Mathematical Methods in Engineering and Science
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Module - VI
Ordinary Differential Equations
Lecture – 04
ODE Systems

Good morning. In this lecture we will study systems of ordinary differential equation.

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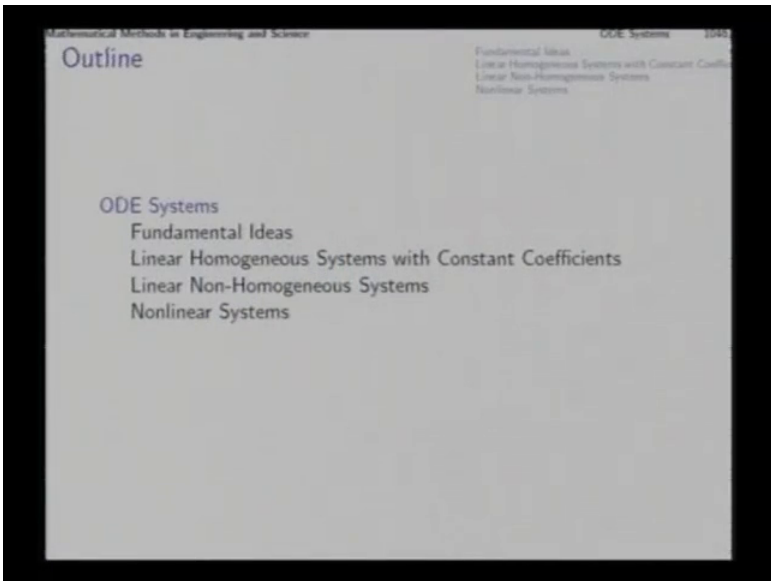

Now, when we have a system of ordinary differential equation or in other word we can say that we have a vector differential equation here the independent variable p is scalar making it ordinary differential equation, but the dependent variable or the unknown function y is a set of functions; that means, it is a vector function of p which is the unknown function and then its derivative is also a vector function of the same dimension.

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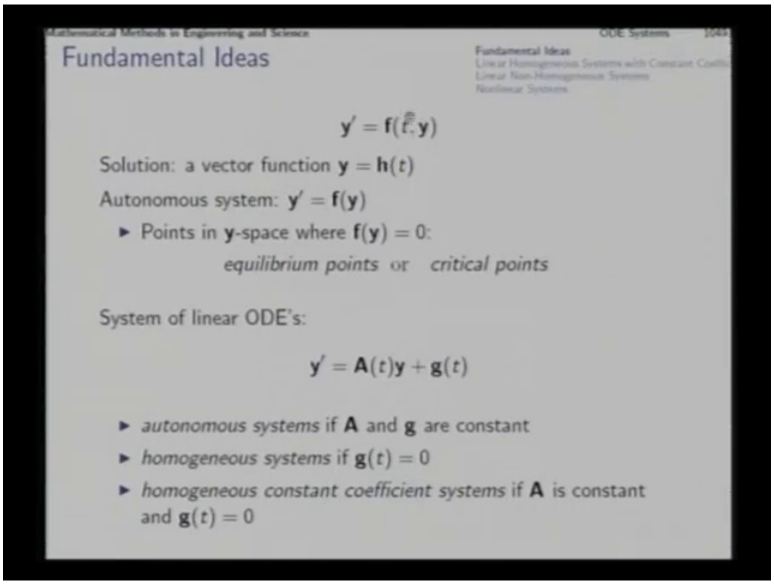

So, you have got a vector function f of t and y; that means, n plus 1 variable it is a function of n plus 1 variable in general. So, as solution you are looking for this vector function h of t. A special case of this vector differential equation or system of ordinary differential equation in which the t term is absent; that means, the system of differential equations which do not include t explicit such distance are called autonomous distance. And why these are particularly important? The special importance of autonomous systems is because of the fact that loss of nature do not change with time; that means, if this particular system of differential equation is representing the phenomena, representing a particular law of nature then it that law of nature will not depend upon time explicitly. That is at whatever time you try to study the system the laws will remain the same.

Time will remain in the discussion because the derivative the differentiation is with respect to time yielding the rates, but then time itself will not appear into the scene this is the typical situation whenever these differential equations try to model a system for which the way the system behave, it governing equations the fundamental law of nature on which it work does not change with time.

So, that is why this kind of a system in which the explicit occurrence of p is not there that assumes a particular significance and in the case of autonomous systems a lot of analysis is possible which will not make even sense for a system of differential equations in which t will appear explicitly. Now, keep in mind that certainly there will be situations where we will be solving differential equations in which t does appear all that is being discussed is that such equations in which the independent variable t appears only through the function as it changes and not explicitly has a special importance for which we will have certain special analysis which we will not make in the case of the general equation in which t also will be involved.

Now, for this kind of a system of equations differential equations which is an autonomous system you can talk of points in the set space, in the space of y where the value of this vector function is 0. Now, those points are called equilibrium points or critical points. Why are they called equilibrium point? Consider for a moment that a system is started with its initial conditions at that kind of a point in y space where f of y is 0. Now, if f of y is 0 at a particular instant of time then that will mean that at that instant of time y prime is 0 which will mean that for an instantaneous for a very small instant for a infinitesimal time from that initial time you will have no change in y; that means, at the next instant of time say t equal to delta t the value of y the point in the y space in the set space will remain the same. And that means, this set of equations this system of equations will be continued to be satisfied which means that that instant of time delta t again the rate will turn out to be 0 and that means, for another small time infinitesimal time the variable y will undergo no change. That means, for eternity the system will stay right there and that is why it is called an equilibrium point.

And why they are also called critical point? This issue will be clear as we proceed forward. For the time being we can just keep in mind that for a general system if we can describe the behavior of the system around these equilibrium points and away from it we can simply extend or generalize these behaviours then in a way we get the complete picture of the dynamic behaviour of the physical system. And in that sense these points have certain important attached to them. That is around these points if we analyze the system completely and then the rest of the picture we can construct very easily that is away from the critical points away from these particular equilibrium points the set space will have no distinct or significant features. The vicinity of a critical point will have certain feature and likewise the neighbourhood of the equilibrium points are worth analyzing and therefore, these points from the, these point in their neighbourhood embody the critical amount of information regarding the behaviour of that dynamic system that s why these are also called critical point.

Now, this is an issue which we will explore in detail later currently consider and the special case in which t is present, but then the form of this function has certain special cases special situation that is if the function f of t y is linear in y see the independent variable can appear in any manner. So, in that case if this function is linear in y; that means it has a y plus g kind of form, this is the linear term and this is the constant term constant. So, for as y is concerned in t it certainly varies that is y independent term is this and this is the linear term in y. So, this kind of a function sitting here will give us a system of linear ordinary differential equations.

Now, this will have 3 sub cases, one is that if this matrix a is constant and then and this g vector function is also constant if a and g are constant, constant matrix constant vector then this side the right side will be independent of t. Then we will have an autonomous system, autonomous linear system; that means, when both of these special cases appear at the same time.

In another special case of this we will have g t equal to 0 if g t equal to 0 this term is absent in the differential equation then we have y prime equal to a of t into y; that means, it will be a homogenous system. This is d v 1 term this is also d v 1 term in y the unknown functions if this term is missing that is a homogeneous system of first order ODEs, linear ODEs. Now, if we have both of these at the same time; that means, if the coefficient matrix is constant and this vector function is 0 then what we have we have y prime equal to a y in which a is constant then we will have a system of homogenous constant coefficient differential equations.

Now, we will study these situations one by one. First this homogeneous system a t y.

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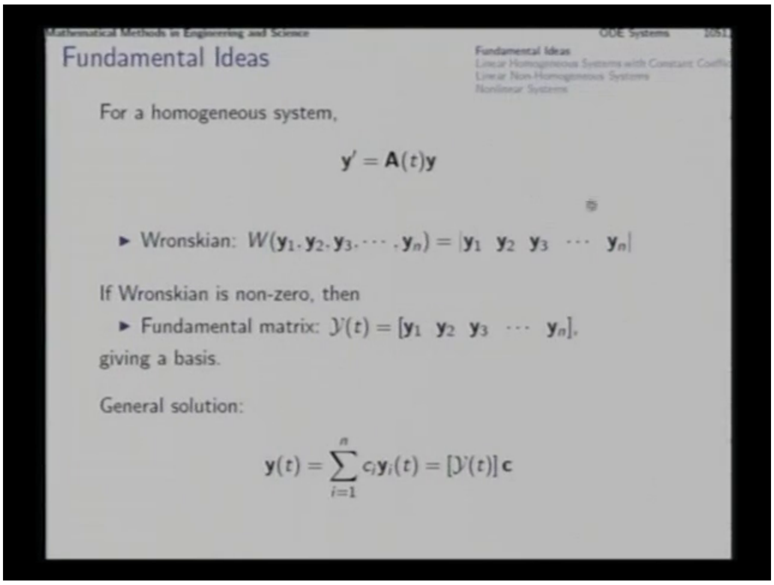

So, for a homogeneous system like this we can if we construct n solutions y 1 vector function, y 2 another vector function y 3 another vector function all satisfying this. Then for these vector functions we can form the Wronskian that is these vector functions each having n components when n such functions I assembled in a matrix we have got an n by n matrix and the determinant of that matrix is the Wronskian. And that matrix itself is called the fundamental matrix having these linearly independent, having these solutions as the columns when the Wronskian is nonzero.

If this matrix if this determinant is nonzero if the Wronskian is nonzero that is if the n solutions here are linearly independent that is if they form a basis then the basis members together like this form the fundamental matrix which we will be representing with this big y. So, this big y is a function of t is a matrix function of t time the independent variable and that will be called as the fundamental matrix and in that case for this homogenous system of equations we can construct the general solution in this manner.

Now, note that this is a matrix of functions of t multiplied with a constant vector which is actually the same as this summation of c i y i where y 1 y 2 etcetera are the n linearly independent solutions of this and c 1 c 2 etcetera are the corresponding coefficients in the linear combination that we will form. So, as a linear combination of these basis members we can get the complete solution of this homogeneous system.

Now, in a particular case when this matrix function of t is constant that is the simplest case that we can think of.

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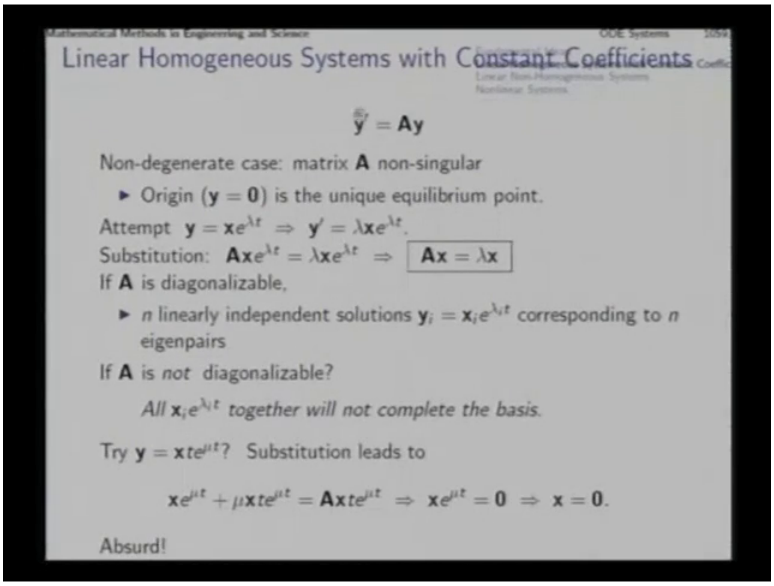

So, we will study only that case when the matrix a is non singular the reason is that if the matrix a is singular that essentially means that we have made a faulty formulation of the dynamic system. There could be another formulation in which case number of the state variables y s y 1 y 2 etcetera could be less that is if the matrix a is singular in this kind of a system of differential equations that singular matrix through its singularity essentially means that one or more of the state variables could be actually dropped from the formulation and we could have made a smaller formulation having a non degenerate case in which the matrix the coefficient matrix in that case would be non singular.

So, solution of or consideration of this case itself in which the matrix A is non singular is good enough. Now, the fundamental reason of vomiting the case of singular A matrix is this; however, as we take this assumption that matrix A is non singular much of our work also gets simplified. That is if the matrix A were allowed to be singular that would mean that there would be infinite equilibrium points of this system because a singular matrix A would have a null space which will mean that many nonzero vectors sitting in the place of y would give this right hand side as 0 and that would mean that complete sub space would be consisting of equilibrium points and that would make the analysis completely haphazard.

So, in the with this understanding that we are discussing only that case in which the matrix a is non singular that is any degeneracy has been handled at the formulation stage itself we will find that with this non singular a the only way the right hand side can be 0 is at the origin at y equal to 0. So, for this linear homogeneous system with constant coefficient origin is the unique equilibrium point that is the only equilibrium point or critical point. So, all that we need to analyze is the immediate neighbourhood of the origin. Now, if we want to and out the solutions of this homogeneous system of differential equations, what we do? Similar to the case of scalar differential equations we note that we are looking for a function of the kind which upon differentiation produces its own kind that is which can be both side put on the both sides of an equation and they balance. So, for balancing if this is a particular kind of function the on the other side of the equality also we should have that kind of a function only right.

So, for that purpose we will choose exponential function. Now, since y is a vector function. So, this scalar exponential function e to the power lambda t this we multiplied by vector say x and then we say we can attempt a solution of this form, the function y is equal to A vector x into e to the power lambda t. Now, this vector x n dimensional ordinary n dimensional vector and this value lambda is the set of quantities that we will try to determine, and how do we determine that after proposing the solution in this manner in this form we differentiate that and insert both the proposed solution and its derivative into the differential equation.

So, the derivative of this is simply x constant vector into e to the power lambda t into lambda; that means, this. So, this y prime the derivative and this proposed function as we insert here we get A x e to the power lambda t is equal to A into y, A into y is equal to y prime that is this. Now, e to the power lambda t will not be 0 in general. So, the way the only way these 2 can be equal is by A x equal to lambda x right. So, since e to the power lambda t in general will not be 0, so we can cancel it and get a x equal to lambda x

Now, what is this? This is simply the eigen value problem of this square matrix a from which we can determine x and lambda. So, with the formal solution of this eigen value problem we can determine the eigen values lambda 1 lambda 2 lambda 3 up to lambda n because this is an n by n matrix and corresponding eigen vectors x 1 x 2 x 3 etcetera, that we can find out if A is diagonalizable. Then we will have a full set of eigen vectors and correspondingly with the help of those eigen vectors and those eigen values we will get n linearly independent solutions of this differential equation.

Now, these linearly independent solutions as y 1 y 2 etcetera will form the basis for the complete solution of this system right. Now, if A is diagonalizable even in the case of repeated eigen values is fine; that means, for example, say for a particular eigen value which is repeated say it is appearing thrice and for that we have got 3 distinct linearly independent eigen vectors then those linearly independent eigen vectors sitting in the place of x here with the same lambda will actually give us 3 linearly independent solutions which will not create any problem. However, if a is not diagonalizable then we will find some lambdas repeated and correspondingly we will not find the correct number of xs correct number of eigen vector and that poses a problem.

Now, in the case of scalar differential equation in the previous lectures in such a situation what we did when we found that e to the power lambda t is repeating and we needed to find additional linearly independent solutions then we tried t into e to the power lambda t right. A similar attempt here we will not tell, that is if we try the same say for example, lambda equal to mu is a repeated eigen value of this eigen value problem and correspondingly we get say only 1 eigen vector x not additional one. Then time x into e to the power mu t will be a solution as proposed, but trying that function into t as another linearly independent solution will not work let us see how.

Suppose we try this with a understanding that x into e to the power mu t is already a solution of this. Now, if we try that function into t then what happens? We differentiate it and try to insert here and see what happens the derivative of this will have 2 parts in 1 t will be differentiated and in the other part e to the power mu t will be differentiated. So, x into 1 into e to the power mu t that is here plus x into t into x into t into derivative of e to the power mu t which is e to the power mu t into mu is equal to on this side A x sorry A y, A into this. So, A x t e to the power mu t.

Now, since x is an eigen vector of this matrix A corresponding to eigen value mu. So, A x is equal to mu x. So, this term completely balances this term which will mean that x into e to the power mu t is 0, but that will mean that since e to the power mu t will not be 0 in general for all time then this will mean that x equal to 0 which is absurd because x is an eigen vector and a 0 eigen vector will make no sense. Eigen vector necessarily is a nonzero vector. So, trying this kind of a solution will not help.

So, here we need to try something else. The reason of why this does not work is that here the notion or the implication of linear independence the necessity of finding a new solution which is linearly independent of the old solution has 2 meanings 2 implications of linear independence is involved here - one is the linear independence in the ordinary vector space which is the linear independence that (Refer Time: 20:54) in x, in the n dimensional space in the n dimensional ordinary sense and the other sense of linear independence is in the function space. So, by multiplying with t in the proposed solution we are catering to the linear independence notion of the function space, but not of the ordinary space. In the ordinary space n dimensional space of x this is not linearly independent of the old solution.

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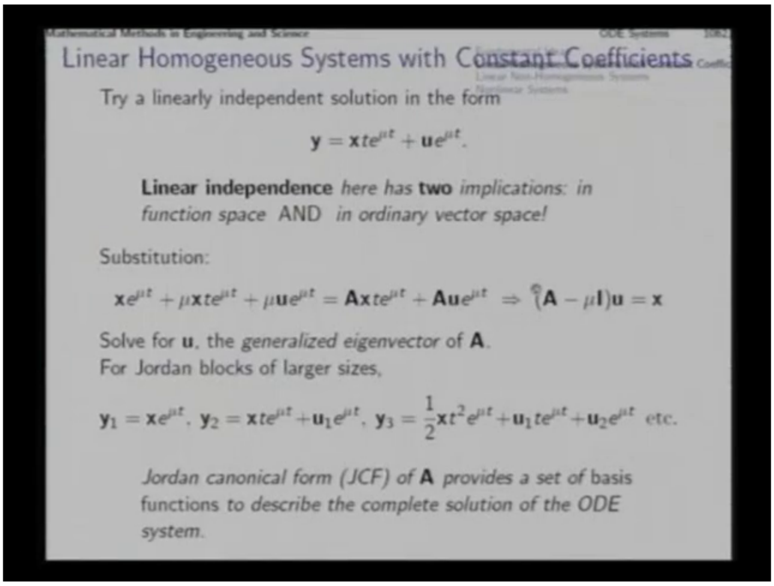

So, therefore, in this kind of a situation to propose a really linearly independent solution we need to make a proposal of this kind where this t here caters to the linear independence in the function space and this u which is not proportional to x will cater to the sense or notion of linear independence in the ordinary vector space of x. So, linear independence here has 2 implications in functional space which is held with the insertion of this t and in ordinary vector space of x which is handled which is catered to by the introduction of a different vector which is allowed to be linearly independent of x.

Now, we try this function and try to insert it in the differential equation y prime is equal to A y. So, derivative of this we will equate to A into this and in from there we will try to find out the vector u. So, as we substitute the derivative of this. So, x into 1 into e to the power mu t that is here and then x into t into derivative of e to the power mu t that is here x into t into derivative of e to the power mu t that is e to the power mu t into mu that is these 2 terms together constitute the derivative of this then the derivative of this, vector u into the derivative of e to the power mu t there is vector u into e to the power mu t into u into mu.

So, from this point to this point we have got y prime is equal to A into y. So, a multiplied with this whole thing A x t e to the power mu t plus A u e to the power mu t (Refer Time: 23:18) these two that is we force this proposed function to satisfy the differential equation system and from there we hope to determine u. Now, note that here x is an eigen vector of a corresponding to eigen value mu. So, A x is mu x. So, this term and this term is exactly equal, so this two vanish. Then e to the power mu t here and here and in the remaining term here it is common and that will not be 0 in general. So, we can divide by that.

So, here we will get x and whatever rest we get from here that we will take on the other side. So, other side what we will have we will have A u minus mu u is equal to x, A u minus mu u equal to x. So, the entire term rewrited to u are taken as one side and we have got this system of equations in u. Now, remember from the discussion of our eigen value problem module in linear algebra that as x is an eigen vector of A corresponding to eigen value mu then the solution of this system of equations will give us actually a vector u which will be the generalized eigen value of a corresponding to this eigen value and this eigen vector right.

So, we get u as the generalized eigen vector of the matrix and that generalized eigen vector put here will give us a new solution of the differential equation of the homogeneous system of differential equation which is linearly independent in both the senses. It is linearly independent in the sense of the ordinary vector space of x because of this vector which is linearly independent of x and also in the sense of function space because of this inclusion of t. So, this way we can construct additional linearly independent solutions to complete the basis.

Now, if the multiplicity is larger that is if the multiplicity mismatch between the algebraic multiplicity and geometric multiplicity of the matrix A, turns out to be more then similarly the way we found u here we can find u 1 and then next u 2 and next u 3 as many of them are required. That means, as many generalized eigen vectors appear to be there in the case that many we can determine and then compete the basis and therefore, we find that the Jordan canonical form of a will provide a complete set of basis functions to describe the complete solution of the ordinary system.

Now, with this method of determining the complete solution of the homogenous system in which we will have only y prime equal to A y after we have got that, then we can corresponding we can find out the complete solution of the corresponding non homogenous system also with a non homogenous term g t involved here.

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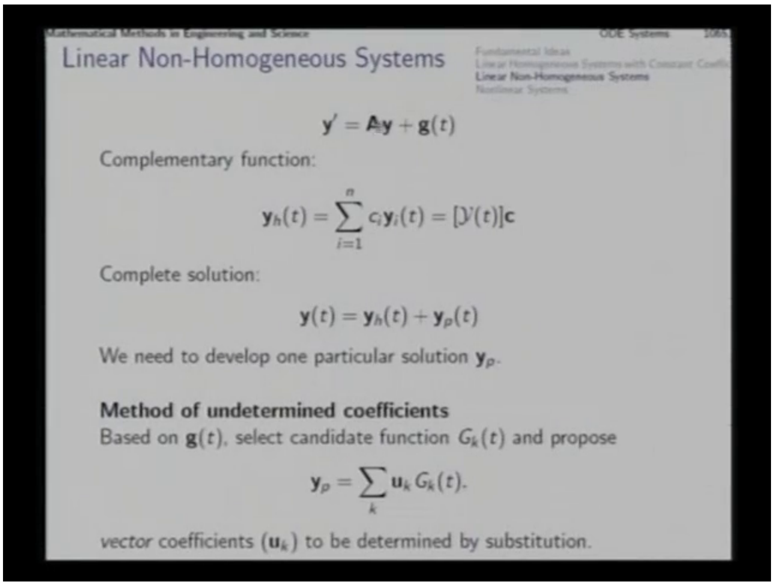

And now, for solving the non homogenous system as in the case of the ordinary one single differential equation scalar case we can have we can use the method of undetermined coefficient which has limited scope or we can use the method of variation of parameter which is more general. Here, there is a third method also which is called the method of diagonalization which will handle the matrix A through diagonalization if it is a diagonalizable matrix.

So, for the solution of this non homogeneous system of equations we will get the complementary function as usual that is the solution of the corresponding homogenous system which has no g t term right, so that will form the complementary function for the solution of this. That means, anything from here we can add to a particular solution of this to get another particular solution right. So, the complete solution will be this entire stuff entire complementary function plus 1 particular solution of this which we now try to find next.

So, for the method of undetermined coefficients of course, the understanding is that the constant coefficient matrix must be constant and as g t only certain classes of functions can appear for undetermined coefficient method to operate. Those special classes of functions are polynomial, exponentials, sinusoids and their combinations may be two product and some certainty. Now, based on g t we select candidate, candidate functions this capital g k t in the same way as we did in the case of single ordinary differential equations and then propose the particular integral like this.

Now, note that if it happens that g t this function has a polynomial in one of the entries an exponential function in another a sin function in a third. Then as proposals or proposal for the particular solution we need to include all of them a polynomial, a sinusoid, combination of sin and cos and an exponential all of these we need to include. Not only for that particular component of y, but for all of them that is why this vector is general. So, those particular types of candidate functions will appear here and unknown vector will be put here and then this entire huge sum will be taken as the proposed solution and we will differentiate this vector function and put it here and together we will try to solve for all these coefficient vectors. So, as many terms like this are there that many coefficient vectors we need to solve for. So, through substitution we can solve for this, but note that the method of undetermined coefficients is severely limited by the type of functions that can appear here and the matrix that must be constant.

Now, the second method, method of diagonalization will operate in somewhat generalized case, but only or those matrices which are diagonalizable.

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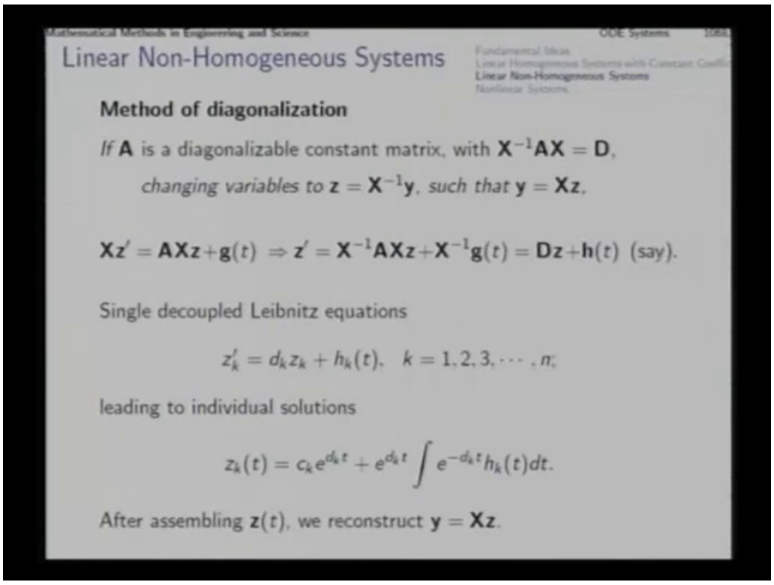

Now, if the matrix A is diagonalizable say with the matrix x. Now, here also it is constant matrix that we are talking about A and if it is diagonalizable that means, there is a basis x in which the representation of the same linear transformation is to a diagonal matrix. Then what we do we make a change of basis through this basis matrix x; that means, from for the unknown functions for the y variable we change the basis x inverse y gives the new unknown functions z and; that means, y is x z.

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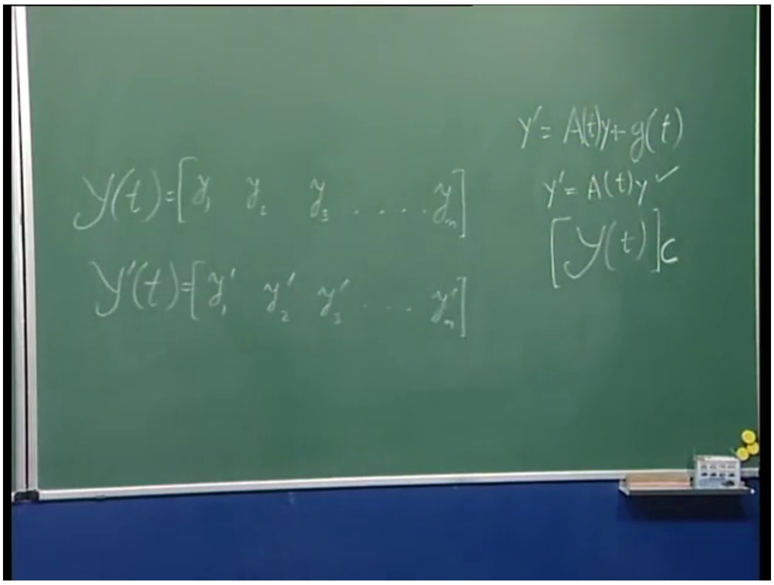

Now, this y equal to x z if we insert in this original differential equation then what we will get, in place of y if we put x z capital x matrix into z then here we will get x z prime right. So, x z prime will turn out to be A x z plus g. And then through the pre modification of x inverse we will get z prime is x inverse A x z plus x inverse g. Now, if a is diagonalizable with this basis matrix x then x inverse A x is d a diagonal matrix. So, this whole thing turns out to be d and x inverse g t will simply give us in its components linear combinations of earlier members of g t that is g 1 g 2 will appear as linear combinations now in h t. So, h t is a new function of t which is x inverse g t and d is a diagonal matrix and this is the prime achievement that we made through the basis change.

Now, this equation z prime is equal to d z plus h t with this matrix d diagonal what we have got is differential equations which are decoupled that is now, we can write it term by term component by component and we will have single decoupled equation - z 1 prime will be d 11 into z 1 plus h 1 and so on. So, individual equations will get decoupled in this manner. So, each of these is a simple first order Leibniz equation right. So, z k prime is d k z k plus h k. So, these individual Leibniz equations can be solves like this which we studied a few lectures back and then after getting z 1 t, z 2 of t, z 3 of t and so on the entire z of t that vector function is found and that can be then multiplied with x which gives us y of t. So, this is the method of diagonalization.

Now, this also applies in the case where A is constant and diagonalizable the only generalization from the method of undetermined coefficients is that as g t we can have more general kinds of functions rather than only combinations of polynomials exponential and sinusoid. So, that is the only generalization that has appeared from the method of undetermined coefficients into the method of diagonalization the method of variation of parameters in contrast is completely general.

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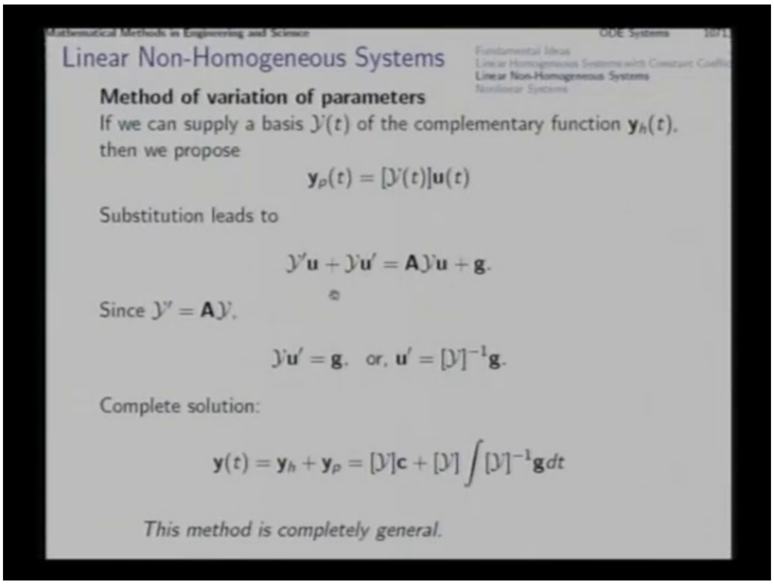

There we can talk of even the variable coefficient systems of differential equations in which the differential equation here has a matrix function sitting only requirement is of course, that beforehand we have solved the corresponding homogenous system y prime equal to a t y that is the corresponding homogenous system and we have listed out all its basis members. So, basis members of all its solutions, y 1, y 2, y 3 etcetera and we have formed the fundamental matrix.

So, if we can supply a basis y t of the complementary function then in any situation we can develop the complete solution of this system of differential equation with the help of variation of parameters. How do we do that? We propose this, that is the way we develop the complete solution of the homogenous equation that is this with the help of this matrix multiplied with a constant vector that is c 1 y 1 plus c 2 y 2 plus c 3 y 3 and so on, like that in place of this constant vector if we now, put a put a vector function u 1 u 2 u 3 etcetera then we propose the solution as u 1 y 1 plus u 2 y 2 plus u 3 y 3 and so on and that is the proposed function which we will put in this to find a particular solution of this non homogeneous system.

So, we propose this form of the particular solution of the non homogeneous system of differential equation. After proposing the particular solution in this manner then we will differentiate this function and substitute in the differential equation system in here. Now, as we differentiate this we get y prime into u plus y into u prime that is the derivative of this, that is y prime this side is equal to A y that is A y, A y p that is A, this y into u plus g of t that is this. Now, note that what is this that we are getting from here what is y? y is this large y this is essentially the matrix formed by y 1 y 2 y 3 etcetera as columns right. Now, this will be then the matrix with columns as y 1 prime, y 2 prime, y 3 prime and so on right and now, we know that each of these vector functions forming the columns of this matrix each of them is actually a solution of the (Refer Time: 37:59) homogeneous equation, homogeneous differential equation that is homogeneous system.

So; that means, this y 1 prime is A into y 1 that is this is nothing, but a into this similarly this is A into this similarly this is A into this and so on; that means, this entire matrix column by column will be found to be A into this matrix column by column; that means, y prime is a into this matrix y. So, in this equation here y prime u will cancel exactly with a y u because this matrix y prime this entire matrix is nothing, but a into this matrix y. So, this term and this term will cancel out and then after these 2 terms are gone we have capital y u prime equal to g and here u prime is the unknown function that we would like to determine y is the fundamental matrix that we already know, g is the function here that is that has come with the differential equation.

So, from here we can solve for u prime which is simply this the inverse of the fundamental matrix multiplied with g. This way we get the rate of the coefficient function u. And then the solution we will get after we integrate this and find the vector function u which we need to insert here. So, then this integral of y inverse g will give us this u and that multiplied with y will give us y p a particular solution and that added with the complementary function which is y c will give us the complete solution of this system of ODEs and this method is completely general.

The only starting point for this method, method of variation of parameters is this complete basis is the fundamental matrix of the solutions of the corresponding homogeneous equation.

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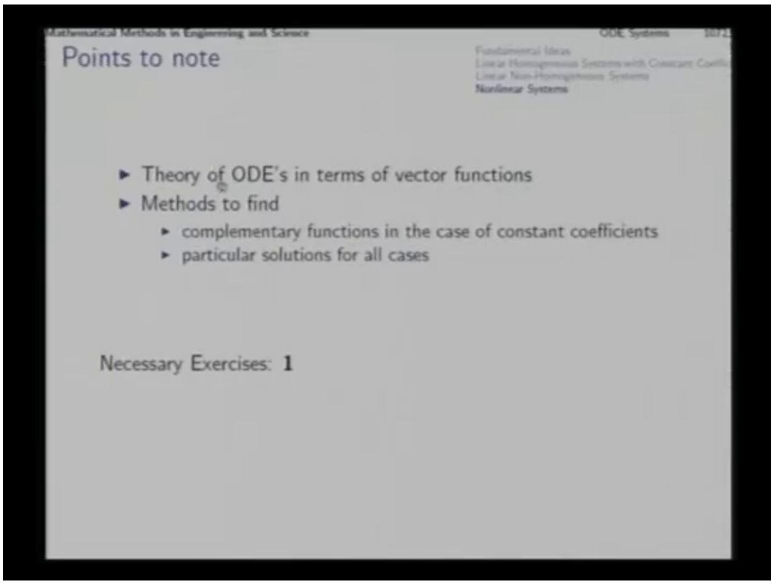

Now, here we have, in this lesson till now we have discussed the theory of ODEs in terms of vector functions and we have studied the methods to find complementary functions in case of constant coefficients, in the case of variable coefficients with A as a function of t there is no general method. But in some cases exploiting the particular situations of the coefficient function coefficient matrix we can we may be able to determine the complementary function and then we can find out particular solutions for all cases after the fundamental matrix is in hand.

Now, for our purposes systems of special interest are those which have only 2 state variables that is the second order system. Now, before going into that we can briefly discuss 2 important issues one is the situation with non-linear systems. Now, if the system is non-linear like for example, if the general situation that we consider at the very beginning.

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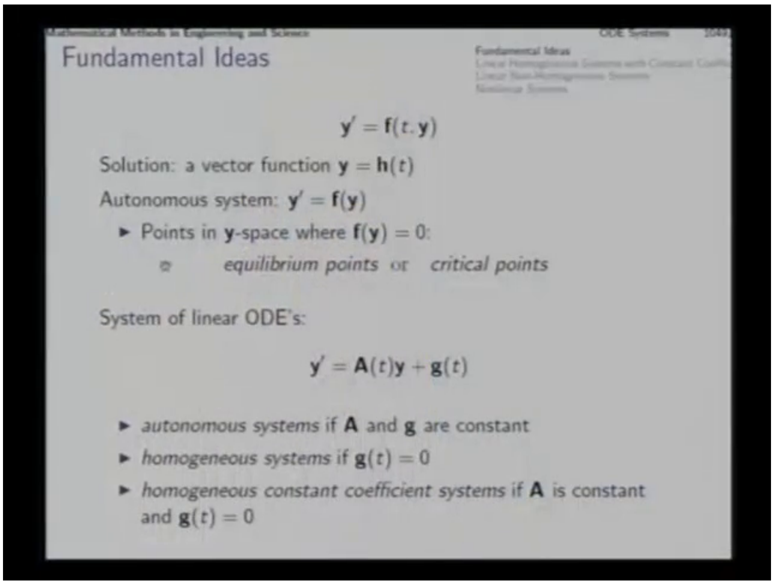

Say, if we have got a system of ordinary differential equations which is in this form and not in this form; that means, the system of differential equations is not linear. So, till now whatever we have discussed is the solution of linear systems like this.

Now, if that is not the case if the system is general which involve the unknown functions y in all kinds of ways not only in a linear manner, then what do we do? There are two things we can do one of course, is the method that we studied earlier that is we can seek a numerical solution for which we have to say the initial conditions first and then we get the solutions in terms of numbers in terms of values. There is another possibility which is often reserved at two in the case of non-linear equations and that is through linearization. So, for that what we can do we decide on an operating point around which we want to describe the solutions and in that case what we can do is that around that point we try to make a first order truncated Taylor series. That is f p y we try to expand as the value of f p y at the reference point y 0 plus the Jacobean of this matrix this vector function with respect to y into y minus y 0, and then we truncate the Taylor series.

So, that gives us the first order representation of the same dynamic system and the resulting differential equation turns out to be in this form and then we can analyze that and the result of that analysis will be valid in the immediate neighbourhood of that particular reference point y 0, this is one issue. Now, there is another important issue and that is if the function here is non-linear that is if it is a non-linear differential equation.

Now, if it happens to be an autonomous system that is if e does not appear explicitly then we have this case rather than having the case in which y prime is equal to A y we have y prime as a general non-linear function of y. And in that case as the solution of f y we can have multiple equilibrium points isolated yes, but multiple equilibrium points not only the origin, but several equilibrium point we can have in fact, origin need not be an equilibrium point in that case. Only in the case of A y sitting here we had origin as the equilibrium point.

Now, if it is a general function then all solutions of this system of non-linear equations will be equilibrium points. So, for example, in that case suppose we have got some five different equilibrium points then around each of those equilibrium points we can conduct a linearization of this function, this function rather and then say that around that particular isolated equilibrium points we can make independent analysis and for each of these independent analysis there will be a case of this kind, let us see how.

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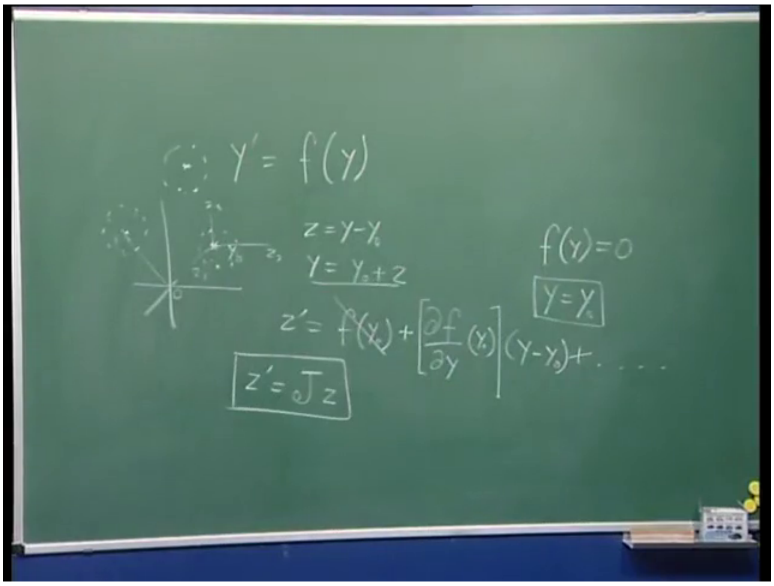

Say this is the system of differential equations describing an autonomous system describing a physical system which do does not have explicit dependence on time.

Now, suppose we separately solve for this system of non-linear equations and find out that y equal to y 0 happens to be one solution of this right then what we can say is that we make a. Now, this is suppose a frame of reference of y. So, y equal to 0 is here and y equal to y 0 is here. Now, if we shift the frame of reference here and say that this is z 1 z

2 z 3 and our new set of variables new set of coordinates of the state space is z that is z 1 z 2 z 3 then we say z is y minus y 0 then we want to find out what is z prime.

So, then y is y 0 plus z we insert this here. So, if y is y 0 plus z then y prime is same as z prime y 0 is constant. So, here we will have z prime and that is equal to f of this f of y that is this. So, that will be f of y 0 plus first order term del f by del y evaluated at y 0 into y minus y 0 that is plus higher order term. Now, see f of y 0 is 0 because we found y equal to y 0 through the solution of this system of non-linear equations. So, this is 0 then y minus y 0 is z. So, then what we get we get z prime is equal to this matrix which is a Jacobian into z. that means, around this point we have the same phenomena described by this system which is linear and therefore, a linear analysis the way we discussed can be conducted which will be varied in the vicinity of this point similarly if this system of equations had another solution at this point. Then for this point we could have another shift of coordinate frame and have another linear system like this which will be based on the Jacobian at that point and then we can conduct another linear analysis which will be valid in this vicinity and so on and so on.

Now, the sense in which these equilibrium points are the critical points in that sense if we conduct analysis around these critical points and then extend the findings of these analysis over the entire rest of the space rest of the y space then we essentially capture the entire behavior of the dynamic system all over its state base. That is the idea which is used in analyzing the non-linear system

Now, after we will apply some of the methods that we have studied in this particular lesson into the special case of state vector being of dimension 2. Now, why dimension 2 in particular because there will be many physical systems of enormous importance which have their state space of dimension two only and the reason for that is that our nature follows by enlarge a second order dynamics. You will note that the Newton’s laws give you a relationship a governing equation which is second order in nature the force the effort is related directly to acceleration which is the second derivative of the position.

So, in this manner the dynamics of the complicated systems also will be typically framed in terms of second order differential equations because of the typical way our nature behaves Newton’s laws giving the dynamic equations of many systems c f d, c f d equations typically they are second order differential equations then in electro dynamics electro dynamics you will find that Maxwell's equation is also second order differential equation. So, in many cases in the study of natural phenomena you will find that the second order differential equations appear in enormous lead diverse situations and therefore, second order differential equations turn out to be very important in the analysis of systems appearing in physics and of course, in its off shoot which is engineering.

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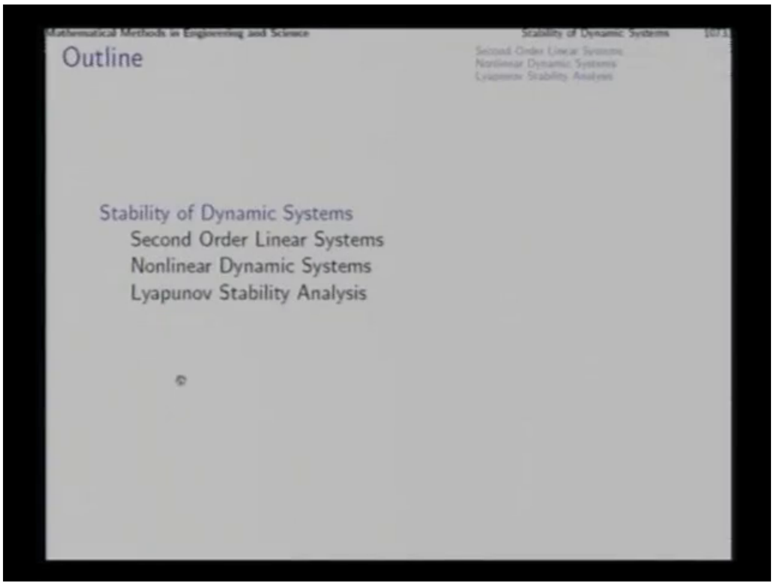

So, in the next lecture we will discuss the stability of dynamics systems which are described with the help of second order linear systems first. So, second order one single second order differential equation means the same as two first order differential equations because we can always represent the second order differential equation in the state space which will have two state variables. So, second order linear systems we will discuss in detail the stability of such systems and what kind of such critical points we can get in the case of dynamic systems and then we will consider the situation where we can analyze non-linear dynamic systems as well and then we will consider an alternative route of stability analysis which is called the Lyapunov Stability Analysis, Cartesian.