation will be its own adjoint, if this condition is satisfied that is if F prime is equal to F p; that means, if the coefficient of the y double prime is differentiated to get the coefficient of y prime, then we have a self adjoint second order differential equation. Same thing happened when we tried to solve the Labseque equation in that case we had y prime plus P y equal to something and we multiplied it with F and then we said F y prime plus F P y equal to something, and in this left side had the condition that F prime must be equal to F P the same condition we get here ok.

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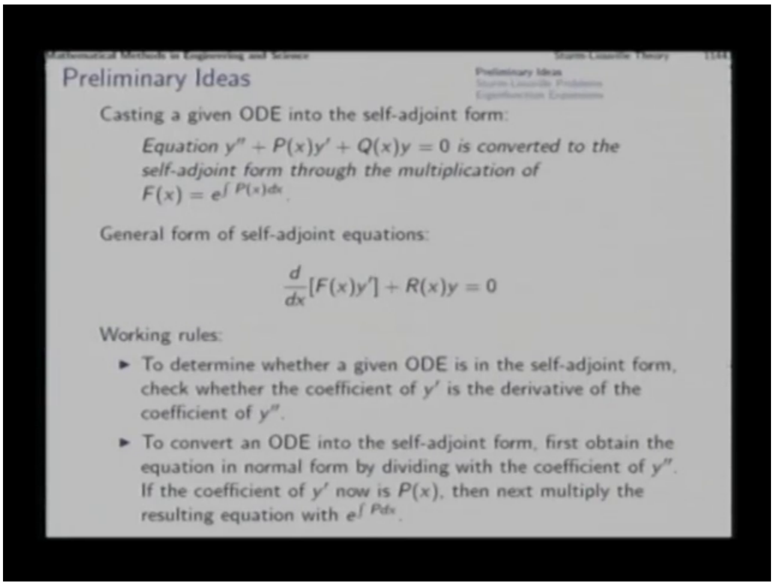

So, we can cast a second order differential equation into the self adjoint form by multiplying with F x, which is this, the same thing which we found as the integrating factor in the case of Labseque equation. So, when we have a second order differential equation like this and we want to convert it into the self adjoint form, all that we need to do is to get this function P integrate it and put that integral over the exponential in the index and get e to the power integral P d x and that turns out to be the F, F of x which we should multiply it is to convert it into the self adjoint form.

So, once we multiply this equation with this particular F x, then we find that the here we have got F and here we have got F P and then it is certain that the derivative of this F will turn out to give us first this and then into the derivative of this derivative of this is P and that P is simply sitting here. So, F will come here P will come here.

So, automatically the derivative of F will turn out to be F P. So, to cast the ODE into the self adjoint form, all that we need to do is to multiply the equation in this form with this F x right. So, in that case we find that these 2 terms together will become the exact derivative of an expression like this, F y double prime will come here and then F prime y prime will come here because here we will get F P which is same as F prime. So, this entire term these 2 terms together will be sitting here the exact derivative of this kind of a combination plus whatever comes here into y ok.

So, this will be the general form of the self adjoint differential equations that we are going to study. Now there are 2 small working rules which we can immediately appreciate and remember, one is to determine whether a given ODE is in the self adjoint form, we simply check whether the coefficient of y is the exact derivative of the coefficient of y double prime if the coefficient of y prime is the derivative of the coefficient of y double prime. If it is not so, and we want to convert it into self adjoint form then first we convert it into the standard form by dividing throughout with the coefficient of y double prime, if it has any non trivial coefficient which is not one.

So, to convert an ODE into the self adjoint form first we obtain the normal form the standard form by dividing the entire equation with the coefficient of y double prime. After that whatever appears as the coefficient of y prime call that P x and then multiply this new equation throughout with this integrating factor this F x, and the resulting equation is in the self adjoint form. Now with this self adjoint form what can we do? With this self adjoint form in our hand, we have a certain an interesting theory which is the basic issue in Sturm Liouville theory. This way when we define the equation and in the next step we are going to expand this and have a lot of other terms involved in it including the so called Eigen value.

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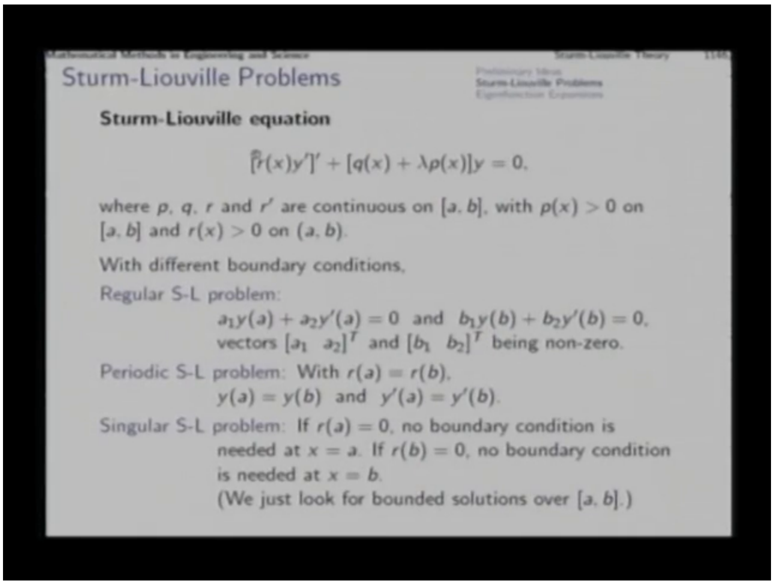

In the place of F x now we will have this small r x and in the place of this capital R x we have this large thing, in which this lambda that is getting involved here will be termed as the Eigen value and this equation which is a little bigger version of the previous equation that we studied is called the Sturm Liuoville equation.

In which the functions p q r etcetera have certain properties, what are those properties? First of all p q r and r prime we need all of these to be continuous over the close interval a b and p x must be a positive definite function on a b and r x is also a positive definite function on this interval a b. In some of the variations so the Sturm Liouville problem, this also is needed to be on the close interval, but there are situations where it is required only in the open interval because at a and at b, r x is allowed to be 0 in one version of the Sturm Liouville problem.

So, only this much is required for the general case, in which this is the Sturm Liouville equation. Now with this Sturm Liouville equation we can define 3 different classes of Sturm Liouville problems with different kinds of boundary conditions. If we define the boundary conditions in this manner, that is at x equal to a at one boundary we have got a condition like this, that is this linear combination of y and y prime is 0 and at the other boundary this linear combination of y and y prime is equal to 0 in which both a 1 and a 2 should not be 0 together that is a 1 and a 2 together is a nonzero vector one of them could be 0 but not both, because if a 2 is 0 that will simply mean that at that boundary the boundary condition is y a that is y a equal to 0.

Similarly, if a 1 is 0 that will simply mean that at that boundary y prime a is 0, but if both a 1 and a 2 are taken to be 0, then this will mean nothing. So, therefore, a non trivial boundary condition is given when a 1 and a 2 are not both 0 at the same time. So, a 1 a 2 is a nonzero vector similarly b 1 b 2 is a nonzero vector. So, if the boundary conditions at 2 ends are like this, then what we have got with this Sturm Liouville equation is a regular Sturm Liouville problem it is a boundary level problem. Now in a very specific case where this function r has equal values at both the boundaries at a and b. In that case another special kind of boundary condition is also will define a Sturm Liouville problem and that is if y a and y b are equal similarly, y prime a and y prime b are equal.

So, this will define what is called a periodic Sturm Liouville problem, this makes sense direct sense in a situation where the variable under question say x is a cyclic variable like the angular position theta on a cylinder. So, in that kind of a situation r 0 and r 2 phi will be same. So, in that case this is a natural condition y 0 and y 2 phi is same, y prime 0 and y prime 2 phi will be same. So, this is the periodicity involved. So, over that interval a to b if we try to define the problem, then this will be the boundary condition which is periodic in nature.

So, we define we can define a periodic boundary problem with this kind of a condition which is fundamentally different from this kind of a condition, and this will define a Sturm Liouville problem the periodic Sturm Liouville problem when r a is equal to r b, that is this function has same value at the 2 end points of the domain interval. In a singular Sturm Liouville problem, which is a very special critical kind of a case in which if this function r a is 0 at one boundary say at x equal to a, then to define the Sturm Liouville problem at that boundary we do not need a boundary condition.

Similarly, if r b is 0 then at that boundary at x equal to b, we will not need a boundary condition and in that case while solving this boundary value problem, we will just look for bounded solutions over a b, that is solutions y of x we will look for in this interval which are bounded functions of x that is which are continuous and bounded do not get undefined that kind of a solution we will just look for over the interval.

Now with this differential equation with this kind of boundary conditions or with lag with no boundary conditions in the case of singular problem, as we define a boundary value problem, like that what are the solutions of such a boundary level problem defined as the Sturm Liouville problem and what are the applications of them. The great property of the solutions of such a boundary level problem is the orthogonality. And the way we have defined the problem itself the proof of the theorem turns out to be quite straight forward if y m and y n are 2 Eigen functions or solutions of the Sturm Liouville problem defined as such defined as earlier corresponding 2 distinct Eigen values lambda m and lambda n that is corresponding to different lambdas lambda m and lambda n if these are the 2 Eigen functions 2 solutions.

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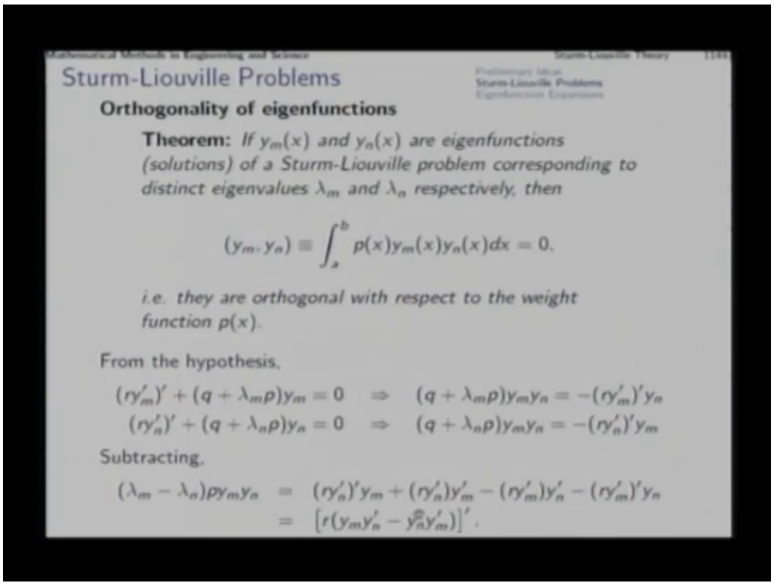

Then they turn out to be orthogonal with respect to each other with the weight function p x here. So, this is the definition of orthogonality of 2 functions y m and y n over this interval a to b i-th weight function p x. So, mutually the 2 functions will be orthogonal if they are the solutions of the Sturm Liouville problem corresponding to different Eigen values distinct Eigen values. Now this can be proved quite easily, what we do for that is that if y m and y n both are solutions of that boundary value problem that we have defined; that means, they are in particular the solution of the differential equation also and that will mean that y m is a solution of the differential equation corresponding to Eigen value lambda m will mean this and when we multiply this with y n we get this ok.

So, this multiplied with y n is here and this term with multiplied y n has been taken on the other side of the equality. Similarly y n is a solution of the differential equation with the Eigen value lambda n that will be this, and this equation we multiply throughout with y m and then we get this. Now simply we subtract these 2; as we subtract these 2 equations q terms will cancel out, and we will have lambda m minus lambda n into p y m y n that is the integrants here. So, we will get this and on this side we will have this minus this right so; that means, this will be positive that is here and this will be negative which is here and in between these 2 terms are actually same we have added and subtracted.

Now, these 2 terms together will be the exact derivatives of r y m y n prime, see r y n prime derivatives into y m plus y m derivative into r y n prime and similarly these 2 terms together will give us the derivative of this r y n y m prime. So, together the 4 terms will be the derivative of this whole thing and that is equal to this. Now we integrate both sides from a to b and these are this is constant. So, this stays outside the integral and we have got this which is equal to this is exact derivative of this bracketed expression. So, its integral will be this bracketed expression with the limits at x equal to a and x equal to b subtracted that is that entire expression at x equal to b minus that entire expression at x equal to a.

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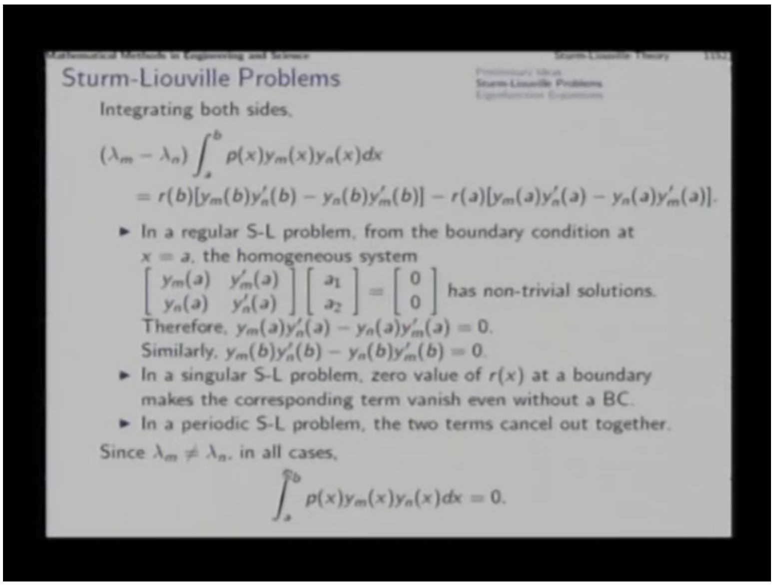

Now, note the 3 kinds of boundary conditions, in the regular Sturm Liouville problem we had this boundary condition a 1 y m plus a 2 y m prime, at x equal to a is 0 and a 1 y n plus a 2 y n prime at x equal to a is 0. Since y m and y n both are solutions of the Sturm Liouville problem. So, that condition at x equal to a is satisfied or satisfied by both y m and y n. So, they constitute these 2 equations and now since a 1 and a 2 are both not zero; that means, this system of 2 equations has a non trivial solutions, which means that this coefficient matrix is non this coefficient matrix is singular, which means y m y n prime minus y n y m prime at x equal to a is 0. Similarly the other boundary condition satisfied by y m and y n will mean that this is also 0; that means, this is 0 and this is 0 and; that means, this entire side is 0 and since lambda m and lambda n are distinct Eigen values they are not equal; that means, this is nonzero which will mean that this integral is 0.

Now, in a singular Sturm Liouville problem r a and r b both are 0 both are same. So, now, r a and r b are both 0 in a singular problem in periodic problem they will be same. So, r a and r b both are 0 in a in a singular problem or if the singular problem has a singularity only at one end and not at the other then at whichever end it has the singularity that is say only r a is 0, then at x equal to a we do not need a boundary condition if r b is 0 then at that end we do not need the boundary condition; that means, if this is 0 then we do not need this to be 0.

Similarly, if this is 0 then we do not need this to be 0 if at both ends of the domain we have r x equal to 0 that is r a and r b both 0, then we do not need boundary conditions at either of the end anyway this will this entire expression will become 0. In a periodic Sturm Liouville problem r a and r b are not equal, but are not zero, but they are equal to each other r a is equal to r b and in that case y a and y b same y prime a and y prime b same will constitute the periodic boundary condition, which will mean that r a and r b are same and this bracketed expression and this bracketed expression will turn out to be the same and in that case also though individually the terms are not zero, but their difference will vanish.

So, in all the 3 cases of the Sturm Liouville problem defined regular singular as well as periodic, we find that this right hand side vanishes and lambda m not being equal to lambda m lambda n that will mean that this integral will vanish in each of the 3 cases. So, we find this and this shows us that y m and y n are orthogonal functions with respect to the weight function P x over this interval and that shows that if the 2 Eigen functions 2 solutions of the Sturm Liouville problem that we take corresponding to distinct Eigen values, then they must be orthogonal to each other. Similar is a situation of the Eigen value algebraic Eigen value problem of the symmetric matrix, when we took the 2 Eigen vectors corresponding to distinct Eigen values and found the 2 Eigen vectors must be orthogonal similar is the case here.

You can extend this and also say that Eigen values of a Sturm Liouville problem are necessarily real and the way you prove that is also similar to the algebraic Eigen value problem, you assume a complex Eigen value and then force Eigen function corresponding to that to satisfy the Sturm Liouville problem and then you can show that the imaginary part turns out to be 0, the way we found the corresponding result for symmetric matrices. Now as an example of this orthogonality let us take the Legendre equation which we studied in the previous lecture previous lesson.

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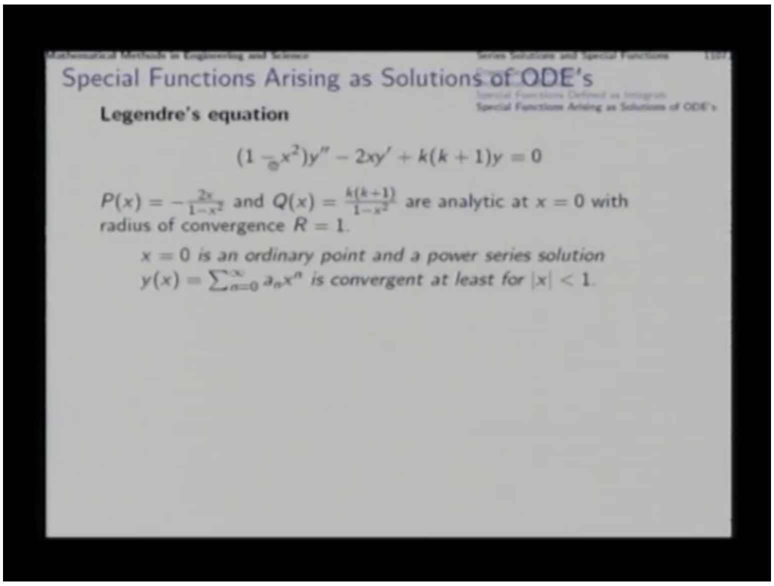

You will recall that this is the Legendre equation in which case this is already in the self adjoint form why because the derivative of this coefficient of y double prime is minus 2 x which is exactly the same as the coefficient of y prime.

So, it is already in the self adjoint form and these 2 terms together can be written as the derivative of 1 minus x square into y prime and that is exactly what we will do when we consider that in our context here, say we take the Legendre equation which is this.

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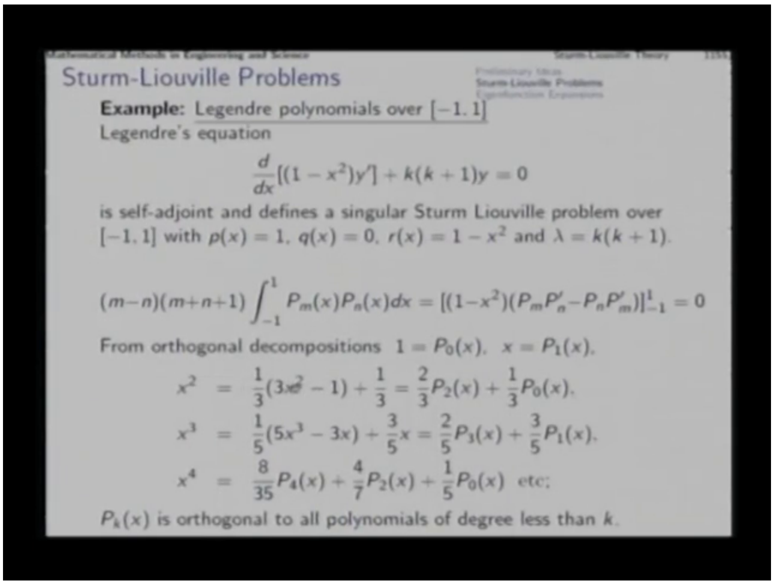

The 2 terms having y double prime and y prime we have put together here and k k plus 1 is sitting in the place of lambda the Eigen value. Here in place of r x we have got this function in place of q x we have got 0 in place of p x which is here we have got 1 and in place of lambda we have got k k plus 1. Now if we find 2 solutions of this corresponding to different values of lambda; that means, different values of k k plus 1 lambda m is m m plus 1 lambda n is n n plus 1.

Then we can write down the corresponding solutions as y m and y n and continue with a with an analysis which is similar to the general case that we considered here, but we can also observe that this turns out to be a singular Sturm Liouville problem over the interval minus 1 to 1, because this function r x 1 minus x square is singular is 0 this is 0 at x equal to minus 1 and x equal to 1 that is at a and b a equal to minus 1 b equal to one at these 2 values this function is 0.

So, over this interval from a to b, this differential equation without any boundary conditions defines a singular Sturm Liouville problem and as such the Legendre polynomials P 0 P 1 P 2 P 3 P 4 etcetera that we will get we will turn out to be mutually orthogonal with the weight function p x in this case it is 1; that means, that other such families of solutions or Eigen functions of Sturm Liouville problems will turn out to be mutually orthogonal with respect to the weight function p x, which appears in the place here in the general Sturm Liouville equation format here that function is 1; that means, the Legendre polynomials will turn out to be orthogonal with respect to one; that means, we will have this integral equal to 0 for every case of m not equal to n and if so, then we already know that Legendre polynomial of order k turns out to be a case (Refer Time: 48:48) polynomial expression right and then if we have P 0 as 1, P 1 as x, P 2 which is this ok.

So, then we will find actually P 2 is 1 by 2 of 3 x square minus 1 as we found in the previous lecture. So, then we can see that 1 can be expressed in terms of P 0, then x can be expressed in terms of P 1, x square can be expressed as a linear combination of P 2 and P 0, x cube can be expressed as a linear combination of P 3 and P 1 and so on; that means, that all ordinary polynomial terms 1 x x square x cube x to the power 4 etcetera etcetera can be expressed as linear combinations of P 0 P 1 P 2 P 3 P 4 etcetera. So, x to the power 4 will consist up to P 4, x to the power k will be expressed as a linear combination of P k, P k minus 2, P k minus 4, P k minus 6 up to either P 1 or P 0 and so on.

If these turn out to be expressed in terms of the Legendre polynomials up to that order that will mean that suppose we have gone up to till this point, then we say that P 5 that will define will be orthogonal to each of these expressions because this is a linear combination of P 0, this is linear combination of P 1 that is up to this 1 x square, x cube, x to the power 4 all these are linear combinations of P 0 P 1 P 2 P 3 P 4 then P 5 will be orthogonal to P 0 P 1 P 2 P 3 P 4 as we have found from here and that means, P 5 will be orthogonal to any linear combination of these 5 Legendre polynomials P 0 to P 4, which will mean that these 5 will be orthogonal to all these 5 functions and any linear combination of themselves; that means, that P 5 will be orthogonal to all polynomials up to the fourth degree.

Similarly, P 7 will be orthogonal to all polynomial expressions of the sixth degree and so on. So, P k will be orthogonal to all polynomials of degree less than k. So, this gives us a handle to represent the polynomial polynomials or other functions in terms of linear combinations of Legendre polynomials. Other such families of Eigen functions also can be used for similar purposes and that is the great use of the solutions of Sturm Liouville problems or Eigen functions of a Sturm Liouville problem.

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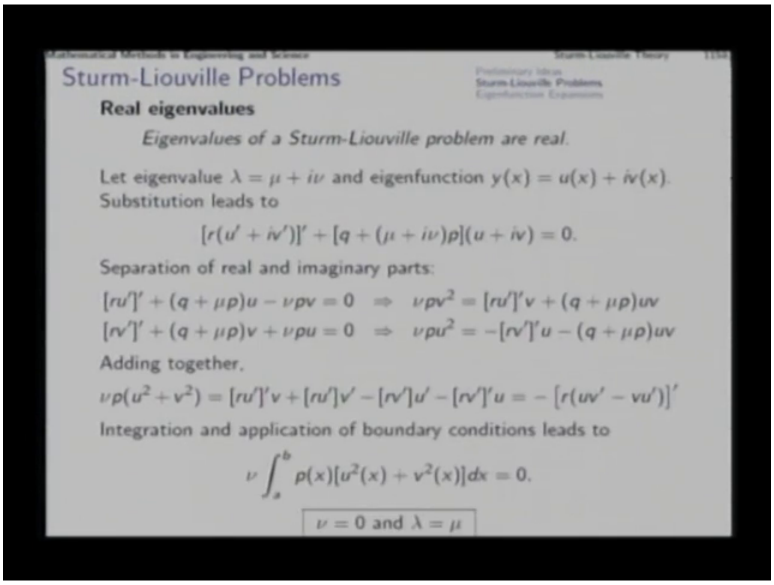

Now, this case of real Eigen values we can vomit the proof of this, but you can try to establish this small result in the same manner as we did in the case of symmetric matrice.

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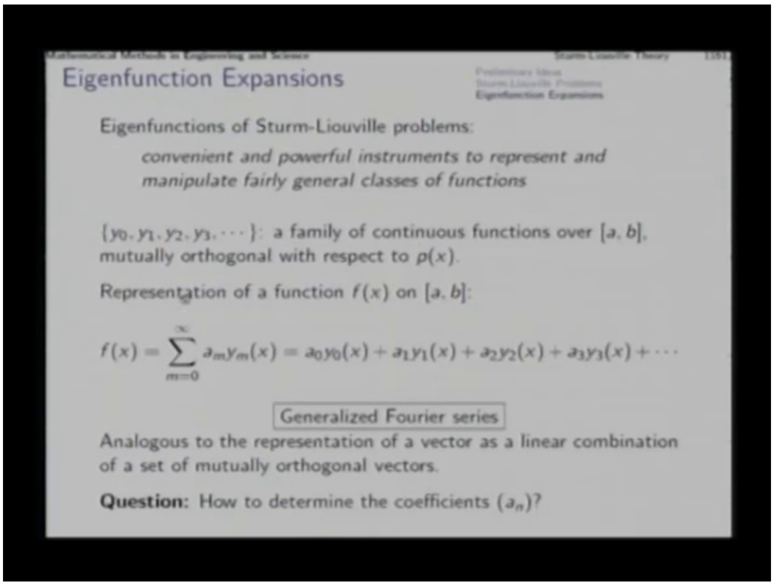

Now, we come to the most important application of the Eigen functions of Sturm Liouville problems, and that is for function representation. So, Eigen functions of an sl problem give us a convenient and powerful tool to represent and manipulate fairly general classes of functions. So, for function representation if you select a family of functions y 0 y 1 y 2 y 3 etcetera which are continuous functions over an interval which are usually orthogonal. Then a representation of a function over this interval could be made as a linear combination of this family of course, this will be an infinite series in general and this kind of a series is called generalized Fourier series, which is analogous to the representation of a vector as a linear combination of several basis vectors orthogonal basis vectors.

Now, how to determine the coefficients? To determine the coefficient a n we can do a similar thing as we could do in the case of ordinary vector spaces and that is we take the inner product of this f with the n-th function in this for which we are seeking the coefficient and if we construct this inner product, that will in the case of functions the inner product is given like this which we have studied earlier in the context of function space.

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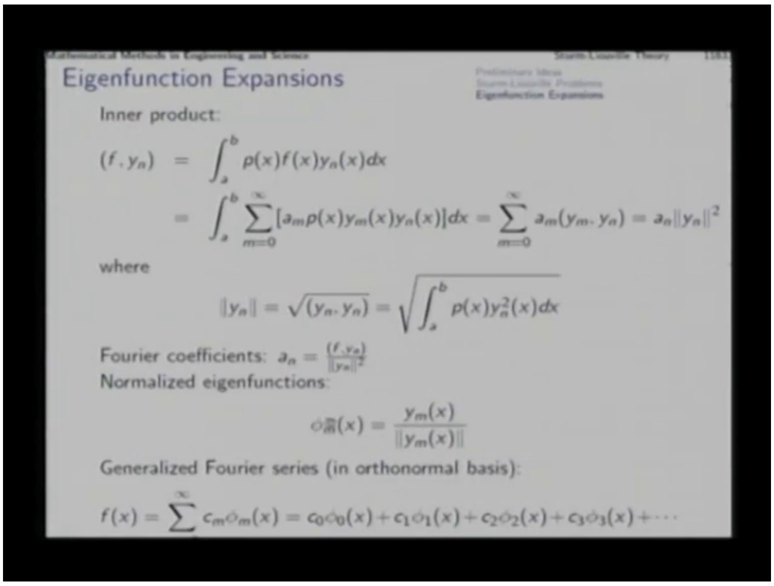

Now if we substitute the expression the summation expression for f x here sum of a m, y m then we will find that this integral will give us a series of inner products like this.

Now, since y m and y n are orthogonal corresponding to m not equal to n therefore, in this sum only that particular term in which m and n are equal will survive all others will become 0. So, you will get this and y n norm turns out to be the square root of the inner product of y n with itself which is this. Now these this a n which we get out of the division of this inner product with this norm square turns out to be the Fourier coefficients.

Now other than the orthogonal family of functions y 0 y 1 y 2 y 3 we could also say that we look for orthonormal family of functions, in which all functions are mutually orthogonal and each of the basis functions has a unit norm, in that case from the same y 0 y 1 y 2 etcetera we could have defined phi 0 phi 1 phi 2 etcetera like this which will be an orthonormal basis and in that case the Eigen function expansion in orthonormal basis will be given like this.

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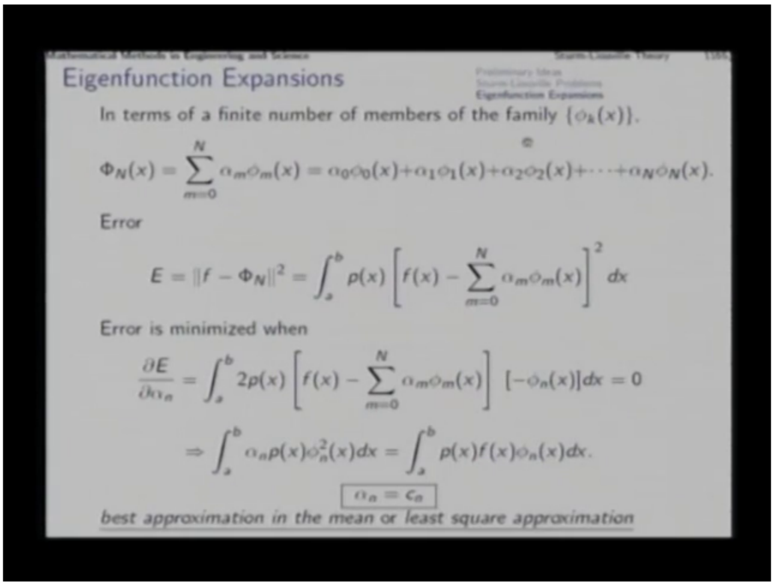

Now, the question arises what c 0 c 1 c 2 etcetera will give us the best representation of the function and that is very interesting result that is, if we say that we are looking for an expansion of this kind and then up to capital N times if we take a finite sum a finite number of terms if we keep in hand, then we will get this phi n that is sum up to capital N. Then in that case the error will be this actual F minus phi N norm square, and when we try to put that f x and this phi n here and try to find out which values of alpha 0 alpha one alpha 2 etcetera this norm will be this error norm square will be minimized, then we find that the first order condition del E by del alpha n equal to 0 will give us alpha n equal to C n.

That means the previous method of finding the coefficients based on the inner products will give us that particular composition for a finite number of terms in the series, which gives us the least square error. And that is why the least square approximation or as it is called the best approximation in the mean involves these orthonormal coefficients, orthonormal components and the coefficients based on the inner product as we have seen earlier.

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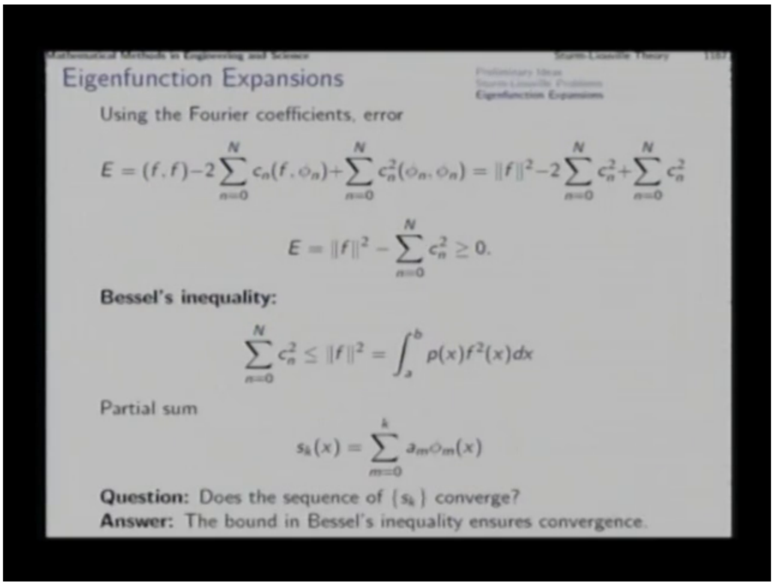

Now, using the Fourier coefficients you can find out the error the least error that you get as this, this is something like the vector error that is for this vector if you try to represent with the help of several vectors in the space several basis vectors, then if you take a partial sum not taking all the basis vectors, then the error that remains will turn out to be this square plus whatever minus whatever components you have accounted for. So the size of the function square minus whatever components you have accounted for will turn out to be the error square that has remained.

Now since this is greater than equal to 0. Therefore, you find that the sum of the squares of the coefficients is always less than equal to square of the function norm and this is called the Bessel s inequality, and if you if the number of terms capital N tends to infinity, then the sum converges to the function norm and that is called the Parseval's identity.

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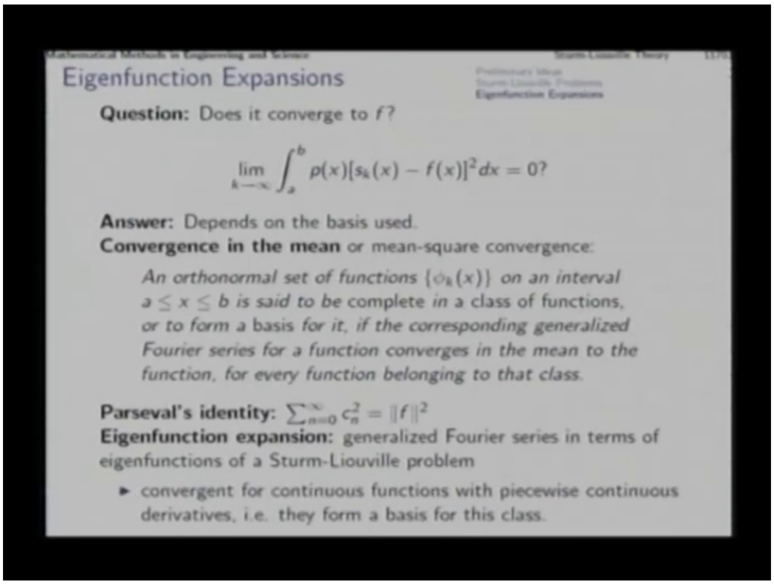

Now, we can omit this steps and the most important result out of this entire Eigen function expansion is that this is Parsevals identity. So, we say that an orthonormal set of functions on an interval is said to be complete in a class of functions or to form a basis for it, if the corresponding generalized Fourier series convergence to the mean in the mean to the function. That is as you include more and more components as you include more and more terms, the sum of the terms converges to the actual function then you will say that for all functions in that class.

Then you say that that set of basis functions is complete in that class of functions and here is the great use for the solutions of a Sturm Liouville problem, that is the generalized Fourier series in terms of Eigen functions of a Sturm Liouville problem, turn out to converge for all continuous functions with piecewise continuous derivatives. That is functions of this kind with which are continuous and the derivatives also are continuous piecewise for all such functions, the solutions of a Sturm Liouville problem turn out to form a complete class of basis functions and therefore, they serve to expand the arbitrary functions of this class in order to represent and manipulate them.

In the coming two lectures we will consider a few cases of such representations and that will include Fourier series and its generalizations in terms of Fourier integrals and Chebyshevs polynomials.

Thank you.