Mathematical Methods in Engineering and Science
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Module - VII
Application of ODE's in Approximation Theory
Lecture - 01
Series Solutions and Special Functions

Good morning. In this lecture we will study power Series Solution of Differential Equations. Actually there will be two series solution one is power series and the other is another method which also gives the solution of a differential equation in terms of an infinite series. Let us first try to understand why it is important and for that let us revisit this discussion that we made earlier regarding second order differential equations.

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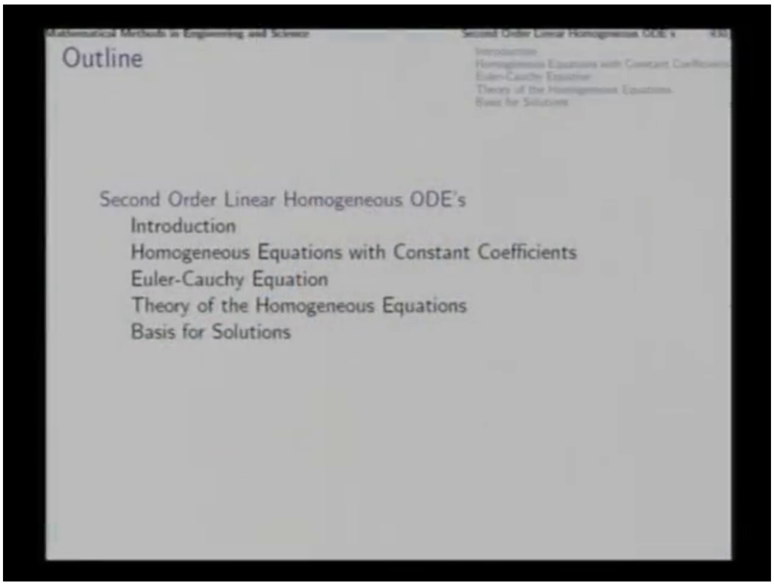

We had already discussed that second order ordinary differential equations are quite important in terms of applications and their solutions is are very important for describing a lot of natural and engineering systems.

Now, we discussed earlier.

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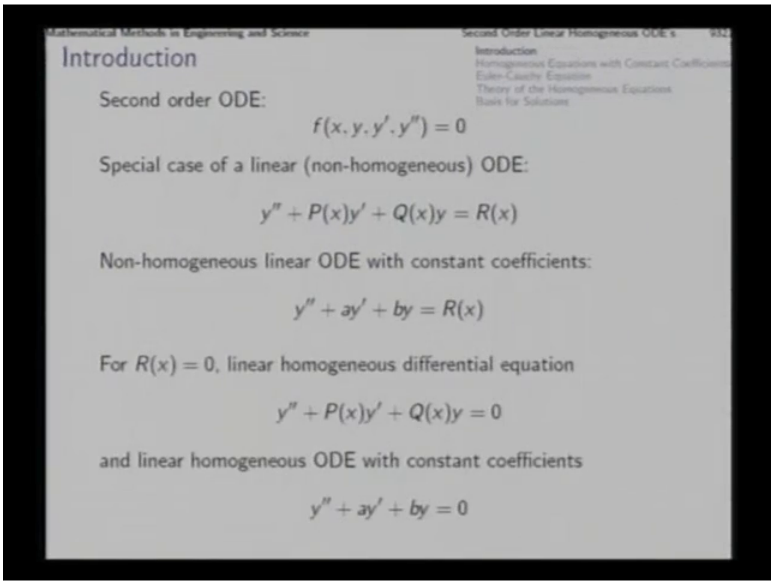

That the solution of this general second order linear differential equation, we can make completely we can form construct completely if we can solve the corresponding homogeneous equation first. That is we already have a general method the method of variation of parameters to solve this differential equation in general for any continuous and bounded right hand side vector R x, provided that we first have a basis for all solutions of the corresponding homogeneous equation.

Now; that means, that if we can find a complete solution of this homogeneous equation, then with the help of method of variation of parameters we can always find the complete solution of this. Now when we try to solve the homogeneous equation completely then we find that for variable coefficients P x and Q x there is no general method which will give us 2 linearly independent solutions of this. All that we could make out from the reduction of order method that if one solution of this equation is available then with the help of reduction of order we can find out another linearly independent solution, but then for variable coefficients P x and Q x there is no general method to find that first solution itself

In particular cases if we are lucky then we might be able to find both solutions of this and in some cases if not both then at least one and then reduction of order we can solve we can find out the second linearly independent solution. But in general we cannot solve this equation completely that is we cannot we do not have a general method which will give us a solution of this whatever be the functions P x and Q x. In a particular situation with P x and Q x constants we know that we can all the time find its solution; now what stops us from finding the solution general solution of this.

Now, in this case you are lucky in the sense that we already knew certain functions exponential functions and sinusoid, in the terms of which we could find out the solution of this. Now the question is that after conducting the theoretical analysis of this equation we could say a lot of things about the way the solutions of this behave. If this equation has 2 solutions y 1 and y 2 and which we know then we can find out a lot of their properties. Now quite often we can say that even if we do now know those solutions precisely, but we can tell a lot of things about their properties. For example, if y 1 and y 2 are 2 linearly independent solutions then we could say that their (Refer Time: 04:39) will be always positive or always negative and so on.

So, based on such observations a lot of theory has been developed on the way the 2 solutions behave and many of the facts can be studied and analyzed even without knowing the solution precisely, and from such studies a lot of information regarding those solutions can be arrived at now if. So, then even without knowing the 2 solutions completely and in closed form in terms of analytical expression, we can say a lot about their properties the way they behave. And if we need to find out the value of the function y 1 and function y 2 and their derivative then as an alternative route to the closed form expression we can try to find their expressions in terms of infinite series, and this gives us the motivation to see series solutions for those cases where we cannot solve this analytically in terms of elementary functions like sinusoids or exponential functions or logarithmic functions or polynomials or such thing.

Now, when we say that when this is the course of action we can take, when we cannot find the solution of this equation in terms of elementary functions then we may as well ask; what are these so called elementary functions? (Refer Time: 06:12) sinusoids are elementary functions why exponential functions are elementary functions. There is no absolute reason why these are called elementary functions these are called elementary functions because we know these functions from mathematics which is more elementary to the study of differential equations. And therefore, we call them elementary functions. Any series solution which we might develop by expanding the solution in terms of infinite series could also be considered as a function just like sin x or cos x or e to the power x if we had known their properties beforehand.

Similarly, if we do not know sin x and cos x as functions to begin with, then we can develop those functions from series solution of a differential equation. To understand the idea behind this we can consider this differential equation.

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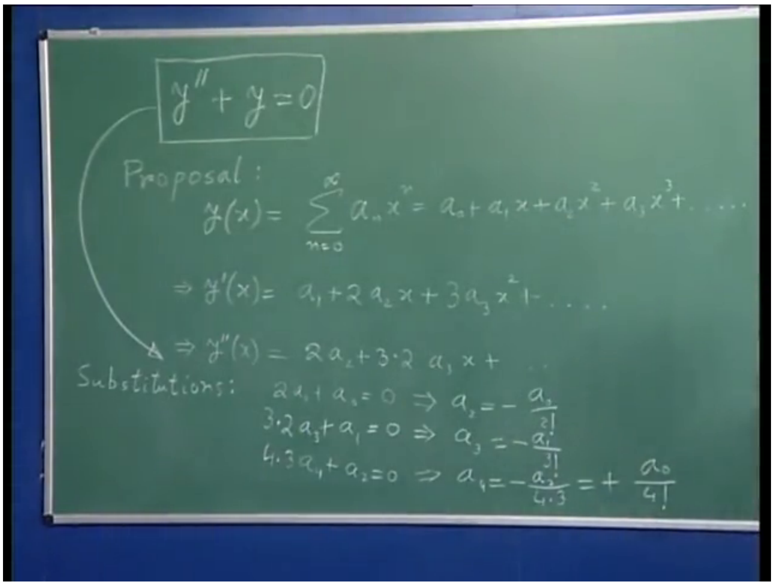

We know that cos x and sin x are 2 linearly independent solutions of this, but suppose we did not know that, in that case we could have said that let us propose a solution in this manner in the form of an infinite series.

Now, if we propose this solution as an infinite series then we can differentiate it. Now as I differentiate we get the first derivative as a 0 gives constant. So, from here we get a 1 from here we get 2 a 2 x from here we get 3 a 3 x square so on, we can differentiate once more and this gives 0, this gives twice a 2, this gives 3 into 2 into a 3 x plus and so on right.

Now, after having these expressions for y, y prime and y double prime these expressions we can substitute in the differential equation, in order to find these coefficients now in this particular case y prime is absent. So, the substitution will involve just the sum of this y and y double prime. Mow sum of these 2 and say that that has to be 0 for all x; that means, we will look for term by term equality. So, the constant term will be twice a 2 plus a 0. So, when we substitute first we get twice a 2 plus a 0 equal to 0 and that will give us a 2 in terms of a 0, a 0 by 2 which is same as minus a 0 by 2 which is same as minus a 0 by factorial 2, then we get the coefficients of x together. So, we will get this 3 into 2 a 3 plus a 1, 3 into 2 a 3 plus a 1 equal to 0. So, from here we get a 3 in terms of a 1 that is minus a 1 by 3 into 2, minus a 1 by 3 into 2 which is same as 3 factorial. Then we will equate the x square term from here we will get a 2 and from here we will get 4 into 3 a 4 four into 3 a 4 plus a 2 equal to 0. So, this gives us a 4 in terms of a 2 minus a 2 by 4 3 minus a 2 by 4 into 3 now a 2 is already known in terms of a 0. So, when we substitute this also will get in terms of a 0 and that is this

Now, you see all the even order coefficients will be found in terms of a 0, a 2 is minus a 0 by factorial 2 a 4 is plus a 0 by factorial 4, a 6 will be again minus a 0 by factorial 6 and so on. Similarly all the odd order coefficients will be found in terms of a 1 a 3 is minus a 1 by factorial 3 similarly a 5 will be plus a 1 by factorial 5 and so on. So, first you substitute these coefficients back here in all the even order terms a 0 will be common, from here we will get 1 from here we will get minus x square by factorial 2 next from here we will get plus x to the power 4 by factorial 4 and so on.

Similarly, the odd order terms will again come together with a 1 common. So, then we will get this solution.

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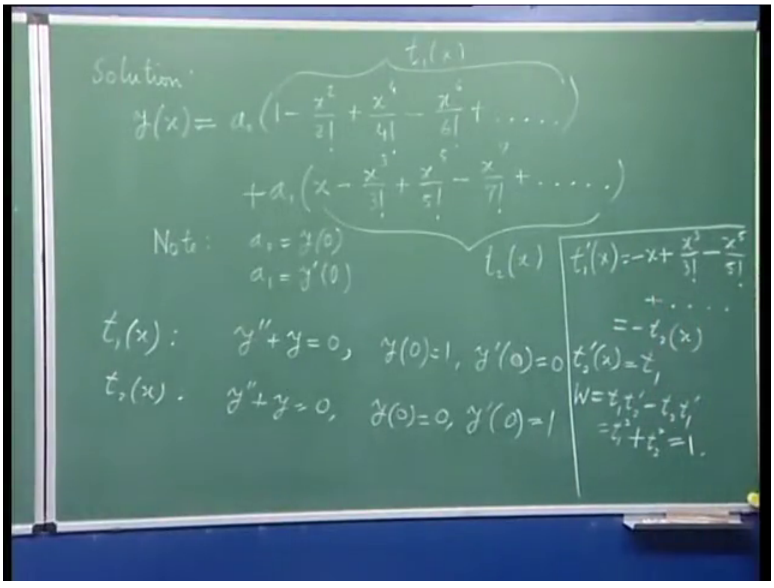

A 0 into even order terms plus a 1 into odd order terms and you can say see very clearly that these 2 coefficients which remain are actually the 2 arbitrary constants, which will appear in the solution of any second order differential equation right and in particular you can see proposed solution itself that a 0 happens to be the initial condition y of 0, y at 0 at x equal to 0 y is a 0. So, this is actually the initial condition and the other initial condition that is y prime at 0 will be found from here, here you try to put x equal to 0 you get y prime as a 1.

So, these 2 coefficients turn out to be actually these initial condition at x equal to 0. So, now, and why x equal to 0 is important, because this infinite series is actually the (Refer Time: 11:45) series of y x around x equal to 0, similarly we could find out the (Refer Time: 11:53) series around any other point x 0. So, x minus in that case the solution would be proposed in terms of powers of x minus x 0 ok.

So, now from here you can say that suppose this function which is in this parenthesis and this function which is in this parenthesis, we will give them some name. You can see that this function this series is actually the (Refer Time: 12:24) of cos x similarly this is (Refer Time: 12:27) around x equal to 0 or sin so, but suppose we did not know this 2 functions we did not have any prior knowledge of these 2 functions then this entire function suppose we give a name t 1 of x. Similarly this entire function we give another name say t 2 of x we could have done that and once we give them 2 names then you can also see that this function itself turns out to be the solution of the differential equation with these initial condition and this is the solution of this initial value problem.

Similarly, t 2 x will be the solution of the same differential equation with initial conditions right. Now, you can see that we have got a series solution and to that we have given 2 names, and if these functions this function and this function turn out to show certain interesting pattern and turn out to appear in the solution of many systems then we would call these 2 functions as special functions.

Now, after defining these 2 function, we say that these are 2 function 2 solutions of this differential equation; one with this set of initial conditions and the other with this set of initial conditions now you want to ask whether they are linearly independent. Of course, you can see that they are linearly independent very easily this precise analysis we conducted while we were studying this particular that particular lesson. Apart from that after having the x 2 solutions in hand you can see that you can multiply this with no constant to get this. So, from there you can see that they are linearly independent suppose you want to do a better job at verifying the linear independence of this, then you try to find out the Wronskian of these 2 functions, then you will need derivative and let us quickly do the derivative calculation in this window t 1 x is this, ok.

So, what will be t 1 prime? You will find the derivative of this is 0 derivative of this is minus 2 which will cancel this. So, x you will get you will get minus x then from here you will get 4 x cube by factorial 4. So, 4 will cancel the component 4 from here and what will remain is factorial 3. So, plus x cube by factorial 3, and some here you will get 6 x to the power 5 by factorial 5 factorial 6. So, 6 will cancel 6 from here and what you will get is this and so on do you notice that we get exactly the negative of this minus x plus x cube by factorial 3, minus x to the power 5 by factorial 5 and so on. So, this is minus t 2.

Similarly, if you differentiate this t 2 if you differentiate, then you will find the derivative of this is here, derivative of this is here derivative of this is here and so on. So, you will find that t 2 prime will turn out to be simply t 1 and then when you try to find out the Wronskian; Wronskian will be t 1 into t 2 prime, minus t 2 into t 1 prime. Now t 1 into t 2 prime t 2 prime itself is t one. So, this is t 1 square minus t 2 into t 1 prime t 1 prime is minus t 2. So, minus t 2 into minus t 2; so you get this. So, Wronskian of the 2 solutions here will turn out to be t 1 square plus t 2 squares.

Now, you find that if you try to differentiate this Wronskian, then you will find that a Wronskian derivative turns out to be (Refer Time: 17:29) that means, Wronskian turns out to be a constant function. If it is a constant function then its value the value of the Wronskian will be the same everywhere. So, as a function it will be a constant function and. So, its value can be found out from any point. So, at x equal to 0 t 1 is 1, at x equal to 0 t 2 is 0. So, from x equal to 0 we evaluate the value of the Wronskian and since Wronskian prime derivative of the Wronskian is 0. So, Wronskian is a constant function. So, everywhere it will be 1.

We find out a very valuable piece of information for this pair of function this fellows derivative is negative of this fellows derivative is exactly this not only that the 2 functions when we square them and add we will always get 1 this is the Pythagorean property cos square x plus sin square x equal to 1.

Now, this property of these 2 functions we arrived at without any regard to Pythagoreans theorem, without any regard to the way we understand trigonometric ratios. Now this way we can go on and develop the entire trigonometry just by studying the 2 functions the series solutions that we have got and with the reference of that equation there. So, this is the way lots of functions get developed which was not known earlier.

So, let us try to summarize what we have been discussing all these while.

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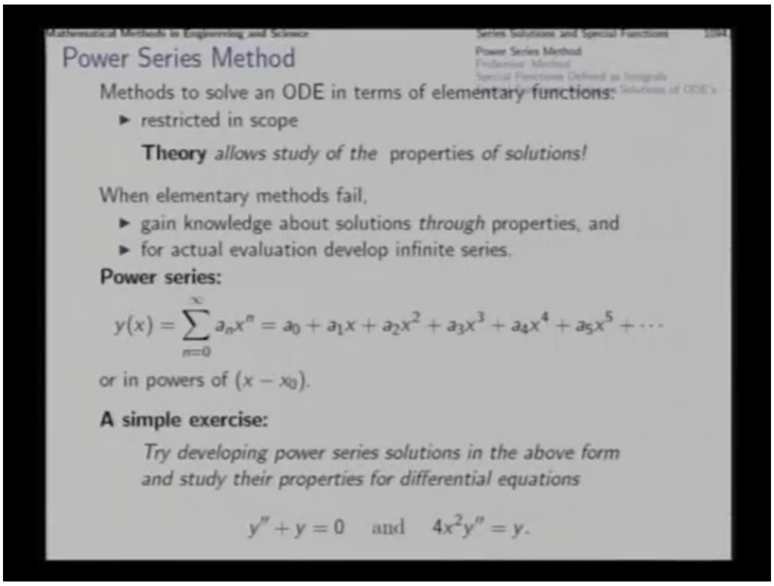

Methods to solve an ODE in terms of elementary functions are restricted in scope; however, the theory developed for the solutions of these linear differential equations allows a lot of study of properties of the solutions. Now if we can study the if we can study the solution with their properties, then when elementary methods fail to find the solutions of the differential equation for variable P x and Q x, then we can gain lot of knowledge about solutions through their property and for actual evaluation of the function value, value of the solution or that derivatives etcetera we can develop infinite series a power series of this kind in powers of x or in powers of x minus x 0, that is the power series can be around x equal to 0 or around x equal to any other point say x naught.

Now, a simple exercise to understand the idea behind it is to try developing power series solutions in this form for some simple differential equation like this, which we just now did and you can try to develop such solution for this case also in this case this method will fail and we will go to the second possibility very soon, before that let us exhaust this side here.

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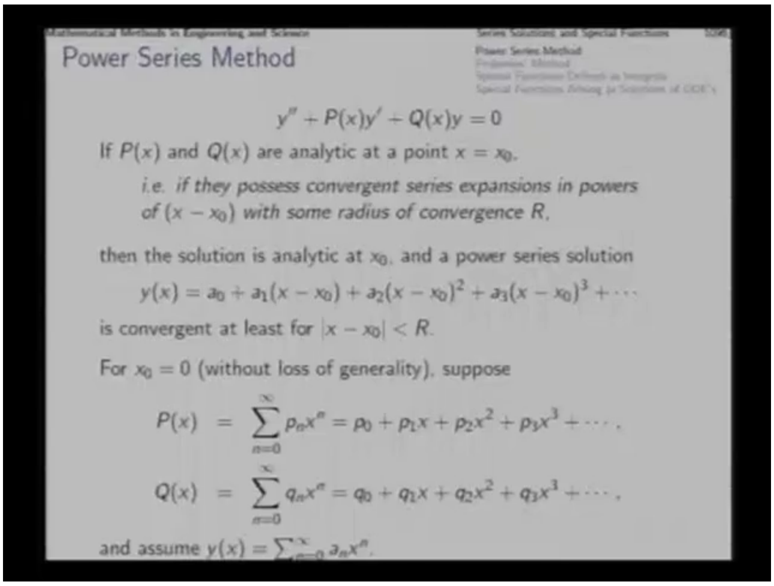

Now there is a little bit of theory behind it, to develop this kind of a solution called power series solution. So, suppose our differential equation is this and then for a solution of this kind to be valid to make sense it will be necessary that the functions P x and Q x are analytic at the point around which we are going to develop the series solution

And what is the meaning of P x and Q x being analytic at this point? The meaning is that P x and Q x possess convergent series expansions in powers of x minus x naught with some radius of convergence, that is P x and Q x must have a power series expansion in terms of an infinite series of powers of this, which is convergent with some radius of convergence that is within some distance the series of P x and Q x should be convergent and in that case the solution that we developed like this will be convergent series with a radius of convergence of at least R, it could be larger. But at least R is then guaranteed that is if P x. And if the coefficient functions are analytic that is if they have convergent series representations within a distance of R then a solution that is developed like this will have meaning that is will be a convergent series for all x calling within that distance at least it could be better.

Now, whether we do it do the solution around x equal to x naught or whether we do it around x equal to 0 for conceptual purposes no (Refer Time: 22:33) is lost because the coordinate shift of x equal to x naught to x equal to 0 can be always made we can always call x minus x naught as another variable say z and carry out the entire analysis for z. So, to keep the (Refer Time: 22:53) of expressions simple.

Let us discuss here the case of x 0 equal to 0. In that case if P x and Q x have convergent power series solutions power series expansions like this then after assuming y in this manner. The way we did in this particular example we can substitute y, y prime y double prime, P x and Q x in terms of such series and put everything in the equation.

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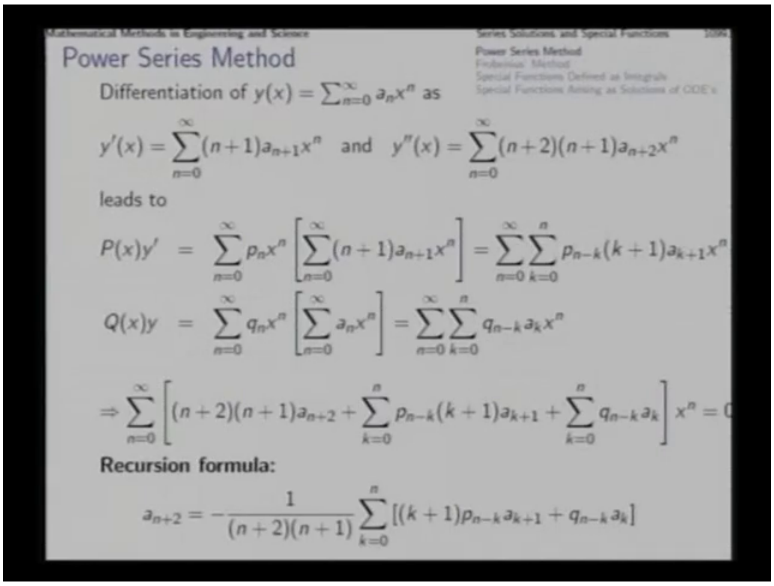

If we do that then we will have y prime as this y double prime as this just differentiation and then P x multiplied with y prime. So, this is P x this is y prime and then this double summation we can write like this and here the terms of powers of x x to the power 0 x to the power 1, x to the power 2 etcetera etcetera having club together. And accordingly the summation indices have been adjusted a little bit to make this summation this double summation the same as this product of 2 series similarly for Q x into y. Then we take this P x y prime and this Q x y and add to this y double prime from there we can find out all the coefficients and equate them term by term to 0. So, when we equate the constant term that is n equal to 0 term to 0 then we get one equation in the coefficients.

Similarly, when we equate the coefficients in the sum of x to the power 1 we get another equation and so on like this, this is the sum of this and this and from there term by term when we equate the coefficients, we get this whole thing equal to 0 called n equal to 0 1 2 3 and so on and these give us a recursion formula like this. A n plus 2 is expressed in terms of the previous coefficients in terms of a 0 a 1 a 2 a 3 up to a n plus 1. So, from here what we can do is we can find out a 2 in terms of a 0 and a 1. And then we can find out a 3 in terms of a 0 a 1 a 2 which means in terms of a 0 and a 1 and all higher coefficients then find out all in terms of 0 and a 1. The 2 first coefficients 2 basic coefficients remain and they should remain because it is a second order differential equation that we are solving and 2 arbitrary constants will remain.

So, now this method can be utilized to solve the case when P x and Q x are (Refer Time: 25:46) something of that sort we did here, in this particular case P x was 0 Q x was 1. Now if P x and Q x are not analytic then this method will not succeed and that you will notice if you try to find out a solution of this differential equation by this same method. The way this attempt fails will tell you why this kind of a situation cannot be handled with a power series solution like this. For this another special kind of series solution has been developed and that is called the Frobenius method.

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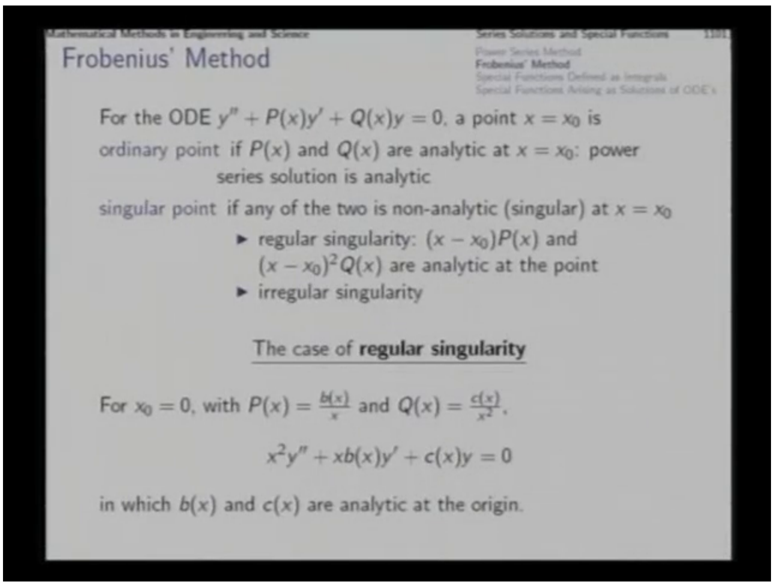

Say we take this differential equation and for that differential equation a point x equal to x 0 is called an ordinary point if P x and Q x are analytic at this point, that is if P x and Q x have convergent series representation around x 0; that means, series in terms of powers of x minus x naught. If it is an ordinary point then the power series solution method will succeed, on the other hand of any of the 2 P x or Q x is not analytic then power series solution method will not succeed.

Now, see what we call P x and Q x are the coefficient functions when the coefficient of y double prime has been reduced to 1 reduced to unity, and when we do that for this differential equation when we try to reduce this differential equation to the standard form in which that coefficient of y double prime is 1, then basically we need to divide by 4 x square. And therefore, the coefficient of y written in the standard form will turn out to be minus 1 by 4 x square, and this Q x will be that case minus 1 by 4 x square and around x equal to 0, it will not be analytic it will be singular, that is it will not have a power series expansion that coefficient function Q x will not have a power series expansion in terms of powers of x that is powers of non negative powers of x. So, therefore, this particular method will not work in that (Refer Time: 28:11) kind of a differential equation.

However, when P x or Q x or both are singular functions at x equal to 0, that is at that point they are undefined. So, they do not have a radius of convergence within which a power series representation will converge. So, that point we call as a singular point and at that point any of the 2 functions P x or Q x or both are non analytic that is singular.

Now, such singularities appear in 2 ways, one is regular singularity in which even if P x is not analytic, x minus x 0 into P x is analytic and even if P x is not analytic x minus x 0 whole square into Q x is analytic. So, that particular situation is called a regular singularity and if even after multiplying P x with 1 power of x minus x 0 and Q x with x minus 2 powers of x minus x 0, even after that if singularity persists then that is called irregular singularity. And this is the case which cannot be really handled in the form of a power series in the form of a series solution, but if we find that this product and this product are analytic then we can still avoid the singularity problem in a certain manner and that manner is the center point of Frobenius method. Let us see how it works the case of regular singularity.

To avoid the (Refer Time: 29:49) without loss of generality, we can consider x 0 as 0 and P x is represented as b of x by x and Q x is represented as P of x by x square, in which b of x and c of x are analytic. In that case x P will be analytic and x square Q will be analytic what is demanded here for regular singularity and then the differential equation which is here will get as x square y double prime plus b x y prime plus c x y equal to 0.

Now, see this is a differential equation in which b x and c x are analytic, now this resembles a particular equation which we have solved earlier a few lectures back and that was Euler Cauchy equation. In the case of Euler Cauchy differential equation we had a similar situation in the coefficient of y double prime we had x square, in the coefficient of y prime we had x, in the coefficient of y we had a constant. Now only difference is that in the case of Euler Cauchy equation b x and c x were constants here they are functions, but good news is that they are analytic functions. So, b x and c x being analytic at the origin will constitute a case of regular singularity and in this case Frobenius method will work.

Now, in the case of Euler Cauchy equation, which was exactly this with b x and c x constant what we did? We suggested solutions in the form of x to the power a here also we will do something similar, but since b x and c x are not constant, but analytic functions. So, we will represent we will propose the solution as x to the power r into a power series in this manner.

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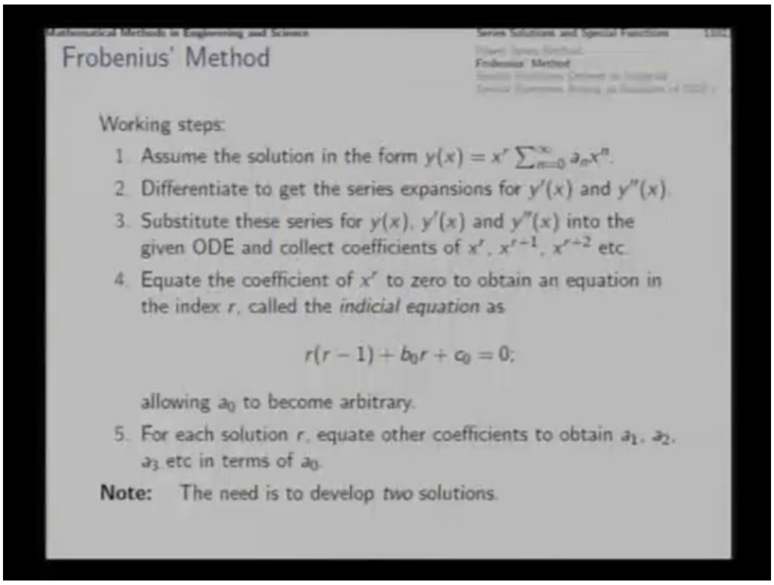

We suggest we propose x to the power r into a power series, then we differentiate it get y prime and y double prime substitute this into the differential equation and that will have powers of x the sum will have powers of x as r, r plus 1, r plus 2, r plus 3, r plus 4 etcetera r could be any number it need not be an integer r could be 1.2 or some such thing now what we do? We equate the coefficient of x to the power r the lowest power to 0 and that will give as an equation in the index r itself called the indicial equation that will look like this you get when you put these expansions into the differential equation.

Now, an indicial equation like this we got in terms of in the case of solution of Euler Cauchy equation also, from there we got a quadratic equation here also we get a quadratic equation and this quadratic equation will give us 2 solutions for r. And as we get these 2 solutions for r, then that will mean that for those 2 values of r we will get a solutions like this which will satisfy the differential equation.

So, for each solution r we equate the other coefficients and obtain the coefficients a 1 a 2 a 3 etcetera in terms of the first coefficient a 0, that first coefficient a 0 is indeterminate because that will constitute the arbitrary constant in the solution of the differential equation. Now for each of the 2 values of r we get 1 such solution with an arbitrary a 0. So, if we take a 0 as 1 we get a solution for r 1, we get another solution like that for r 2 and then when we take these to solutions and combine them linearly that is as if we are allowing the 2 a zeros to be different c 1 and c 2; then the 2 solutions from there we can linearly combine and get the general solution.

Now, there is a particular case here in which the 2 solutions would differ by an integer, now note that if they are if the 2 values of r, r 1 and r 2 they differ by a real number that is both the solutions are real, if they differ by a real number which is non integer then we get 2 linearly independent solutions like this and we can combine them linearly to get the general solution. On the other hand if the difference of 2 r's r 1 and r 2 is imaginary that is if both the solutions are complex, than twice the imaginary part will be their difference. If that is also non integer then also we can find 2 solutions in that case we will always find the solutions.

If the 2 solutions of this quadratic equation are real and if their difference is a an integer then you might find a situation where the 2 solutions are not linearly independent, in that case also through reduction of order from a single solution in hand you can also develop the second linearly independent solution and then you can combine the 2 linearly independent solutions to get the general solution. Now this way quite often it happens that you come across infinite series which have certain interesting properties and they appear again and again in many different diverse applications. Then you get then you extract some suitable function out of it and give it a name such a function is then defined as a special function.

Now, the solutions of ordinary differential equations are not the only source of such special functions in applied mathematics; there have been other sources also. For example, there have been certain interesting integrations integrals which often arise in different field and they get certain special names and they are also called special functions, some of those special functions you know it is gamma function.

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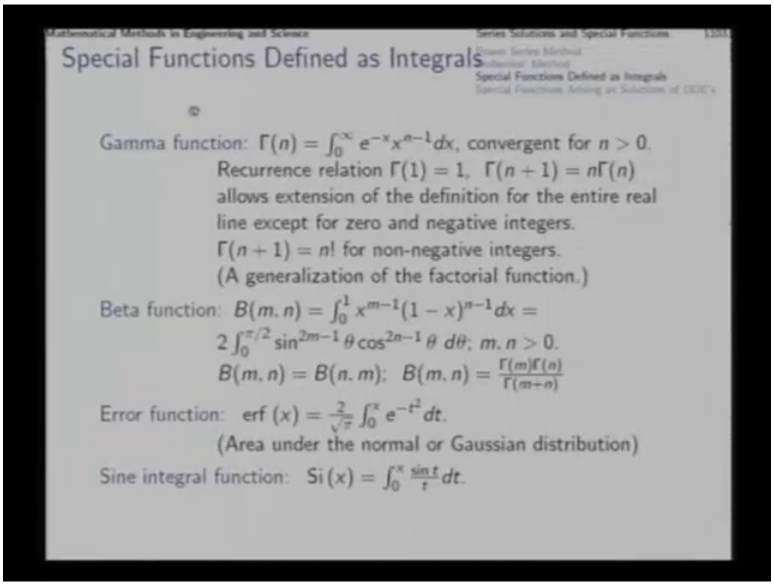

Beta function error function sine integral functions these are special functions which arise out of certain integral.

And similarly out of the solution of certain differential equations which are functions which appear in many situations, they are given certain names.

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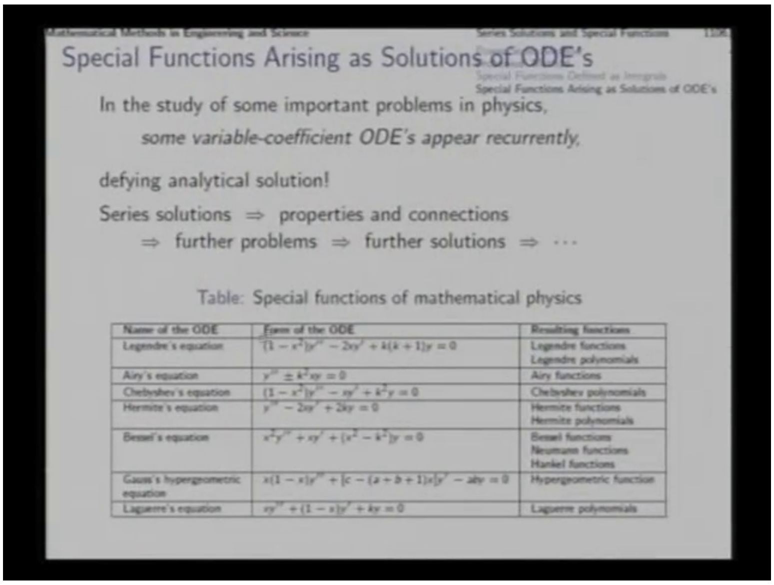

A lot of such special functions have been needed and defined and developed by physicists in because in the study of some important problems in physics; they appear variable coefficient ordinary differential equation recurrently in several different types of problems and when physicist try to solve them then it was not possible to solve them analytically in terms of known elementary function. And therefore, people look for series solutions of such differential equations, because the solutions of those differential equations were necessary to proceed forward in the study of those physical systems. And as they found series solutions they studied the properties of those solutions and their interconnections the relationships among themselves.

And quite often such relationship such properties give rise to new further problems, which give rise to further solutions and further interesting special functions. This way they have developed a very voluminous store house of special functions and a small list a little snapshot of those special functions arising out of differential equations is here, this is called the Legendre equation and the solution of this gives rise to certain special functions called Legendre functions and then out of them also some very special functions which are Legendre polynomials. Similarly from Airy's equation you get airy functions, from Chebyshev equation we get these from this Chebyshev equation we get certain polynomials certain interesting functions which are called Chebyshev polynomials similarly Hermite functions, Bessel functions, hypergeometric functions and so on.

So, these are these functions are all special functions arising out of the solution of certain ordinary differential equations, which arose in applications. And later their application was found in several other areas in particular for function representation function approximation. That is for approximation of other functions quite often these work as basis functions. We will continue into this study for quite a few lectures to come now let us see 2 important cases of this 1 is Legendre equation and the other is Bessel equation and later we will also consider the Chebyshev equation for a particular purpose.

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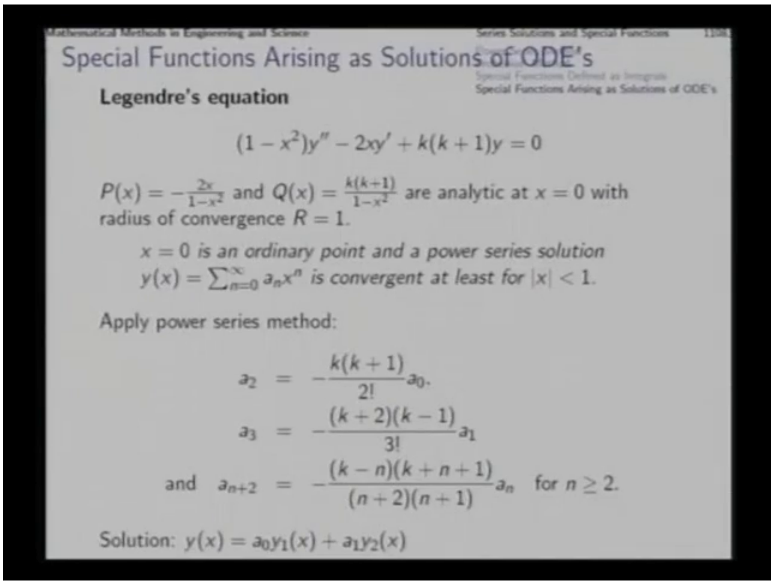

This differential equation is called Legendre's equation, here if you write the differential equation in the standard form then you will need to divide the entire differential equation with 1 minus x square to get the coefficient of y double prime as unity and in that case in place of P x you will get minus twice x by 1 minus x y that is this, similarly Q x will be found as k into k plus 1 by 1 minus x square. Both of these functions are analytic around x equal to 0 with a radius of convergence 1, because they encounter their first singularity at x equal to plus 1 and x equal to minus 1 and therefore, up to that point that is with a radius of convergence equal to unity they are analytic.

That is if you try to work out a representation of this function as a power series of x that is constant plus constant into x plus constant into x square plus constant into x cube and so on, then that infinite series will be valid within the interval x equal to minus 1 to 1 beyond that it will not be valid similarly for this. So, these 2 functions coefficient functions P x and Q x are analytic at x equal to 0, with a radius of convergence R equal to 1.

And. So, you find that x equal to 0 is an ordinary point and a power series solution in this form is convergent at least with a radius of convergence 1 it could be more, but that is not guaranteed. So, you try the power series solution directly.

So, as you apply the power series solution, you get a 2 in terms of a 0, a 3 in terms of a 1, and then a 4 in terms of a 2 which itself is expressed in terms of a 0 and so on the way we found in the first example of this lecture Eigen value we will find a n plus 2 in terms of a n for n greater than 2 greater than equal to 2 that is n equal to 2 onwards; a 0 and a 1 will remain indeterminate because they are the arbitrary constants for the solution in the general solution and a 2 onwards you can determine in terms of a 0 and a 1.

So, all the even coefficients even order coefficients will be found in terms of a 0 and odd order coefficients in terms of a 1 and if you then (Refer Time: 42:07) together a 0 into 1 series plus a 1 into another series and then these 2 series if you call as y 1 and y 2, then this will turn out to be the general solution of this Legendre equation.

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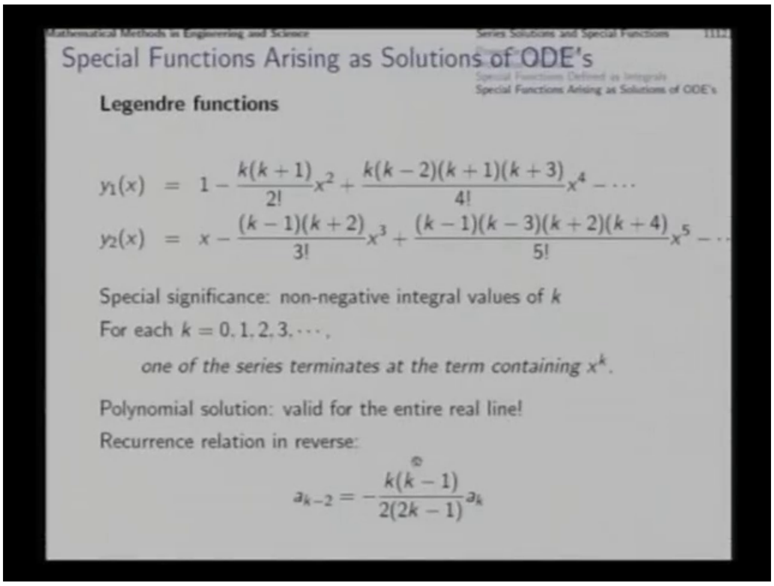

Now if we expand these terms then you will find that in this the even order and odd order terms are separated and one of the solutions is this and another solution is this. Now these 2 are the Legendre functions. Something interesting happens for non negative integral values of k non negative integral values; that means, k equal to 0 1 2 3 4 these are of special significance currently y 1 and y 2 both are infinite series and we have we are calling them Legendre functions.

Now, suppose you take k equal to 0, in that case what happens with the first series in the first series? You find that putting k equal to 0 will mean that this term will drop out because k equal to 0 will make this term 0 this term also will drop out and in the first function in the first infinite series nothing of the series will remain except the first term which is a constant 1 and that is x to the power 0 that is x to the power k. So, this first Legendre function will terminate at the first term itself and that will be just unity.

Similarly, if you put k equal to 1 then you will find that here you get k minus 1. So, this will become 0, here also you will get k minus 1 and it will be 0; that means, in the second polynomial the polynomial with odd powers all the times except the first one will become 0. So, this will be just x for k equal to 2 this term in the first series this term will remain this term will remain here this term has a k minus 2. So, for k equal to 2 this term will vanish and the next term will have k into k minus 2 into k minus 4 into k minus 6. So, k minus (Refer Time: 44:24) also. So, that term will also vanish. So, all these terms will vanish and this will be found k equal to 2. So, you will have 1 minus 2 into 3 by 2. So, 2 2 will cancel 1 minus k x square. So, it will be a finite polynomial, it will not be an infinite series at all. So, again for k equal to 3, this onwards these terms will vanish. So, this will turn out to be a cubic polynomial and so on.

So; that means, for each of these values 0 1 2 3, one of the series either this or this for k even this one for k odd this one will terminate at the term containing x to the power k. So, that series will give us a polynomial of degree k and since it is polynomial solution therefore, it will be analytic with an infinite radius of convergence it will never become undefined. So, it will be valid not only from x equal to minus 1 to 1, but for the entire real line. And therefore, with none negative integer values of k these solutions have special significance, because there will be they will be valid for the entire real line all values of x and these will be of particular interest for us.

So, in this case the same recurrence relation which we earlier wrote in terms of a n plus 2 equal to something into a n, that same recurrence relation if we write in reverse and then the corresponding polynomial if we write then.

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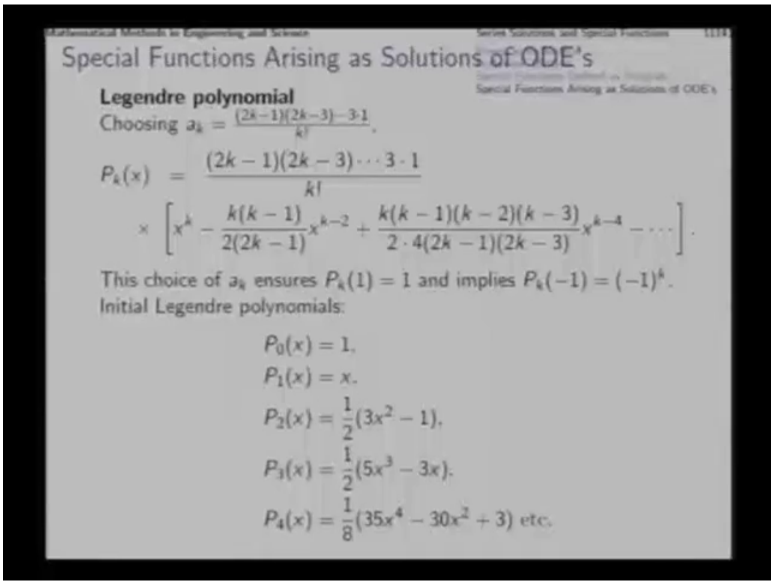

We will get this kind of polynomial and now here what we can do is that we choose a k to have this value which will make a certain point of x equal to 1 where all these polynomials will have the value 1 and that will imply that at minus 1 they are minus 1 or 1 depending upon whether k is odd or even. So, where is the constant to be chosen for defining these polynomials now as the so called Legendre polynomials?

So, we find that the Legendre polynomial of order 0 will be a constant, Legendre polynomial of order 1 will be a linear function, Legendre polynomial of order 2 3 4 5 will respectively be polynomials of degree 2 3 4 5 and so on. So, these are those initial few Legendre polynomials, P 0 is of 0 degree, P 1 is if 1 degree, P 2 is of 2 degree and so on. So, these now the coefficient that you get here outside that is because of this special choice of a k. Now these that particular choice will manifest itself in these coefficient here the leading coefficient here and that will ensure that these polynomials all these Legendre polynomials so defined will have a value of 1 at x equal to 1.

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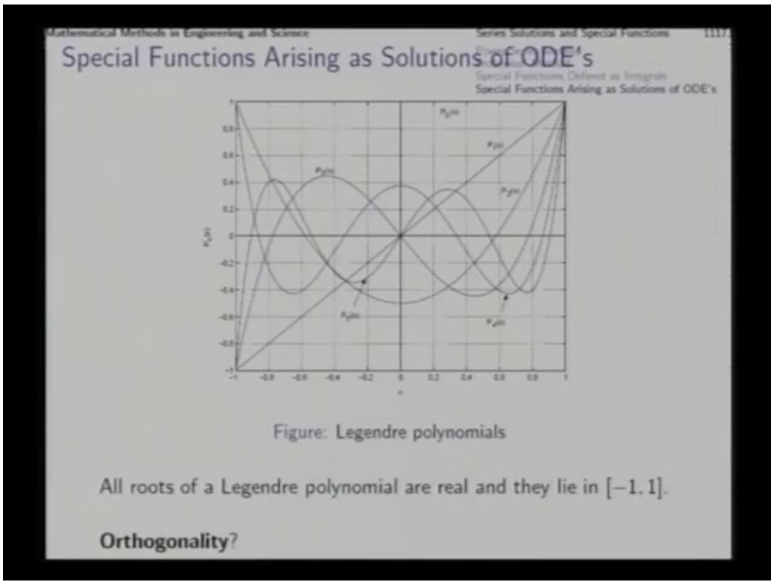

And that will also mean that at x equal to minus 1 the odd order Legendre polynomials will have value minus 1 and even order ones will have value plus 1.

Now, you see here P 0 x is 1 this constant P 1 x is this x, P 1 x is simply x P 2 x which is this parabolic function which is this, this is P 2 x, P 3 that will be a cubic function which is this P 3 going like this, going like this up and coming like this P 3 odd order. So, it will terminate here P 4 even order. So, all start here and they end either here or here depending upon whether the order is odd or even. So, P 4 will be this P 5 is next. So, all of them start from here P 0 goes like this, P 1 goes like this P 2 down, P 3 further down P 4 even steeper P 5 even steeper P 6 P 7 all of them like that and this family of functions family of Legendre polynomials you can go on defining for all non negative integers k.

Now, there are interesting properties that these polynomial functions display. One point you have already noticed that P 0 is constant P 1 is linear and its route is here, its 0 it is here P 2 is quadratic. So, it has 2 routes. So, one is here one is here symmetrically placed around x equal to 0 P 3 has 3 route one is here the other is at the origin and the third is here symmetric. Similarly you can go on noticing go on verifying or you can actually prove that all routes of Legendre polynomial are real and all of them lie within this interval minus 1 to 1. That means, all Legendre polynomials even order which oscillated like this and went up to here in this interval, beyond this interval they do not go back beyond x equal to 1 the values the function will go continuously up.

Similarly, on this side beyond x equal to minus 1 the value of the Legendre polynomial will go continuously up. For odd order polynomials it will look like this. So, here you will have the value minus 1 and beyond x equal to minus 1 on the lower side it will go down and here it will go up. So, all the routes all the zeros of the Legendre polynomials will fall within this interval minus 1 to 1.

So, apart from this there are a lot of other interesting properties of Legendre polynomials which make them very attractive choice for applications in many situations in particular in approximation theory due to also another interesting property which is orthogonality this is a property which we will examine in a little more detail in the next lecture currently have another quick look at another interesting differential equation from which we get another family of functions and that is Bessel's equation.

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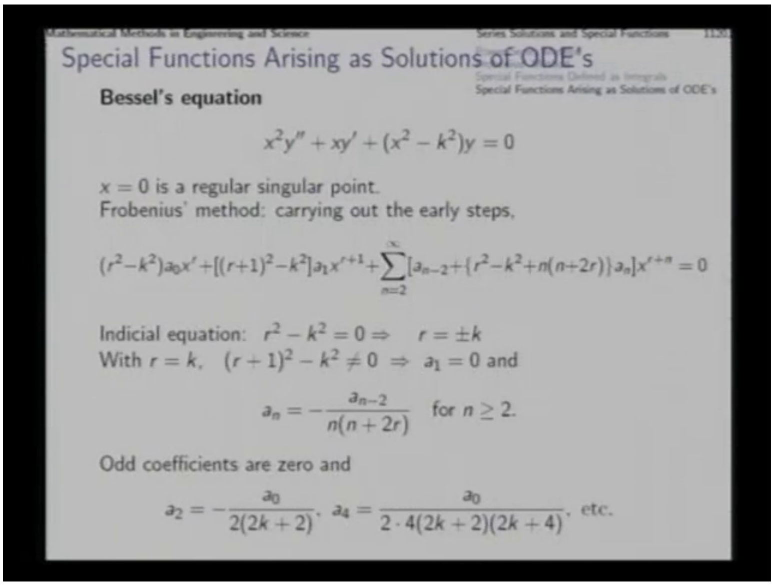

Now, in this case you will notice very quickly that if we try to frame this differential equation in the standard form then for that we have to divide it with x square. As we divide it with x square we will find P x as 1 by x which will not be analytic and will get Q x as x square minus k x k square by x square which will also be not analytic at x equal to 0. Now you can always say that why you are interested we will find P x as 1 by x which will not be analytic and we will get Q x as x square minus ks square by x square which also will not be analytic at x equal to 0.

Now, you can always say that why you are interested all the time in finding the solution around x equal to 0, try to find the solution of that around some other point. Say in terms of powers of x minus something around that point it will be analytic, but quite often the solution is of interest around that singular point itself for practical purposes of the system which we are actually studying.

So, around that point itself we want to really get the solution. And therefore, we cannot use power series solution we use Frobenius method and that we can do because here x is analytic x square minus k square is also analytic and that is the result of multiplying the equation with x square overall.

So, here 1 by x we were getting as P x; so when we multiply that with x we get 1 x square minus k square k square by x square Q x, when we multiply that with x square we get this which is also analytic so; that means, that x equal to 0 is not an ordinary point power series solution is not possible, but it is a regular singular point. So, it is a case of regular singularity. So, it is a case of regular singularity Fresenius method will be applicable.

So, when you apply Frobenius method we find that the solution of initial equation is just plus minus k, and for each of the 2 value is as we make the substitutions we can find the solution and we find that the odd order coefficients are 0 and even order coefficients are found in terms of a 0 in this manner and when we assemble all this we get what is called Bessel function.

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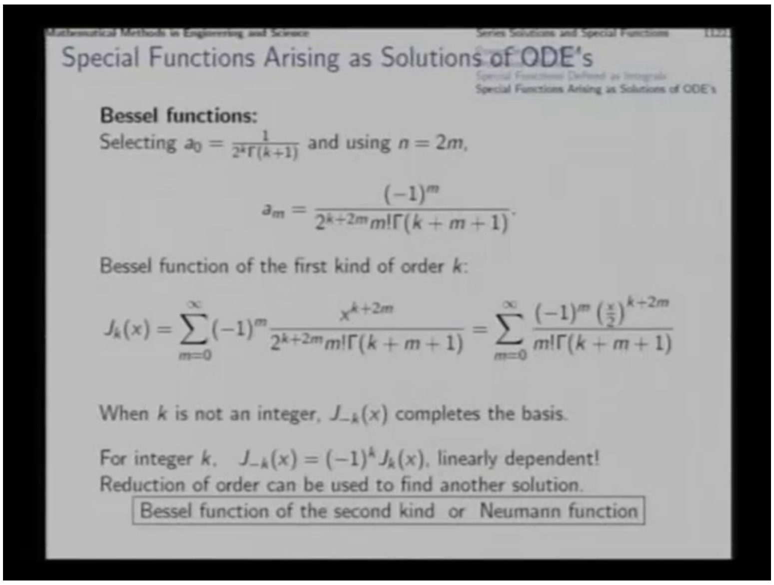

And that involves the gamma function in its expression for coefficient, and this way what we define is the Bessel function of the first kind of order k.

Sometimes if the plus minus k differs by an integer then we will find only one solution out of this and the other solution will not be linearly independent then. So, the application of the reduction of order method, we can find out the other linearly independent solution and that will give rise to another special function that is called the Neumann function.

In any case through the Frobenius method we will be always able to complete the basis either with j k and j minus k or with j k and what is called Neumann function; so this Bessel s equation solution through Frobenius method, i k the Neumann function. So, this Bessel s equation solution through Frobenius method gives us another very rich family of functions called special functions which also have very interesting property and they are applicable in many situations in physics and applied mathematics and engineering systems.

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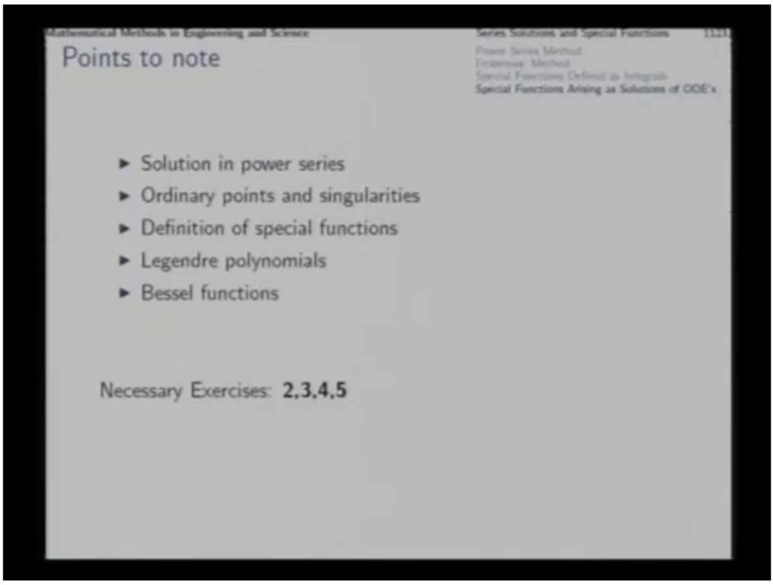

So, in this lesson the important points that we learn; the points that we need to note is solution in terms of power series is possible in many cases. And the other conceptual issues of ordinary points and singularities. We studied, we define some special functions the way the definition of special functions are made we discuss that. And in particular we studied two particular special functions, two particular families of special functions, how those special functions are generated they are Legendre polynomials and Bessel functions. Some of these we will discuss in some of the coming lectures as well.

Thank you.