Mathematical Methods in Engineering and Science
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Module – VI
Ordinary Differential Equations
Lecture – 05
Stability of Dynamic Systems

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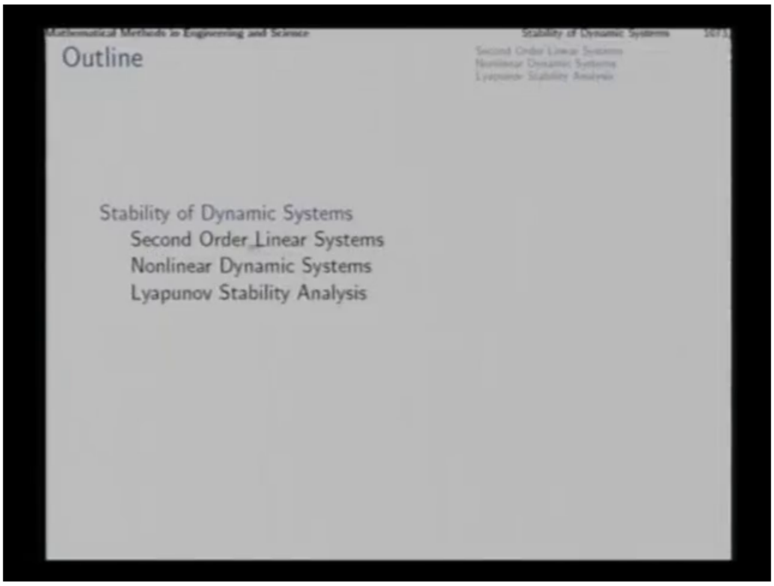

Good morning. In the previous lecture, we studied solution of systems of ordinary differential equation. Today we will apply that knowledge, apply those method to the problem of stability analysis of dynamic systems. At length, we will discuss the special case of second order linear systems, because of two reasons; one reason for by second order, and the second reason for why linear. As I outlined in the last lecture a predominant number of dynamic systems appearing in nature are follow a second order dynamics. And therefore, the analysis of second order dynamic systems becomes very important. And apart from that up to second order analysis is to a good extent possible and a lot of analysis is possible because you can show that analysis on a piece of paper. Any plot that you make on a piece of paper or the one that we will make on the blackboard once in a while all of that is actually two-dimensional. So, if we have two straight variables then we can represent the behavior of the dynamic systems in a two-dimensional plot.

Now why linear because it is first of all easy to analyze linear systems to a great extent for non-linear systems the analytical procedures get blocked after a point. So, for linear systems, we can analyze to a great extent, and theoretical predictions or theoretical study which have far reaching consequences we can draw in a major way in case of linear systems. And many actual systems are either linear or to a good extent they can be approximated by linear approximation. Therefore the second order linear systems have got enormous amount of research focus for quite a few centuries and the basic facts basic analysis regarding the stability of second order linear dynamic systems is what we will consider for the major part of this lecture. And after that we will consider a few issues regarding higher dimensional or higher order systems and non-linear systems how we go about stability analysis of those systems beyond second order and beyond linear.

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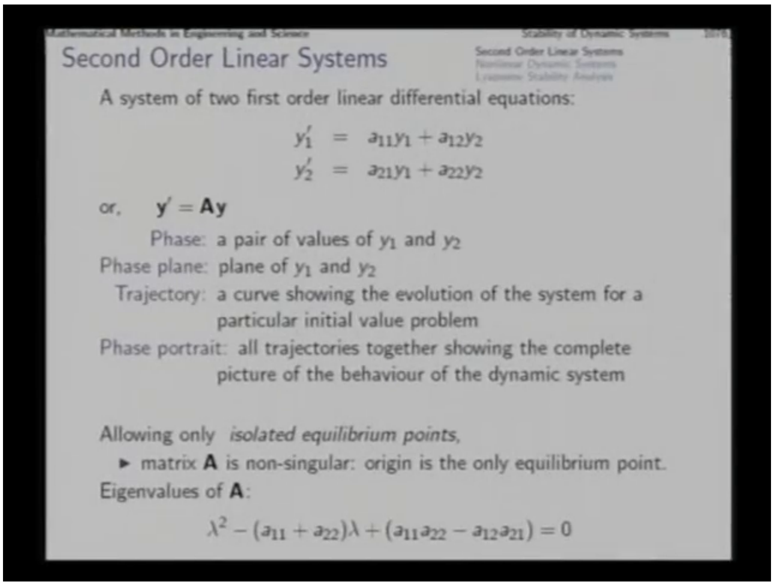

So, first we go to second order linear systems; and among them particularly we will consider the autonomous system, second order autonomous linear system. Now, the point is that why we are considering autonomous because stability analysis will make particular sense in the case of autonomous systems where there will be certain equilibrium points around which we will discuss the issue of stability. So, suppose we take a system of two first order linear differential equations like this. Now, note that we are talking about second order, but we are actually considering two first order differential equations. Now, there is no discrepancy here, there is no mismatch in the two issues, because a single second order differential equation can be always broken down into two first order differential equations as we do in straight place.

So, in straight place a single second order differential equation will also be broken down in this manner in which there will be two straight variables. So, whether the system originally consists of two first order differential equations or one second order differential equation broken down into two first order differential equations for our purposes, there will be no difference between the two. Now, in this case we have this vector equation with in which this entire thing has been written together. And here the matrix A is 2 by 2 and we have got a system of two first order linear homogeneous differential equations which are with constant coefficients. So, a 11, a 12, a 21, a 22 these are constant. Now and it is autonomous also, so anyway these things had to be constant.

In the case of second order systems a lot of terminology got developed when this entire study was in the hands of people, who were primarily physicists and quite a few words related to phase have entered into the (Refer Time: 05:25). So, in this kind of a situation phase means a pair of values of y 1 and y 2 that is the actually the state what we otherwise call state in this discussion quite often we will be calling it as phase. So, one value of y 1, and one value of y 2 consists of one state of this system and in this discussion quite often it is also referred to as phase.

Now, if we plot y 1 in one of the axis in our graph paper, and y 2 along the other axis then the plane in which we will be making the plot that plane is called the phase plane, the plane of y 1 and y 2. Now, in the phase plane in the plane of y 1 and y 2 from one point if we start and then that can be considered as the initial condition for the system of differential equations. And then from there if we draw curve draw a curve, which obeys this differential equations then that curve is called a trajectory that is as we consider the independent variable as time then this curve this so called trajectory will show along which path in the phase plane the system will evolve. So, this trajectory is a curve showing the evolution of the system for a particular initial value problem that initial value is demarcated by the initial point with coordinate y 1 and y 2 which we can put at a point.

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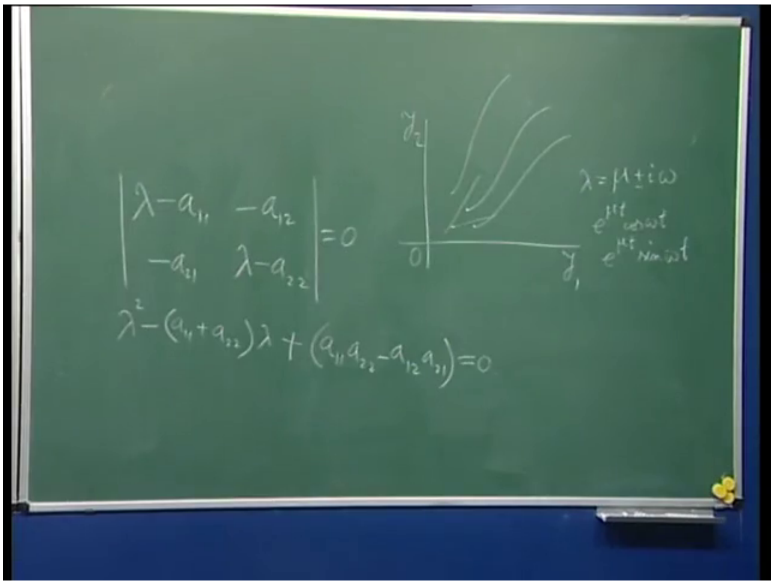

Now, in the phase plane then if we start from say this point then the curve can go in some way. So, this will be a trajectory. Now, if there is another point from where we start we will get some other trajectory, if we start from another point we may get some other trajectory right. Now, these trajectories when we put all trajectories that is a number of trajectories which will represent all trajectories in the sense that how much is the density of trajectories, you plot that is up to you. But if you go on plotting dense enough set of such trajectories then together they will show the dynamic behavior of the system starting from all possible initial conditions, so that picture is called the phase portrait. All trajectories together showing the complete picture of the behavior of the dynamic system that is called the phase portrait.

Now, in this analysis we will be considering the case of non singular A that means, no degeneracy only the non degenerate case as we did last time in the previous lecture; similarly here also we will consider non singular coefficient matrices in this place A. So, that will mean that the equilibrium point will be only isolated equilibrium point. And in this particular case, where it is linear homogeneous then the only equilibrium point possible will be the origin because with non singular A, A y will be 0 only at y equal to 0 that is y 1 equal to 0, y 2 equal to 0, this will be the only equilibrium point. And around that if we can complete the analysis then in a way we would have finished analysis for this particular system. So, we will be allowing isolated equilibrium points and matrix A is the non singular, origin is the only equilibrium point.

And then how is the behavior of the dynamic system round this equilibrium point will be governed by the Eigen values of this, because the two Eigen values will show will give us two components of the solution both exponential in general and then the combination of that the linear combination of that we will show the complete behavior. So, if we try to find out the Eigen values of this matrix coefficient matrix then as you know we will be first trying to solve this. And from there we will get a quadratic equation in lambda so that quadratic equation will be this-this minus this-this right. So, lambda square minus the sum of these two lambda plus this product minus this product equal to 0. So, this will give us the characteristic equation and that is what we have got here.

Now, we will represent for a particular reason, this sum has p and this quantity this value as q that means, this is a trace and this is the determinant of the coefficient matrix. So, trace of the negative of the trace, negative of the trace of the coefficient matrix is this that we will be presenting as p and this determinant we will be representing as q.

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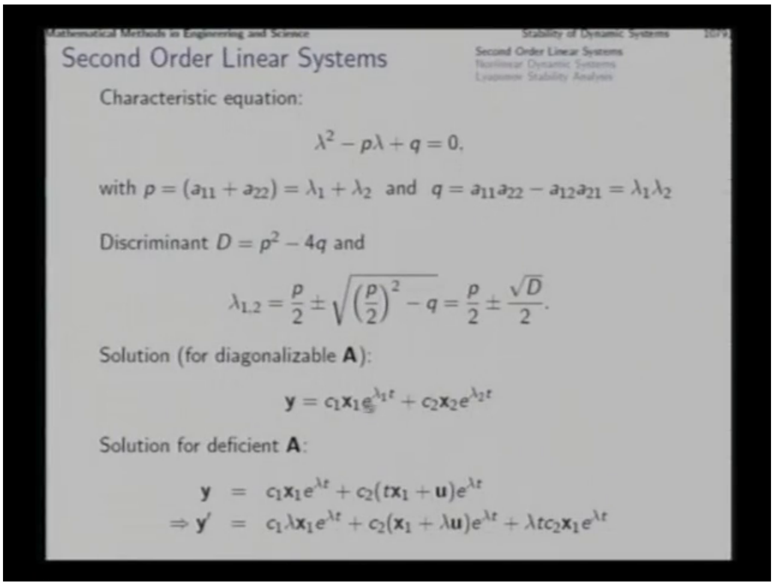

As we do that we get this as the characteristic equation right with p as a 11 plus a 22 which is the sum of the Eigen values and q which is the determinant which is the product of the Eigen values. Now, we will consider different cases that may arise. We know that discriminant of this quadric equation is p square minus 4 q, and depending upon whether this is positive or negative we have different cases of the way we will get the two Eigen values. Now, first consider this issue that is of course, the two lambdas will be given by this standard formula for the quadratic equation solution. Now, a particular case, where q is negative that is the two Eigen values product, the product of the two Eigen values is negative that is a situation where this discriminant is always positive, because this is p square.

Now, if q is negative that means, whole thing is certainly positive. Not only that it is positive and it is larger than p square. Why is that important because in that case this d will be positive and its square root will be larger than the magnitude of p, and that will mean that this term which will be root D by 2 will be larger in magnitude than p by 2. Now, this term in the two Eigen values will appear with opposite signs plus and minus. Now, if this one if this term has larger magnitude than this then that will mean that when we take the plus sign this entire value will turn out to be positive, this fellow p by 2 of whatever time cannot dominate this. So, the sign of the lambda sign of the Eigen value will be dominated by this term because root over D is larger in magnitude compared to p. So, when we take the positive sign, we get the positive root and when we get the negative sign here we get the negative root because this fellow this term will not be able to dominate over that.

So, in the case of negative q, two things will happen one that this will be positive and therefore, the roots will be real and this will dominate over this term and therefore, two roots will be of opposite signs. So, when we have that two roots will be of opposite signs. So, now in that case we have got this now there are two different roots distinct roots so obviously, it is diagonalizable matrix. So, in that case we get the two solutions like this. Now, note that one of them is positive the other is negative. Now, the two corresponding Eigen values are Eigen vectors one is x 1, the other is x 2.

Now, if lambda 1 is larger that will mean that and that is say, now one will be positive, the other will be negative. Now, if one is positive and the other is negative that means, with time the positive one will go that means, its magnitude e to the power lambda 1 t will go on increasing exponentially, and in that case this one will go on decreasing exponentially. And that will mean that around origin, note that we are discussing the behavior around origin because if the initial conditions are given at origin then it will the system will remain there that is the characteristic of equilibrium points anyway.

Now, around the origin whatever point we take and we put that point in this plane around origin near origin somewhere here. And in that case if the two Eigen vectors x 1 and x 2 from here are have two directions. Now, one Eigen value is positive, so around that direction whatever little solution whatever value is there, whatever initial condition is there that will keep on growing. So, along that direction the motion with time will go on increasing. So, whatever is the initial position as we decompose that along the two Eigen vectors the component along the Eigen vector with positive Eigen value will go on increasing with time and the component along the Eigen vector corresponding to the negative Eigen value will go on decreasing with time. And that means, that over time the trajectory will get aligned with the larger the positive Eigen value.

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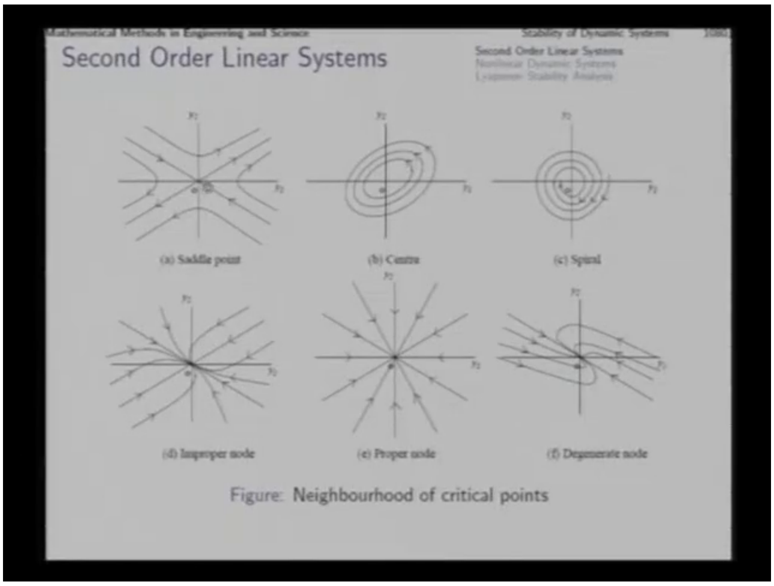

So, see here this will be the case. Now, if these are the two Eigen vectors then if we start from here then with (Refer Time: 16:31) law, we can decompose the position vector of this point into two component. One will be along this direction along which that Eigen value is negative; that means, trajectory is come in; and the other component will be this one this much and that is corresponding to positive Eigen value along which the trajectory go out. So, that means, this component will decay and this component will grow, and that is why wherever we start if we start here then the component along this will decay. So, it is coming like this and the component along this will grow So, it is growing like this. So, all trajectories will move away, and finally, along this direction and it will get as it goes far away the trajectories will get all bunched together with this.

So, the further away they are that much will be the difference, the two trajectories will not cross, but all of them will get bunched along this. And if the starting point is below this line then they will get bunched along this, so either forward or backward. So, all trajectories eventually will grow in this direction and go away from the origin; that means, that this is an unstable equilibrium point. So, this particular equilibrium point in that kind of a situation is unstable because if we start a little away from the equilibrium point, a little away from the origin then the trajectory diverges further and further away from the equilibrium point, so that is the hallmark of an unstable equilibrium point.

So, if q is negative and in that case we get two real Eigen values of opposite sign and in that case certainly the equilibrium point is unstable and such an equilibrium point is called a saddle point. So, one Eigen value positive, the other Eigen value negative. Now, other than that if q is positive then what will be the situation note that q equal to 0 case is not under discussion. Because q equal to 0 would mean that lambda equal to 0 is one solution which is the case when a is singular and that will mean that one complete subspace will be equilibrium point and that case we have omitted So, q equal to 0 case is not in our discussion at all.

So, in the case of q negative, we will get saddle point which will be always unstable. Now, if 1 is positive, now one point is easy to note that the nature of the equilibrium point will be the same irrespective of the sign of p because p is appearing here as p square. So, it will be symmetric with respect to the sign of p. So, let us consider p square equal to 0, larger, larger, larger and so on. So, with q positive, if p square is 0 that is p is 0. If p is 0, then what we are getting we are getting D as minus 4 q and p is 0. So, if p is 0 then this goes off and here we have got a negative discriminant; that means, the Eigen values will be pure imaginary.

So, if Eigen values are pure imaginary that is plus minus omega i kind of Eigen values that will mean that when you decompose that exponential e to the power i omega t plus i omega t minus i omega t. So, you will get basically cosines and sines. So, you will get sinusoidal output and in that case you will get this kind of behavior. So, around that equilibrium point the trajectories will make a closed curve and such an equilibrium point is called a centre. So, this is stable because started close to the equilibrium point, the trajectory will remain close to the equilibrium point, it will never go too far.

Now, when you start at that point whatever is the distance compared to that the distance might increase, but it will again decrease because it is a closed curve. So, this is one particular case that is if p is 0.

Now, if p is greater than 0 that is if p is positive or negative, say p square is positive. Now, when p square is positive then whether p square is less than 4 q or greater than 4 q these two situations will give rise to two different kinds of equilibrium points. If p square is positive and less than 4 q then this discriminant is nevertheless negative. So, this is negative. So, in that case you will but then this is not zero this is not zero. So, there is a non-zero part here and this is negative. So, it will be a full project complex number with non zero real and non zero imaginary part.

And in that case you will get Eigen values which will be like lambda equal to mu plus i omega mu plus minus i omega. So, in that case a solution, this solution these two solutions. So, what you do in that type of situation you reorganize the coefficients and say the solutions will be one solution will turn out to be like e to the power mu t cos omega t; and the other will be e to the power mu t sin omega t. Now, this cos omega t sin omega t term will try to give an oscillatory feature. However, this e to the power mu t part will give the amplitude as variable the amplitude will be exponential.

So, in this kind of a situation, what you will have is that whether this mu is positive or negative, according to that this solution both of them together will grow or decay right, so that sign will be determined from here because in this case this is the only exponential part this part will give you this sinusoid. So, the exponential part will grow or decay depending upon whether p is positive or negative. In any case, it will be if it grows then and this part will provide an oscillatory component. So, you will have either this going inward amplitude decreasing, if mu is negative; and if mu is positive then along a similar curve, the spiral will go out. And because of obvious reasons this kind of an equilibrium point or critical point is called a spiral.

Now, we come to another situation that is if p square is greater than 4 q if p square is greater than 4 q then this D is positive. However, with positive q its value its absolute value will be less than this part. So, if q is positive, so p square minus 4 q is certain less than p square. So, therefore, its square root even if positive will be certainly less than p in magnitude, so that will mean that this plus minus sign will not play a role in the lambda. So, final sign of lambda 1 and lambda 2 will be decided by the sign of p. So, if p is positive then whatever large is this, it will be certainly less than this term. So, then the total will be positive anyway. Similarly, if p is negative then even the positive sign taken here will not be able to make the sum as positive. So, the sin of this entire lambda will be determined by this and not by the plus minus term and therefore, both of them both of the Eigen values will be of the same sign. Either both positive if p is positive; or both negative if p is negative.

And that will save the solution from this kind of a situation where one goes and the other decays in this case if p is positive then both will grow and if p is negative then both will decay. So, that kind of a situation with the two Eigen values having different magnitude will give rise to this situation. You see here there are two Eigen values correspondingly there are two Eigen vectors this is one and this is another. In this case, the sign of p has been taken as negative for this particular plot. In this plot, in all cases, wherever it is possible to have a stable situation, we have drawn the stable case in all these cases. So, here both the Eigen values have negative real part. So, this Eigen vector also comes inward this Eigen vector also comes inward. But then if one of the Eigen values is large, that means, that the rate of approach for that particular Eigen vector say e to the power 7 t and e to the power 2 t.

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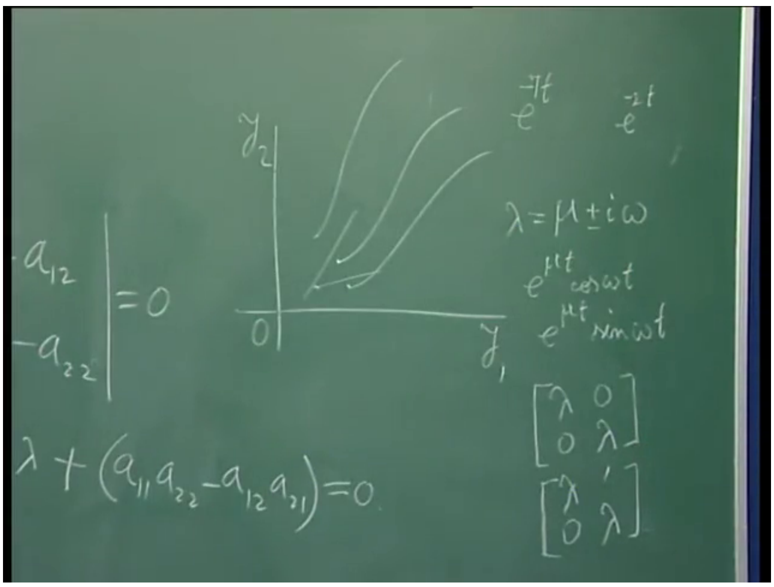

So, this one is large, this one is small. So, as t grows of course, you have negative if you follow this particular kind of plot in which the arrows are inward. So, this is e to the power minus 7 t this is e to the power minus 2 t. Now, this one will decay much faster than this. And therefore, you see that the component along this one will decay much faster. So, from here the component is this big, but this is decaying much faster compared to this component. So, the component along the larger magnitude Eigen value will decay extremely fast compared to the other one. And therefore, finally, all the trajectories will become tangential to this line, and this is called an improper node, there are various other cases of nodes. Now this is the situation where the two Eigen values have different magnitudes.

Now with the same sign if the sign is positive then along the similar trajectories the system will evolve outward. And in that case, it will be unstable. In the case, which you see here it is stable just like spiral it could be outward rather than inward. Similarly, here also now this is called an improper node if p square is exactly equal to four q then what happens then D is zero and then you have got both Eigen values same p by 2. Now, depending upon whether p is positive or negative, you will have the Eigen values positive or negative, but both of equal magnitude as well as sign, so two equal Eigen value.

Now when the two Eigen values of the matrix are equal then you are whether the Eigen vector are full set of two Eigen vectors or only one Eigen vector is there, because with repeated Eigen values the canonical form could be this or the canonical form could be this. In this case, there will be two Eigen vectors linearly independent which will mean that the entire plane is composed of Eigen vectors, all vectors along the plane on the plane are Eigen vectors in this case; otherwise in this case there will be a single Eigen vector. Now, if you have got both Eigen vectors that is if the matrix is diagonalizable then all directions are Eigen vectors and along all directions the behavior will be same and in that case you have got this situation and this is called a proper node.

So, whenever you start the system whatever initial condition you give the system will evolve directly towards the origin in the case where this lambda is positive sorry negative. On the other hand, if this lambda is positive then from wherever you start that is an Eigen vector, so that position vector initial position initial straight vector is an

Eigen vector, so along that with positive lambda it will straight go out with negative lambda it will straight go in. So, this is called a proper node.

The last case where you have got p square equal to 4 q that means, D equal to 0; that means, lambda both lambdas are same repeated Eigen value and the matrix is not diagonalizable. In that case, as we found that in the case of non-diagonalizable coefficient matrix, you get this solution y from which you can determine y prime and you get situation like this. And here you see there is a time element coming here in the coefficient here. So, from that if you analyze then you will find a very peculiar situation of the phase space. If this is the single Eigen vector then all the solutions will approach along this Eigen vector only, this is only one Eigen vector. So, all of them come like this. In the case of negative Eigen value, it will come like this inward; in the case of positive Eigen value, they will go outward with the arrows reversed. So, this is called a degenerate node.

So, what are the types of critical points equilibrium points we found with real and real Eigen values with opposite signs we have got saddle point which is always unstable. And with pure imaginary Eigen values we have got centered, which is always stable, but this is a border line case because any modeling error. And it might fall in the case of spiral that is any modeling error and p turns out to be actually a little positive or negative that will mean there will be an outward unstable or inward stable spiral.

Now, this is the case from p square equal to 0 to 4 q, greater than 0 and less than 4 q; p square equal to 4 q will give these two cases both Eigen values same and Eigen vectors both Eigen vectors existing only one Eigen vector proper node degenerate node. And in the case of p square not equal to 4 q that is p square larger you have got two Eigen vectors, two unequal Eigen values of the same sign, and in that case you have got improper node like this. A summary of all these cases you can see here in the table and also in this plot in the plot of p q.

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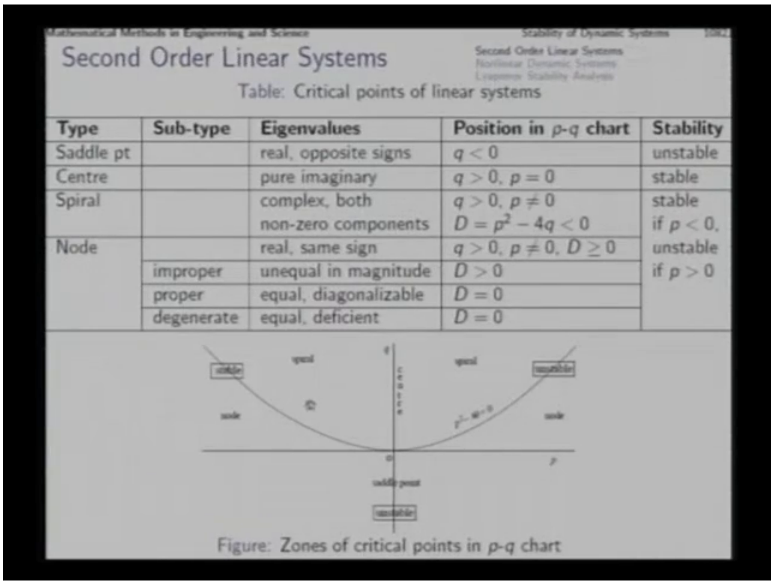

Type	Sub-type	Eigenvalues	Position in p-q chart	Stability
Saddle pt		real, opposite signs	$q < 0$	unstable
Centre		pure imaginary	$q > 0, p = 0$	stable
Spiral		complex, both non-zero components	$q > 0, p \neq 0$ $D = p^2 - 4q < 0$	stable if $p < 0$,
Node		real, same sign	$q > 0, p \neq 0, D \geq 0$	unstable if $p > 0$
	improper	unequal in magnitude	$D > 0$	
	proper	equal, diagonalizable	$D = 0$	
	degenerate	equal, deficient	$D = 0$	

So, you see with q negative this entire part gives you saddle point which is unstable; q equal to zero case we have discarded because that is singular coefficient matrix with q positive that is upper half of this p q plane, you have got saddle point is real opposite sign Eigen value q negative. So, it is always unstable. Above the p axis, with p equal to zero you have got this line long which you will get centered, and Eigen values are pure imaginary this is the case which is stable. In the case of the p q point lying above this parabola above this parabola p square equal to 4 q, you have got spiral. So, Eigen values are complex and both real and imaginary parts have nonzero components. So, you get here negative p stable spiral and positive p unstable spiral.

So, with p equal to negative you get unstable stable points here and with p equal to p positive you get unstable whether it is spiral or node. So, in the case of nodes, you have got several cases all with real same sign. So, they are stable, if p is negative; unstable, if p is positive just like spiral. And here if the Eigen values are unequal in magnitude that will mean D is positive that means, you are here and if you have got you are on the boundary that is it is on the p, q point is on this parabola then that will mean one of the these two cases. In the diagonalizable case, you have got a proper node; in the deficient case, you have got a degenerate node. So, this is the complete breakdown of all the types of equilibrium points that you can have in a second order linear system.

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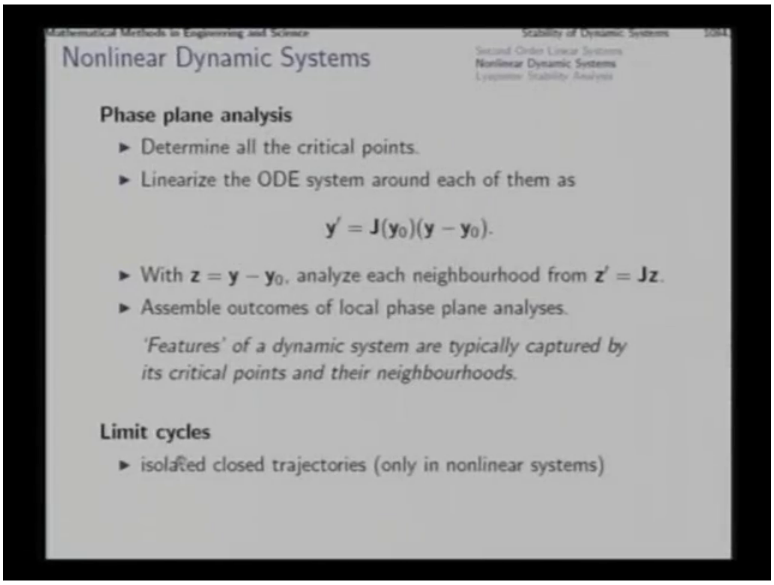

Now, when you get a non-linear system say then what you can do is that around that around every critical point that you get you can conduct an analysis of this kind, and find out how the trajectories around that point will behave. And then you can compose the situations around all the critical points together to complete the face portrait.

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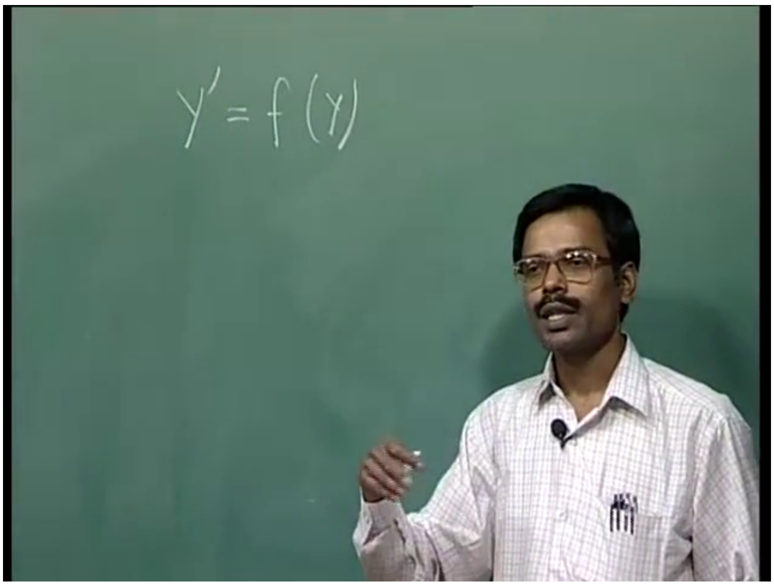

So, for a non-linear system where you will have the differential equations of an autonomous system in this manner then first you will try to find out critical points or equilibrium points. And for a non-linear system origin may not be a critical point and apart from that you will typically expect more than one critical points. Origin may be a critical point among them, but there will be expected other critical points as well. So, what you will do you will first solve f y equal to 0, and collect all the solutions of it. For each solution of it say call it y 0 then around y 0, you will conduct a linearization and you will capture the first order behavior of the system around every critical point with this kind of a differential equation which is linear.

Now, as you put y minus y 0 as z then you will have z prime is equal to this Jacobian J into z, and then this will be a kind of a system which we just analyzed. So, you can make the face portrait of this system and take that portrait a small part of it a small part because this is first order analysis and will be valid only in the neighborhood and that you can plug in at y 0 in the y plane. And similarly, at every y 0, you plug in a small size portrait which you get through the analysis of this kind of a system, and then you can try to assemble the entire face portrait of the original system. So, through the assembly you will find that you are able to picture the entire face portrait even for the non-linear system, so that shows that features of a dynamic system are typically captured by critical points, and their neighborhood.

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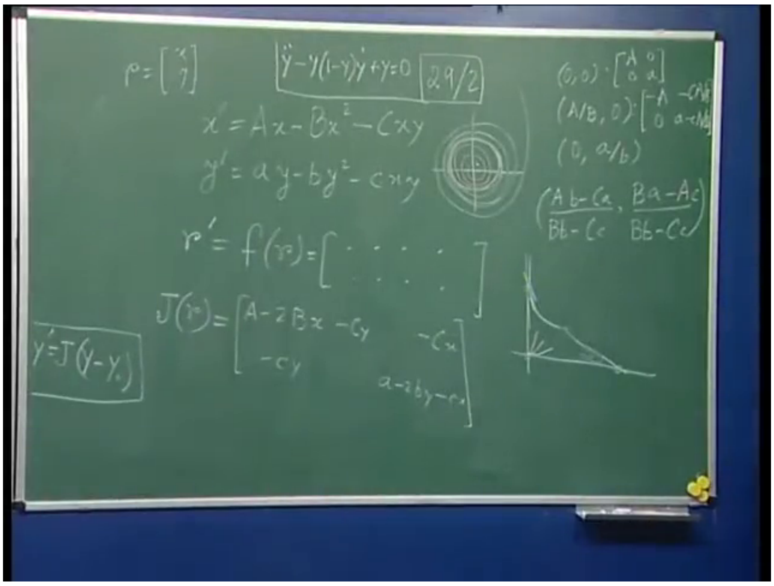

Let us take a small example say we consider this population model of a pair of competing species who depend on similar resources. So, suppose x and y are the populations of the two species as functions of time. And the dynamic system is each term here with its coefficient has a meaning this coefficient a represents the rate of reproduction and growth of species x that is its inherent rate of growth positive. Now, this B represents the result of interspecies rivalry between members of this same species x, so that is why it is the interspecies rivalry. Rivalry basically depends on two individuals, and the more number of individuals that are available as the first party of the rivalry will make the rivalry sharper. Similarly, more number of individuals available to form the second party in the rivalry that will also make the rivalry even sharper therefore, it depends upon x square is proportional to x square. And this term comes as with negative sign because of the interspecies rivalry where both individuals from the species will try to corner the same resource, so that will in some way lead to decay of the species, because they are fighting within themselves to corner the same resource to grow.

Similarly, this term shows the effect of interspecies rivalry with the other species trying to corner the same resources. So, this capital C represents the effect of interspecies rivalry, rivalry with the other species its effect on the first species. Similar, coefficients a, b and c, small a, b and c will represent the similar actions for the second species this is small c. Now, with this you can see that this is a non-linear system.

Now, if you denote r as the vector x y, then this f of r will have two components this and this. So, r prime will be f of r, which will have these two component. Now what will be the Jacobian of this, Jacobian of this you can find out. And before that you can try to find out what are the equilibrium points which are those points where if the population is from the beginning at a particular time t equal to 0, then the population will be constant will never change, what are those equilibrium points. So, for that you can solve these two equations and find out those equilibrium points.

So, if you solve these two equations for x and y, you will get the solutions as one is one solution is origin; obviously. So, you will find the solutions are origin A by B, 0, 0 small a by small b; and the fourth one is complicated a b minus C a divided by d b minus c c, B a minus A c divided by B b minus C c. So, these are the four equilibrium points. You can verify very easily 0, 0 will and these are two quadratic equations. So, in total they will have four solutions. So, origin is obviously, a solution. Now, if you take y equal to 0, then this term goes off this term goes off, and then from here you will get x as capital A by capital B, and y is 0 of course, so that is 0.

On the other hand, if x is 0, then this is 0, this is satisfied and sorry these are y. So, if y is 0, then this is satisfied and this is 0. So, x is a by b, and if x is 0 then this is satisfied anyway and this is 0. So, y will become small a by b that gives you this. If neither x nor y is 0, then the solution is a little complicated, but you can see that substituting this you will find that these two equations are satisfied. So, these four points will fall here origin capital B by capital A 0, 0 small a by small b and the fourth one will be somewhere here. Now, around origin if you now if you can find out the Jacobian matrix, so you will basically differentiate this with respect to x and get A minus 2 B x minus C y, this is x derivative Now, the y derivative of this will be minus C x. Then x derivative of this will be minus small c y and y derivative of this will give you this.

Now, you see at each of these four equilibrium points, you can find out the value of the Jacobian. And then for every such point you will get the differential equation system up to first order as y minus that equilibrium point any of the four, you can put there this multiplied with the J at that point this will be z prime that is y prime minus y 0 that is y prime itself. So, y prime and z prime will be same. So, we can put it like this. So, e around each of the equilibrium points you can find out the value of the Jacobian by putting the x, y coordinates of that equilibrium point and frame a linear system and analyze.

You will be able to see very easily that this point is a node is an unstable node; and because as x and y you put as 0, 0 then this matrix will be A, 0, 0, small a. So, this will be a node both Eigen values will be positive, so this is an unstable node. Why the logically you find that if the initial population of both of them are not 0 that is if any one of them or both have any nonzero small nonzero population that will mean that at that time the resources are plenty because the members in the two species are very small. So, lots of natural resources around. So, they will go on consuming those resources and grow. And at that time with small x, small y values these terms anyway will not play a major role, these will be dominant.

So, growth of both will be supported. Of course, the one which is of larger growth rate depending upon whether capital A is large or small a is large that will grow faster in any case both of them will grow. So, you see that suppose capital A is large that means, around for this the growth will be faster so that means that all trajectories will start moving like this, like this in proper node. Now, at this point what is happening at this point at this point you will find let us mention the Jacobian here, I will give you the Jacobian at this point. The Jacobian at this point, at this point we had simply right. At this point, the Jacobian turns out to be minus A minus C A by B 0 and a minus c a by b.

Now, for this Jacobian, you can see very easily that 1 0 is certainly an Eigen vector of this matrix. Try to multiply this matrix with vector 1 0, you will get minus A plus 0; and in the lower one you will get 0 plus 0. That means, if you multiply this with 1 0 you will get minus A 0 1 0; that means, 1 into this plus 0 into this. So, you get minus A 0. So, that means, minus A is the Eigen value and 1 0 is the Eigen vector. So, at this point, this is one of the Eigen vector and along that the Eigen value is minus A, and that shows that this point along this direction at least is stable.

Now, what is the situation along the other direction, along the other Eigen vector. And that could be positive as well as negative, depending upon what is the other, what are the values of these coefficients. So, one possibility is that this is the other Eigen vector and that means, that if along this it goes in then along this it goes out like this. Similar could be the situation here, this will be one Eigen vector. and this could be the other Eigen vector. So, along this it will go in, along this it will come out, along this it will go in, along this it will come out. So, you see that these directions these trajectories might meet here, because at this point as you come it could be a an inward node that is stable node.

So, try to analyze this particular case, the analysis is there in the book try to in the text book that we are considering. So, the entire analysis is there in the book. So, as you consider different cases you will find that if you draw the phase portrait around these four equilibrium points and try to assemble together, you will find that you get the complete picture of the face portrait of the system.

Now, here we will raise another question other than the non-linearity one question within non-linearity and another question apart from non-linearity regarding dimensions. There is one particular feature, which is not covered in the analysis here. Here we found saddle points, centre, spiral node these kinds of equilibrium points. Now, linear systems can have only this much only these many features such points they can have. There is something else, which is possible in non-linear systems. Non-linear systems also will have in certain points equilibrium points which are nodes, saddle points, spirals and centre points, but apart from that in the case of non-linear systems there is another feature which is possible that is for limit cycle. And in that case you have isolated closed trajectories, this is not like centre point, around centre point centre point is one equilibrium point around which there are trajectories which happen to be close.

Now, limit cycles are completely different. The entire close trajectory is such that the entire trajectory behaves like equilibrium point kind of thing. So, this kind of a feature, this kind of a situation can arise only in non-linear systems. In chapter 29, in one of the exercise problems, this example exercise 29-2. The exercise simply is an exercise on numerical solution of one second order differential equation, but then if you analyze it for enough time or if you study the solution of that particular exercise given in the appendix of the text book, you will find that this case gives you a limit cycle. And that shows that if you start from outside the limit cycle say it is the limit cycle is a close trajectory like this and if you start outside then this is the trajectory and finally, the trajectory merges on the limit cycle does not go in.

On the other hand, if you start from inside say from here then you go outside and then you merge with the limit cycle and do not go out. So, this cycle this closed trajectory is called a limit cycle that kind of a thing is possible only in the case of a non-linear system. This particular example of exercise 29-2 is this non-linear differential equation. This is a non-linear, see this is a non-linear here. So, this non-linear equation will give you a limit cycle of this kind. So, limit cycle is one feature of non-linear dynamic systems, which the linear analysis, linearized analysis will not be able to capture.

Now, one more issue what do you do in systems in the case of systems with higher dimensional straight space, there straight forward face plane analysis you cannot conduct. However, you can always try to linearize the system of equations around every equilibrium point like this and then conduct linearized analysis and then try to work out the features. Now, in the case of non-linear systems quite often you might find that in one sub space of the straight space, you get a spiral like feature where the other sub space shows a node like feature, so that kind of a situation is possible. So, such things might happen in the case of higher dimensional straight space. There is another technique of analysis of stability for non-linear dynamic systems and that is the famous Lyapunov stability analysis.

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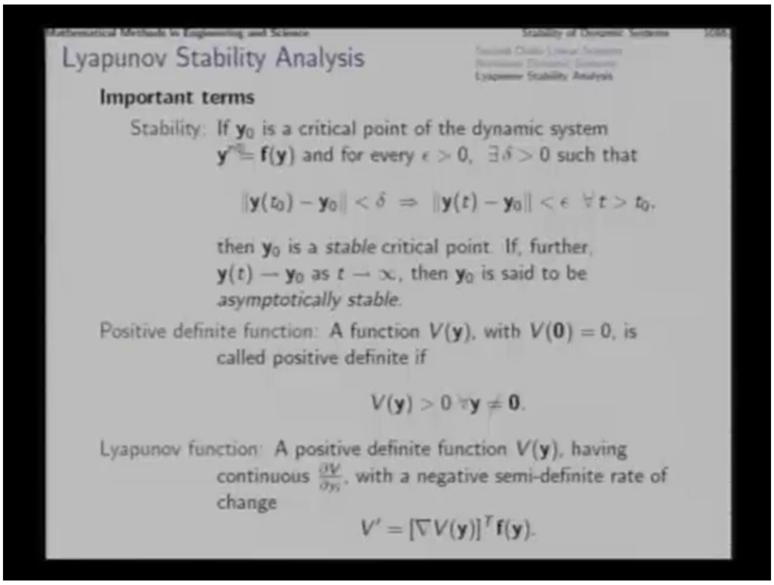

There are quite a few important term as said with Lynpunov method, the precise definition of stability here is this. If y 0 is a critical point of the dynamic system this, and for every positive epsilon if there exists a delta such that the initial point taken within a delta distance of the critical point keeps the system always within an epsilon distance then you say that the critical point y 0 is stable. What is the idea here that is if you prescribe epsilon that is if you say that will your system remain within this much distance of the critical point then you answer that yes it will remain if you start sufficiently close say within delta distance.

If you can say so that if there exist a delta such that if you can say if you can prescribe such delta for every given epsilon then you say that this point y 0 is a stable critical point. That is if you can keep the system close enough by starting close enough then it is a stable critical point. There may be situations where whatever close you start eventually you will go far away from the critical point then that critical point is called unstable. So, this is the criteria for stability. If further not only that it stays close if as enormous time is passed eventually if the point y 0 is approached then you call that point as not only stable, but asymptotically stable.

You would be able to see that in the earlier case, stable nodes and spirals were asymptotically stable. On the other hand, centre points which were always stable were stable, but not asymptotically stable, because in that case the trajectory does not approach the critical point. In the stability analysis, due to Lyapunov there is another important issue there is another important term that is the positive definite function. A function with value zero at the origin, if it is always positive for non-zero points that is away from origin and that is called a positive definite function, as we know positive definiteness in general.

What is a Lyapunov function? If positive definite function of the straight vector having continuous partial derivatives with a negative semi-definite rate of change that is if V is positive definite function, but if its rate of change with respect to time, it was respect to the independent variable T is negative definite or negative semi-definite then you call it a Lyapunov function. A Lyapunov function is actually a an abstraction of the concept of energy in physics. So, energy is can be considered as a positive definite function total energy.

And then the way the system evolves is in order to decrease the total energy, so its rate of change with respect to time is negative it could stay at constant for that matter that is v prime could be 0, so that if I semi definite is there. So, this kind of a function which is positive definite function of y, but its rate of change with respect to the independent variable T is negative semi-definite such a function is called Lyapunov function. With this kind of a definition the actual stability criteria becomes quite straightforward and that is the theorem.

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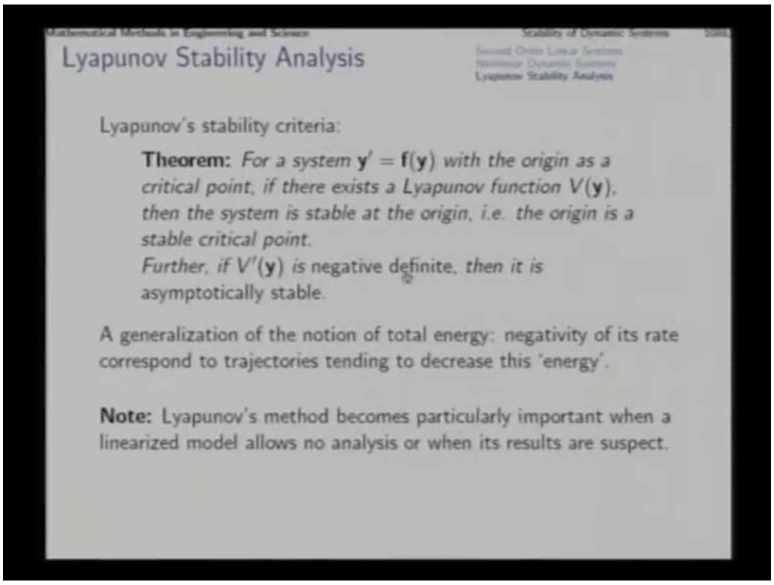

That is for a system like this with the origin as a critical point. If you can construct a Lyapunov function, which is positive definite with a negative semi-definite rate then you can conclude that the system is stable at the origin. Further if v prime is not simply negative semi-definite, but negative definite then that will mean that energy will not only be non-increasing, but it will actually decrease and in that case you can see that the trajectory will actually approach the critical point. And in that case, it is asymptotically stable. So, it is actually a generalization of the notion of total energy. So, negativity of its rate will corresponding to trajectories tending to decrease this energy.

Now, Lyapunov s method is important because in the case in the non-linear case or in the higher dimensional straight space case or in the in those cases where the linearized version of the straight space of the straight equation if the linearized version of the straight equation is unable to allow any analysis in such cases Lyapunov s method turns out to be a useful method for analyzing the stability of a non-linear system. However, caution should be exercised in using this Lyapunov’s analysis because it is a one way criterion only. If you can construct a Lyapunov function then you can say that the system is the origin is a stable point of the system, but if you fail to construct a Lyapunov function that does not mean that the system is unstable. So, it is only a one way criterion.

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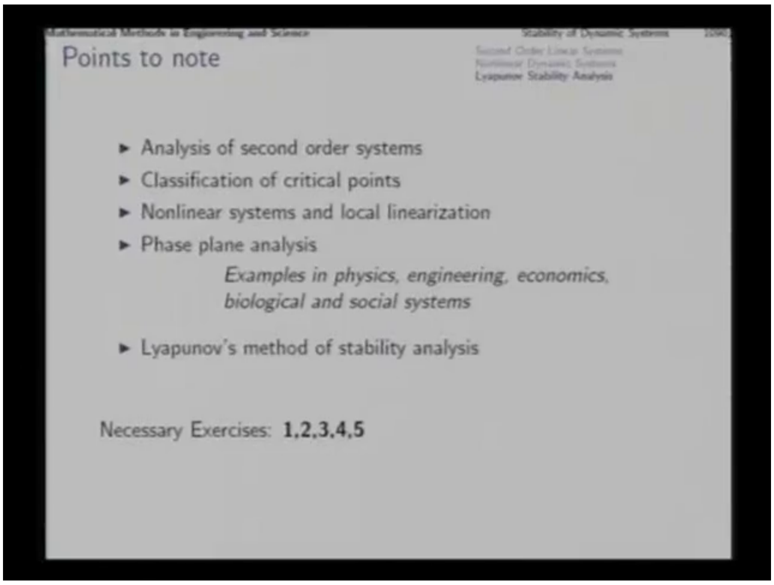

So, in this lesson the important issues that we have studied are analysis of second order systems, critical points of different kinds, and non-linear systems with their local linearization to describe the system, and straight space analysis which has enormous application in branches of science - physics, engineering, economics, biological and social processes. So, this in a way completes one module of our course, from the next lecture again we will go back to solution of differential equations certain situations where analytical method fail, but much better than numerical methods we can use and that is three solution. So, next lecture, we will start three solutions; and in that continuation, we will slowly move from the study of differential equation to approximation theory.

Thank you.