Mathematical Methods in Engineering and Science
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Module - VI
Ordinary Differential Equations
Lecture – 03
Methods of Linear ODE's

Good morning, as we saw in the last lecture the method of undetermined coefficients for solving the linear non homogeneous ODEs is limited by the necessity of constant coefficient and it can handle only if you selected functions as the right hand side. Namely, the polynomials, exponential function, sinusoids and their combinations.

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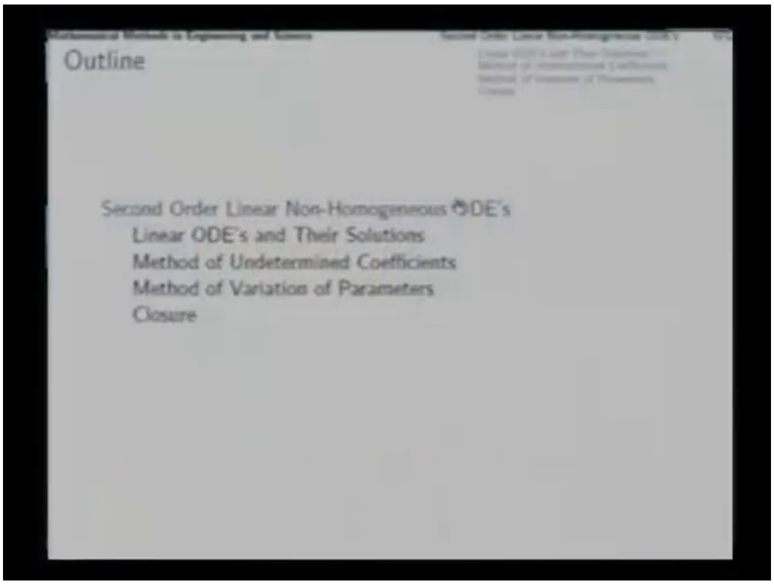

Now, for general situations in which the right hand side function can be anything the input function can be anything, and the coefficients are also not required to be constant if they are functions if they are variable coefficients, then the general method for solving the non homogeneous differential equation is the method of variation of parameter.

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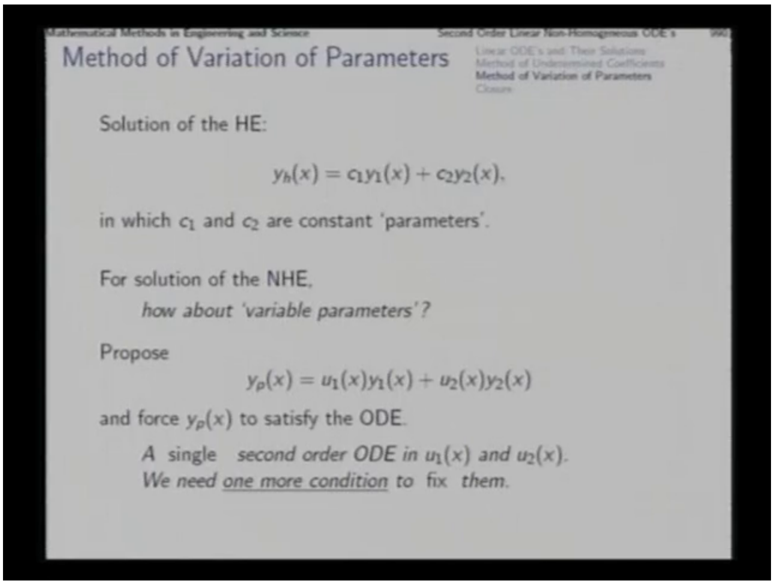

In this we first note that if y 1 and y 2 are 2 linearly independent solutions of the corresponding homogeneous equations, then this combination this linear combination with constant parameters c 1 and c 2 gives as the general solution of the homogeneous equation that is the complimentary function for the purpose of the non-homogeneous equation. Now, for the non-homogeneous equation the solution needs to be linearly independent of these two basis members. Now as long as these parameters are constant we cannot have linearly independent function.

So, what we try to think is that how about making these parameters as variables, that is as functions of x and as we do that and we replace c 1 and c 2 by 2 functions u 1 of x and x u 2 of x then what we are doing we are conducting a variation of parameters and that is why this method is called variation of parameters. So, we consider variable parameters. So, we propose the particular solution of the non homogeneous ordinary differential equation as u 1 y 1 plus u 2 y 2, and then we asked for the functions u 1 and u 2 that will make this yp satisfy the given differential equation.

Now, one point is very clear that we can take this yp and force it to satisfy the differential equation, that will mean that we will differentiate it twice and then take this yp and the resulting yp prime and yp double prime and insert in that equation and we will say that this equation is now satisfied. But that we will give us a single second order ODE in u 1 and u 2. Now there are 2 unknown functions to be determined and for that a single differential equation is not enough; we will need two differential equations to solve for two unknown function. So, one more condition is needed to fix them.

Now, at this stage we need one more condition between u 1 and u 2, apart from the condition that together they have to make yp satisfy the differential equation apart from that we need one more condition and we try to frame that additional condition in such a manner that our work is also reduced our work is also made more easy the way to do that

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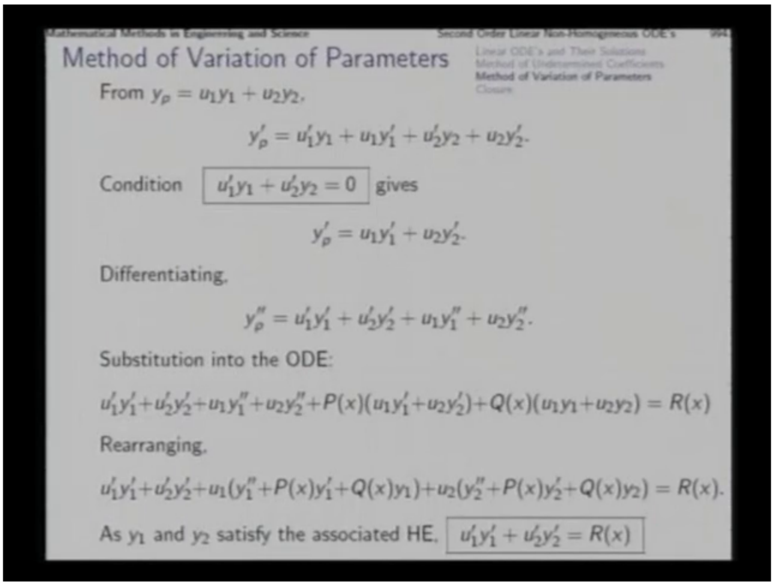

is at first as we differentiate yp this function this proposed function, we get u prime u 1 prime y 1 plus u 1 y 1 prime from the product rule, similarly from here u 2 prime y 2 plus u 2 y 2 prime. So, these are the four terms in the derivative of these 2 terms here.

Now, we know that as we differentiate it further, then u 1 u 2 y 1 y 2 all four of them will generate their second derivatives; and that means, that the resulting expression for yp double prime will have up to second derivatives of all the four functions y 1 y 2 u 1 u 2. With y 1 and y 2 there is no difficulty because they are known functions. But u 1 and u 2 when they are when involved their second derivatives that mean that the resulting differential equation will be a second order differential equation. The additional condition that we need we can choose in a particular manner which will obviate that second order.

So, what we do is that from here we collect the terms which has the first derivatives of the unknown functions u 1 and u 2 so; that means, this term and this term and together we say that we will impose the condition on functions u 1 and u 2 that, this term and this term together vanishes that is we impose this plus this equal to 0. If we do that then yp prime reduces to the sum of the rest of the 2 terms this. Now in this expression in yp prime we have got the derivatives of y 1 and y 2, but not the derivatives of the unknown functions.

So, that way when we now differentiate this, we get the derivatives of the unknown functions only up to first order. So, that is this yp double prime that again with the now ordinary product rule will give a u 1 prime y 1 prime plus u 1 y 1 double prime and similarly 2 terms from here right. Now here again we find that the unknown functions u 1 and u 2 are involved only up to the first derivative; known functions y 1 and y 2 are involved up to second derivatives, but that is not a difficulty.

Now, on this note that another condition like this we cannot impose, because if we impose another condition like this that is if we try to say that these 2 together also vanish then there will not make sense because this condition and that condition together will imply that u 1 prime and u 2 prime are all are both 0 right because these 2 this condition and this condition together will mean that we are looking for u 1 prime u 2 prime, which satisfy a homogeneous system of linear equation is not a singular coefficient matrix. So, that we will give only the trivial 0 solution, u 1 prime u 2 prime will become 0 which means that u 1 and u 2 will turn out to be constant and that will not give us a solution of the non homogeneous equation.

So obviously, only one condition we needed and that we have got the second condition we need to get by substituting this yp double prime this yp prime and this yp sorry this yp double prime, this yp prime and this yp into the differential equation. As we do that yp double prime these 4 terms plus P into yp prime, plus Q into yp is equal to Rx. this is the given differential equation. Now as we rearrange these terms collecting together all terms which have u 1 that is this is the first term, then from here we will get Py 1 prime and then from here we will get Qy 1 right. So, we collect the u 1 terms together and u 2 terms together.

Now, we know that y 1 and y 2 are solutions of the corresponding homogenous equation; that means, that this equal to 0 and similarly this equal to 0. So, then concerning these 2 0 terms we get u 1 prime y 1 prime plus u 2 prime y 2 prime as equal to Rx. Now this condition that we imposed through which we also simplified the expression of yp prime, through which we avoided the second derivatives of u 1 u 2 in the resulting equation here that is one equation and the given differential equation gives us the second equation in 2 unknowns the unknowns are u 1 prime and u 2 prime.

Now, this equation and this equation if we write in the proper matrix vector form, then y 1 y 2 y 1 prime y 2 prime come in the coefficients and we get the system of equation in the standard form.

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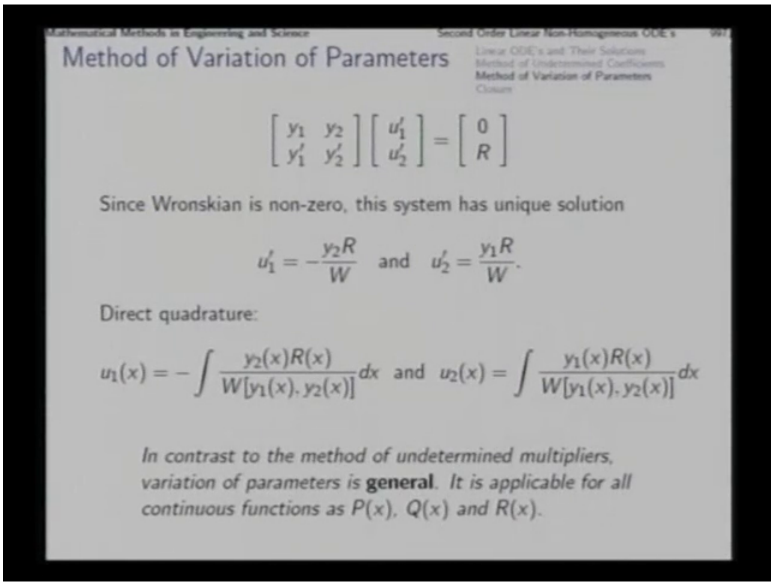

Now if we solve this then we will get u 1 prime and u 2 prime and we already know that this matrix is non singular for all values of x under question in the interval for which we are looking for the solution. Now the determinant of this matrix is actually the Wronskian. So, as long as y 1 and y 2 are linearly independent solutions forming the basis for the solutions of the homogeneous equation. So, we know that these are 2 linearly independent functions. So, this Wronskian will be non zero.

So; that means, that the solution of this we will give us unique u 1 prime u 2 prime in other words this is an invertible matrix. So, you get that solution. As we get that solution from the 2 equations here we get u 1 prime as this and u 2 prime as this, Wronskian goes in the denominator which is not a problem because it is non zero and then this is a known function of x similarly this is another known function of x. So, direct integration will give as u 1 and u 2 these u 1 and u 2 when put in the proposed form of the particular solution will give us the particular solution that we are looking for that is u 1 y 1 plus u 2 y 2 ok.

So, this is the method of variation of parameters to find the solution of the non homogeneous differential equation in a general situation, and the first step even in this is to find the solution complete solution of the corresponding homogeneous equation, that is the first step because they will form this matrix here. So, this is applicable for all functions Px Qx and R x which are continuous and bounded, and it will be valid only for that interval in which these functions remain.

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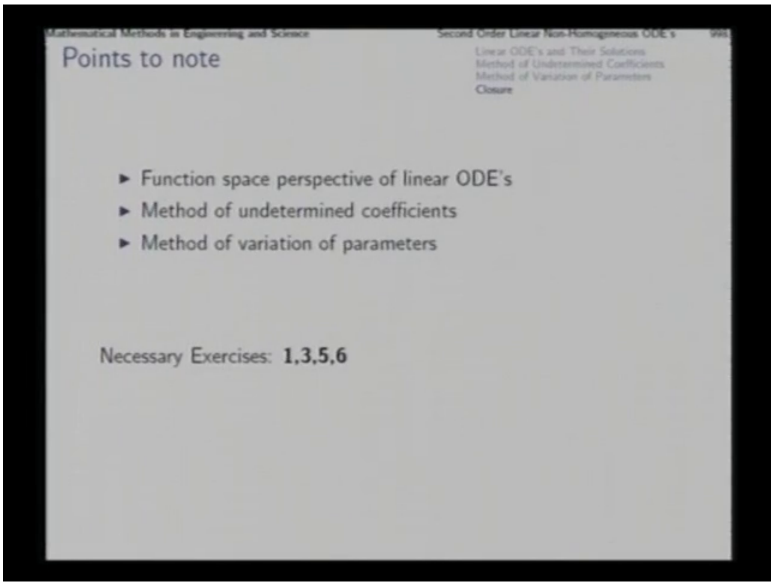

So, in this relation these are the important points that we have studied after looking at the function space perspective of linear ODEs we studied these 2 methods undetermined coefficients and variation of parameters. These same methods and the previous discussions on finding the solution of the corresponding homogeneous equation all these now we generalize for higher order differential equations.

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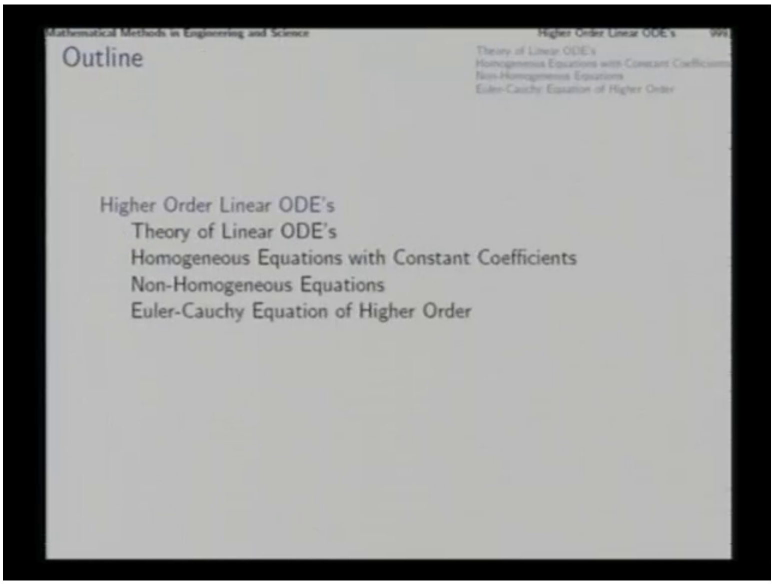

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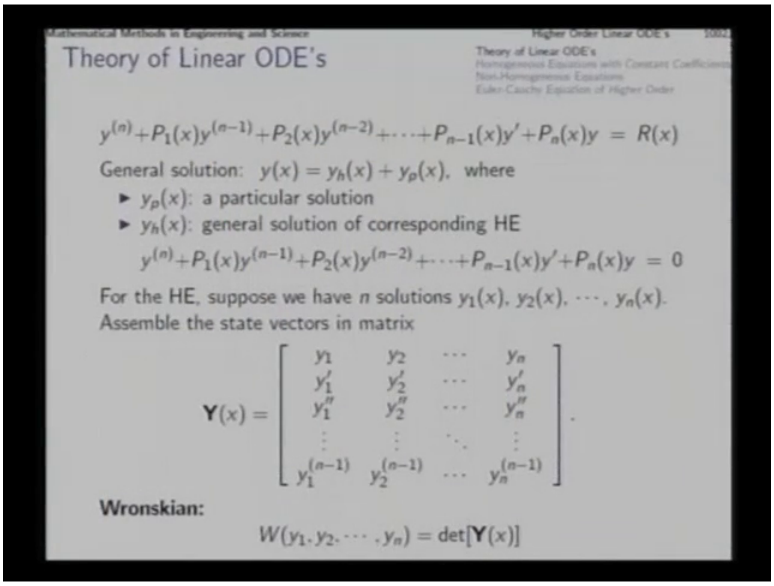

First we consider this equation; this is the general linear ordinary differential equation of say nth order. So, up to the nth derivative is appearing linear and these are coefficient functions which depend on x only.

Now, understand again that we are here going to consider only those cases where P 1 P 2 P 3 up to Pn and R these are functions of x which are continuous and bounded in the interval under discussion. Now again for this will have the general solution in this manner as yh plus yp, where yh is the complimentary function in other words yh is the general solution of the corresponding homogeneous equation that is this equal to 0 that is it is solution it is a general solution of this equation.

Now, for the homogenous equation suppose we have got n solutions, y 1 y 2 y 3 etcetera then the state vectors of these solutions when assembled as column vectors in a matrix, then what we have it is an nth order differential equation. So, state vector will have the function itself its first derivative second derivative up to n minus 1 th derivatives. So, this is the state vector of the first solution similarly this is the second vector second state vector for this corresponding to this and so on. So, when we assemble these as columns then we get an n by n matrix of functions, that is the first row is these functions second row will have its first derivatives, third row will have the second derivatives fourth row will have the third derivatives and so on the last row the nth row will have the n minus 1 th derivatives of these functions.

So, this is the fundamental matrix which will appear again and again. Now first point is what is the Wronskian in this case. So, the determinant of this matrix is the Wronskian in this case, this is a direct generalization of the case of second order differential equation. So, this is the Wronskian of all these solutions all these n solutions. Now parallel to the theory that we studied in the case of second order differential equations here we will have corresponding points.

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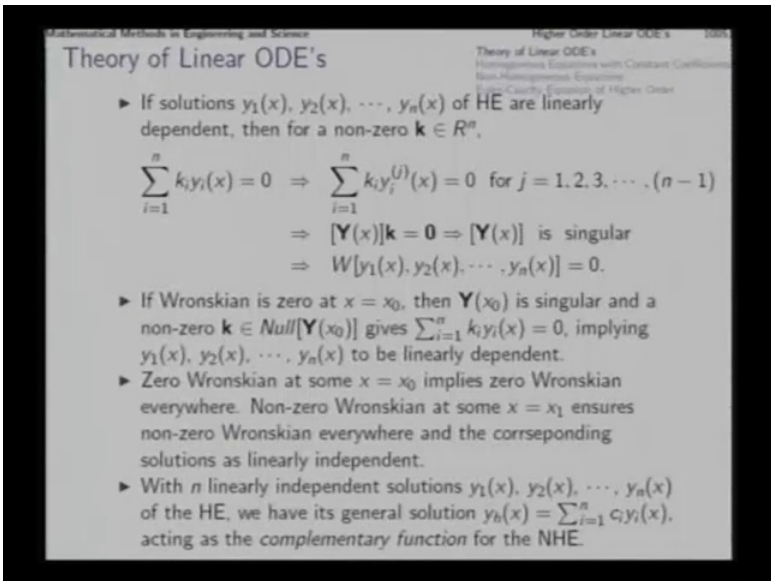

First point is that if these n solutions of the homogeneous equation are linearly dependent then the Wronskian will be 0 how do we prove that? We say that if these are linearly dependent then for a non zero vector k we can find this that is we can have a linear combination of the n solutions 0 even though each contribution is not individually 0 that is for a set of case k 1 k 2 k three up to kn not all 0 together, we can find a linear combination of these y is which turns out to be 0 if they are linearly dependent.

And if so, that will mean that the derivative of this will give us sigma there is some of ki yi prime as 0 as ki yi prime it is j is equal to 1. The derivative of that will give us ki yi double prime equal to 0 that is for j equal to 2 and so on that means, up to n minus 1 we can construct these sums and get them 0 if the solutions are linearly dependent. Now note that this will mean this and this together will mean that the fundamental matrix y into k is equal to 0, which means that its first row is this, second row is this for j equal to 1, third row of this matrix vector equation is this same thing for j equal to 2 and so on.

So, the complete n rows we can call step like this, and note again that if y 1 through the yn are n linearly dependent solutions then we can construct such a matrix vector equation homogeneous equation 0 on this side and have a solution which is not zero that is vector k is not 0, and that means, that this thing coefficient matrix y must be singular because for the homogenous system of equation to have a non trivial solution k it, is necessary that the coefficient matrix is singular. So, Yx is singular which means that it determinant the Wronskian is 0. So, this is the first point that is the n solutions of the homogeneous equation being linearly dependent will imply that the Wronskian is 0.

Now, if the Wronskian now we try to show the converse; if the Wronskian is 0 at some point then y this matrix at that point evaluated at that point is singular and then that will mean that a non zero vector k can be found in its null space, which will in turn give this which will imply that the solutions are linearly dependent. So, this is the converse of this now together they also mean that 0 Wronskian at some point x 0 will immediately imply 0 Wronskian everywhere because 0 Wronskian at some point will mean that the solutions are linearly dependent which in turn will mean that the Wronskian is 0 everywhere this is x and that will again mean that non zero Wronskian at some point will ensure non zero Wronskian everywhere and the corresponding solution as linearly independent.

So, now what we do with those linearly independent solutions if the Wronskian is non zero, we say that with these n linearly independent solutions of the homogenous equation we have the general solution in this manner as we had for the case of second order differential equation and this linear combination, the complete solution of the homogenous equation will serve as the complimentary function for the solution of the non homogeneous equation.

Now, we consider the first case of constant coefficients.

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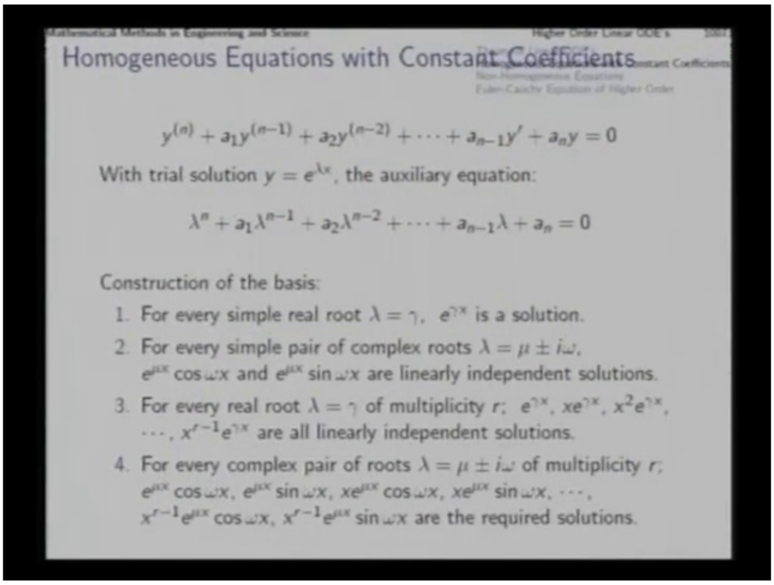

Consider this constant coefficient case for homogeneous equation, for which the complete solution of homogeneous equation we can find. If the coefficients are functions p 1 xp 2 x and so on then in general we are not able there are no analytical procedures to find out all the solutions of this homogenous equation. Now, in this case if the coefficient are constant, then we can do that exactly as the second order case with necessary extensions and changes. Here also the left hand side of this equation has y y prime y double prime in linear combination. So, we expect solutions of this form exponential solution.

So, if we take a trial solution of this kind and insert it its derivatives as many times we differentiate that when you lambdas get multiplied with it. So, we get the auxiliary equation as lambda to the power n plus a 1 lambda to the power n minus 1 and so on till from here we get an. Now this is a polynomial equation of n th degree; that means, it will have n solutions. So, out of from those n solutions for every simple real root that is if gamma is one solution of this polynomial equation appearing only one, that is in its factorization if lambda minus gamma appears only one that is that makes it is a simple root.

So, then corresponding to that e to the power gamma x would be a solution. It is for every simple real root the corresponding exponential function will be a solution of this differential equation, because that is the idea with which this was started. For every pair of complex roots like this mu plus minus i omega, you will say that e to the power mu plus minus i omega will be 2 solutions integers, and in the case of second order differential equation we reorganize it and found the 2 solutions in a different manner there was e to the power mu x into cosine omega x and e to the power mu x into sin omega x.

So, these 2 will be the corresponding 2 linearly independent solutions of this homogenous equation. Now, if we get a real root which is repeated and if its multiplicity is r then we will find that e to the power gamma x x into e to the power gamma x x square into e to the power gamma x and so on up to x to the power r minus 1 into e to the power gamma x all these will give us r linearly independent solutions of this, this is something which you can prove which you can show.

Similarly, for every pair of complex roots which appears repeated, you will have similar additional linearly independent solutions for example, if you mu plus if mu plus minus i omega appears twice. So, that should account for four linearly independent solutions. So, first pair is e to the power mu x cosine omega x, e to the power mu x sin omega x and the next pair will be these two multiplied with x. Similarly if it appears thrice the third pair will appear with x square e to the power mu x cosine omega x and x square e to the power mu x sin omega x and so on.

So, with multiplicity r you will go up to x to the power r minus 1 power. So, this way if you determine the solutions n solutions of this binomial equation, then from that you can determine n linearly independent solutions of this homogeneous differential equation. Now, after having the solutions in this manner, you want to find out the solution corresponding to the solution for the non homogeneous equation with this whole thing equal to some Rx. Now, if the equation is with constant coefficients like this and R x is one of those chosen functions for the right hand side that is one of those special functions, binomial exponential sinusoid or their mutual combinations then the method of undetermined coefficients will give you a solution that is for this.

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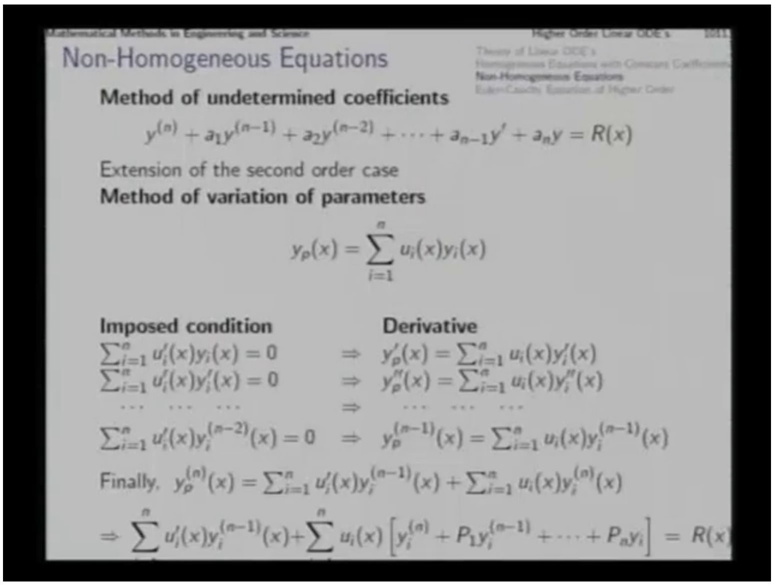

So, method of undetermined coefficients will work exactly as the extension of the second order case. Here also those things those modification rules will apply that is even if this happens to be e to the power gamma x, but then if e to the power gamma x happens to be already there in the basis for solution of this, then rather than proposing e to the power gamma x you will propose x into e to the power gamma x.

Similarly, if e to the power gamma x e to the power gamma x and so on up to x to the power r minus 1 e to the power gamma x up to this if all of these are sitting already in the basis for the solution of this. Then even if e to the power gamma x appears in the right hand side function you will have to propose as yp x to the power r e to the power gamma x there is a next into some constant and so on. So, this is a general rule.

Now, measure of undetermined coefficients is something which will work exactly as in the case of second order equation and as you know that in the general case with general right hand side function and functions appearing in the place of coefficients a 1 a 2 a 3 this will not be sufficient. For those purposes for those situations you will need the more general method which is the method of variation of parameters, which will work in the general case that is in this case that will work in this case as well.

So, for that for solving that we will consider a linear combination like this now let us. So, we propose a function of this kind which is a linear combination of all these n functions y 1 to yn, which are known to be the basis members of the corresponding solution of the homogenous equation, basis members of the solution of the corresponding homogenous equation.

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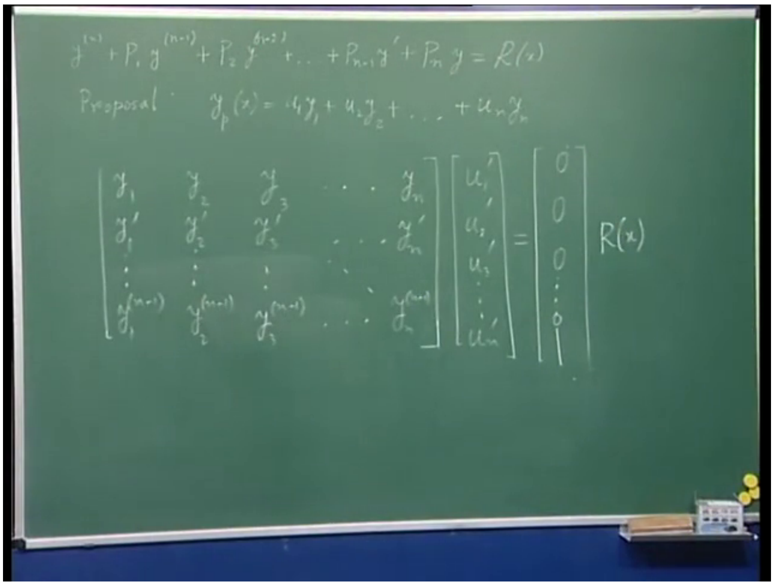

That is the solutions of this. So, n linearly independent solutions of this equal to 0 is y 1 y 2 etcetera. So, yp is proposed like this.

Now, we differentiate this and eventually we will differentiate it n times and insert all the functions and derivatives into this equation, in order to find the solution find a particular solution of the non-homogeneous equation. Now, as we differentiate first time we find that the derivative of these n terms will get us 2 n terms. In the first n terms we will have ui prime yi and the other n terms will be ui yi prime. Now, we try to put additional condition that this is 0. So, when this part is 0 then the first derivative of this will turn out to be as ui yi prime.

Now, this is 0 means basically that the rate of the function u is orthogonal through the vector function in the vector space sense that is if u of x is a vector function with components u 1 u 2 etcetera, then u prime will be its derivative its rate and that rate is orthogonal to the vector function y of x. So, that is the condition we have imposed and that gives us yp prime which is like this. Now note that this condition turns out to be the first row from this matrix vector equation, y 1 into u 1 prime plus y 2 into u 2 prime and so on yn into u 1 un prime equal to 0 that is essentially this condition.

Now, imposing this condition we get the first derivative as this which is free from u prime terms which means that its derivative will be free from u double prime terms. So, we find out its derivative and as we try to put its derivatives in that they will be again 2 n terms n terms together are taken as 0 that is the second condition that we have imposed. So, as we impose that the from the derivative of this u prime y prime terms the sum of those terms vanish that is the condition we impose and the rest of the 2 terms u into y double prime that remain here.

Now, this condition is equivalent to the second row that is sigma yi prime ui prime, y 1 prime into u 1 prime plus y 2 prime into u 2 prime and so on is equal to 0, that is the second row in this matrix vector equation. Similarly we go on differentiating the derivatives further at every step we go on imposing these conditions in the n minus 1s derivative that gets involved here and corresponding n minus 1 th derivative we get here, and this is the point this is the stage where we have to find the last derivative that is needed that is yn ypn.

So, when we differentiate this we get the full derivative in which full 2 n terms from ui prime yin minus 1 plus ui into yn. So, these 2 both the terms will appear here in the n th derivative. So, note that from this point till this point there are n minus 1 conditions that have been imposed with the function the first derivative second derivative up to n minus 2 th derivative. So, these are n minus 1 conditions that have been imposed these are the top n minus 1 rows from here involving in the coefficients here up to the n minus 2 th derivatives.

The last equation we will get when we insert this ypn this expression and all these expressions for the previous derivatives and the function itself also. So, when we put all of these expressions, now into this differential equation given then what we do? We add to this last equation p 1 times this plus p 2 times the previous 1 plus p 3 times the still previous 1 and so on finally, p n minus 2 times this plus pn minus 1 times this plus pn times this that is from here. As we do that and collect the terms together then this term is here this term is here u prime yn minus 1 ok.

And here the other terms that is this into this plus p 1 into this plus p 2 into previous 1 and so on. So, there you will find that u is common in all these even up to this. So, in the that is why u has been ui has been kept common outside and inside the bracket you have yi plus p 1 into the yi n minus 1 th derivative and plus p 2 into yi n minus 2 th derivative and so on that is this actually. Now we know that each of the functions y 1 y 2 y 3 y 4 etcetera are solutions of the homogeneous equation corresponding homogeneous equation that is this equal to 0.

And therefore, for every i separately this bracketed term vanishes. So, this whole thing goes to 0 this remains on the left side left hand side and R x remains on the right hand side and that is why we get this last row from this system of equations and what is the matrix sitting here this is our old friend that is the fundamental matrix capital y that we constructed from the state vectors of these n linearly independent solutions of the corresponding homogeneous equation these are the columns. So, here is our fundamental matrix capital Y of x.

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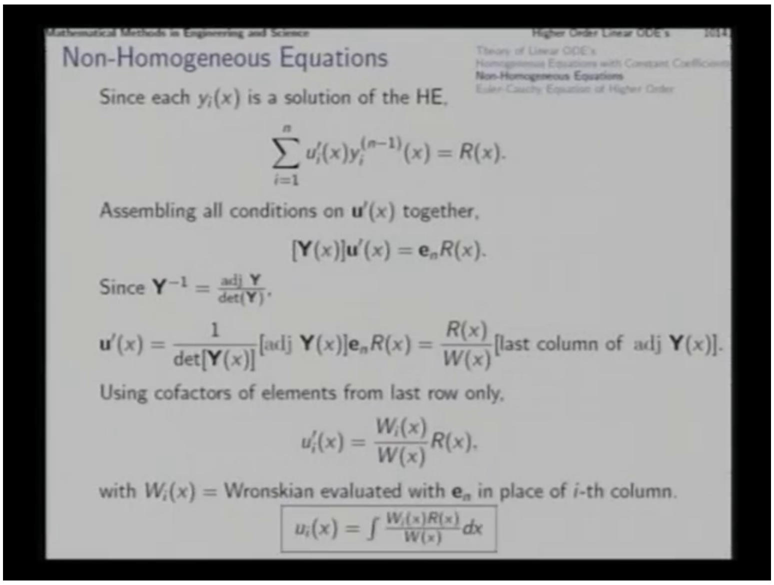

So this is the last equation and this is the assembled matrix vector equation together Y of x this matrix into the vector u prime is equal to this vector with all zeroes and R x at the last entry.

That can be said as en into R x which will mean that last one is one and here we have R x it is the same thing. So, this is the en that is the last natural identity member in the n dimensional rates. So, now, this is the matrix vector equation in the rates u prime and this matrix is non singular because these solutions form a basis; that means, they are all linearly independent. So, this matrix is non-singular as its Wronskian is non-zero. So, this non singular matrix can be inverted and this matrix vector equation can be solved.

Now, usually I have told you earlier that in the context of linear algebra that the computation of inverse numerical computation of inverse with the help of this formula is typically inefficient, but here we are doing the analysis in terms of not numbers, but in terms of expressions and in this context this is this formula turns out to be of advantage. So, inverse of Y is adjoint by determinant right. So, if we use that then the adjoint by determinant will come here and we can calculate u prime x. So, that is 1 by determinant into adjoint into this side.

Now, note that this adjoint matrix will be multiplied with a vector immediately, which has all its entries 0 except the last one. So, from the adjoint only your last column is needed that is adjoint y into en will essentially mean the last column of adjoint y so; that means, when we construct adjoint y we do not construct the entire adjoint of this matrix, but we need to construct only the last column of it. So, last column of its adjoint will mean will require the cofactors of only the last row because adjoint will be the matrix formed by the cofactors of this matrix transposed.

So, the last column of the adjoint will be formed by the cofactors of the last row. So, last column of the adjoint matrix is what we will need that is adjoint into en. So, for that we will find out the cofactors of the elements from the last row only. Now, what is the cofactor of this entry for the cofactor of this entry we will need to remove this column and this row and whatever remains. For that we have to find the determinant and then we will put the sign according to the row number column number of this, that is as which is the row number n and 1 2 3 that is the column number.

So; that means, the position of this will reflecting the sign and other than that this column and this row will be removed. So, the cofactor of this this element we can get by replacing this column with 0 0 0 0 0 0 1 because this column anyway will be removed. So, if we put 1 here then as we remove this row and this column the rest of the determinant will give us the correct cofactor with the sign that we will get from the position. So that means, if we put if we replace this third column with en that is 0 0 0 0 1 here, then the determinant of this itself will be that cofactor because the in that general computation through this column we will get only one term which is 1 into the rest of it which is the correct minor and the position we will get from the expression or the determinant anywhere.

So; that means, that the entries the first second third entry will be given as w 1 w 2 w 3, when Wi is the same Wronskian same determinant of this matrix when evaluated with en in place of the i th column. So, this gives us the expression for ui prime and then when we find that this is a function of x only then we can evaluate ui as direct quadrature direct integration and then this set of u 1 u 2 u 3 u 4 when inserted in the original proposal gives us a particular solution of this non homogeneous equation.

So, this is the way we develop the general solution for a non homogeneous equation that is we first develop the general solution of the corresponding homogenous equation by replacing the r with 0 from that we find out the basis for all solutions of that homogenous equation, that basis yn y 1 to yn we use to propose a solution of the non homogeneous equation including the Rx. And then based on this and based on that means, based on this expression which determine the proposed coefficients coefficient functions u 1 u 2 etcetera and then this gives us 1 solution of the non-homogeneous equation.

And then yi plus yp gives us the complete solution for the non homogeneous equation.

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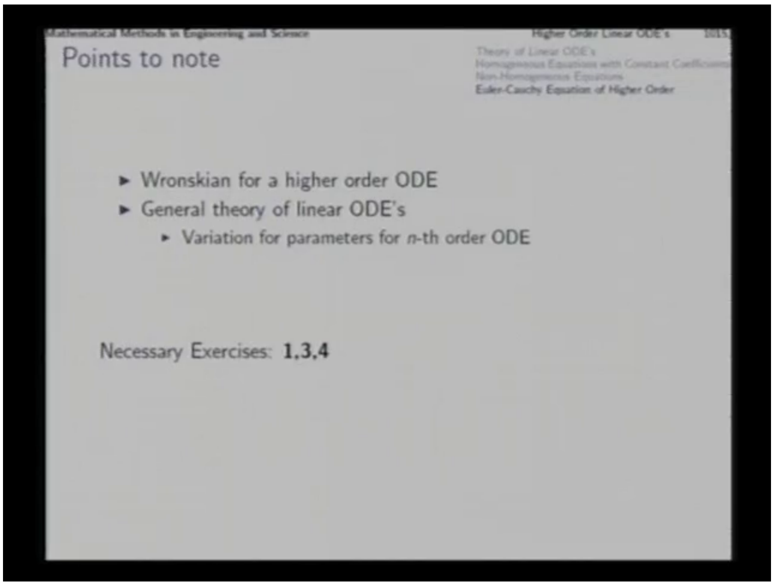

Now many of these issues that is the Wronskian for the higher order ODE or the basis for describing the solutions of the homogenous equation, and then the way these basis members can be used to find a particular solution of the general non homogeneous equation; all these will find similar analogous situations when we discuss the solution of systems of ordinary differential equations in the next lecture. But currently in the remaining time of this lecture we will take a small digression into another topic which is in a way complementary to this classical framework of solving linear ordinary differential equation, and that is the paradigm of Laplace transform method.

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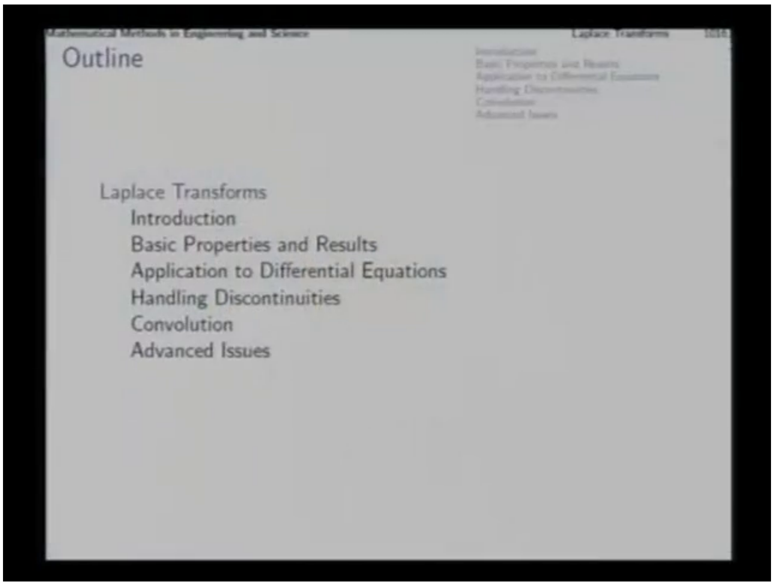

Till know in the classical framework of solution of ordinary differential equations we have typically discussed with the understanding that the entire differential equation is known in advance.

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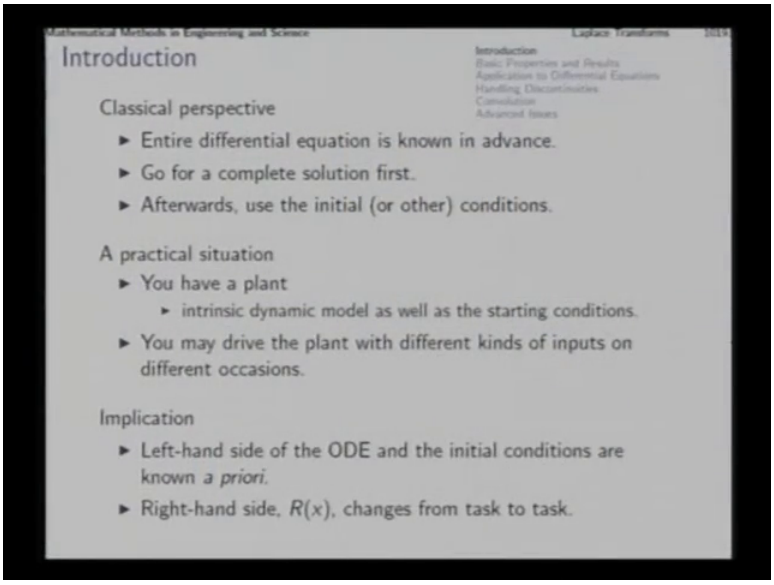

That is both the left hand side which has terms related to y and its derivative and the right hand side Rx. So, the entire differential equation is known in advance, this is the typical understanding with which till now we have developed the methods and the typical way to look for the solution in the previous few lectures including the current one has been to go for a complete solution first. And after getting the complete solution then if the conditions are available initial or boundary conditions then we can impose those conditions to find out the arbitrary constants arising out of the solution process.

Now this is the typical classical perspective; however, there can be practical situations where you already have a plant, you already have a physical system which means that you know the left hand side of this differential equation completely. Now that particular system that particular plant can be operated with different kinds of inputs which means this R x can vary from one application to another that can be a practical situation.

So, you have the plant or system means that its intrinsic dynamic model you know and for that matter you know the starting condition, that is from which initial condition for this initial position, for this initial first rate second rate up to n minus 1 th rate. So, all those things are known. Now as you decide to plug in different kinds of inputs you will be basically changing the right hand side so; that means, that as you drive the plant with different kinds of inputs on different occasions, you will be changing the right hand side without changing the left hand side of the differential equation and without changing the initial conditions.

So; that means, that entire differential equation is not known in advance only the left side of it is known the input function side is not known in advance that can change from one application to another. On the other hand the point of using the initial conditions afterwards does not sound so nice because we know the initial conditions beforehand. So, in such a situation another way of solution which takes care of the left side of the differential equation and the initial conditions first in one framework, and depending upon the right hand side the last amount of last part of the word can be accomplished differently in different situations that is a paradigm shift.

So, with the left hand side of the ODE known, an initial conditions are known a priori. So, right hand side R x changes from task to task. So, Laplace transform methods give you a method which takes care of this part completely and keeps the job half done waiting for the right hand side to appear at any time. So, as a different right hand side appears every time the solution can be changed by another little amount of additional work. Apart from that there is another question that may be asked which is answered properly and adequately by Laplace transform methods and not so, well by the methods that we have been discussing till now.

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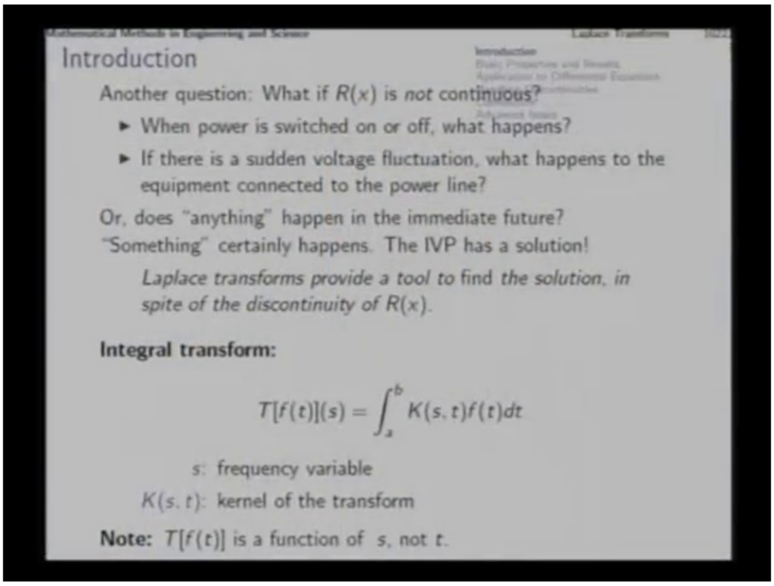

And that is suppose this R x is not continuous, till now we have been relying upon the existence and uniqueness results which use the notion that all these coefficient functions and R x are continuous and bounded. Now suppose the input function with which we try to drive the plant try to drive the system is not continuous. So, for example, when power is suddenly switched off or switched on the if the power if the plug is giving you the input function R x and as it is suddenly switched on or suddenly switched off, there is a discontinuity at R x say x is representing time here ok.

So, as power is switched on or off what happens? What happens is the question that we asked when we posed an initial value problem. So, this initial value problem is it well posed. So, we are asking this question that what happens, what is the future evolution of y of t y of x in this context. Now when we ask is this question is this initial value problem well posed then we say that does a solution to the differential equation exist or does the solution is the solution unique and so on. Apart from switching on or off if there is a sudden voltage fluctuations.

Suppose the voltage is somehow related to this input function input function of time. So, with time as the voltage fluctuate in a in the appliance in some appliance the model, the dynamic model of which is sitting here then we say that the equipment the appliance which is connected to the power line what happens to its state how does the state evolve with time. Now asking does anything happen in the immediate future that is the mathematical the mathematical equivalent of this question does anything happen in the immediate future is basically the question, that for this initial value problem does there exist a solution.

Now, this question is named because something will certainly happen. So, as something certainly happens we say that this initial value problem certainly has a solution. Now depending on the R x being continuous and bounded may not help to find the solution since at situation. So, Laplace transforms provide a tool to find the solution in spite of the discontinuity of the right hand side function R x through certain ways of handling discontinuity. So, let us quickly have a have an overview of the Laplace transform technique the main salient features of it, before we proceed to the system of ordinary differential equations.

Laplace transform happens to be one particular kind of integral transform of this kind, where there is a there is an integral there is a kernel function and ft function for which we are looking for the transform. So, this resulting function t of ft is actually not a function of t because integration with respect to t has been carried over carried out to get that transform function. The transform function will have the other variable s which is typically called the frequency variable.

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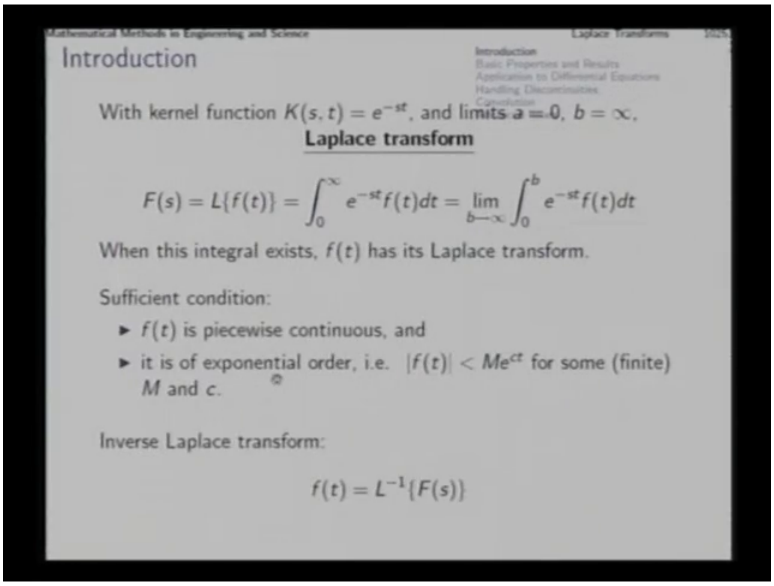

So, with the kernel function like this and limits of integration from 0 to infinity we have what is called the Laplace transform defined by this formula.

In which case this integral this improper integral is evaluated in this now under certain conditions the a function ft has a Laplace transform and the corresponding inverse function is called the Inverse Laplace transform of a function of the frequency variable fs. So, if fs is the Laplace transform of ft then ft is called the inverse Laplace transform of fs. Now with some background work people have developed some long tables of Laplace transforms and inverse Laplace transforms which we can keep as reference.

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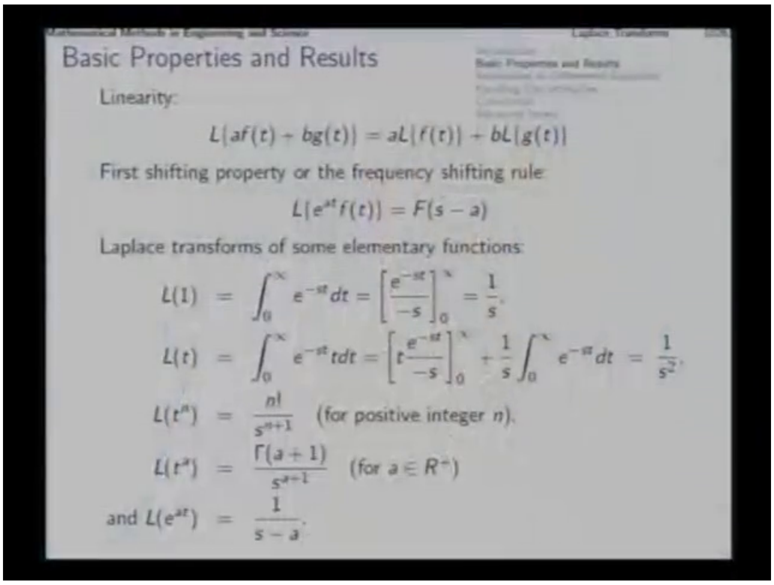

So, right now I am omitting the details of this.

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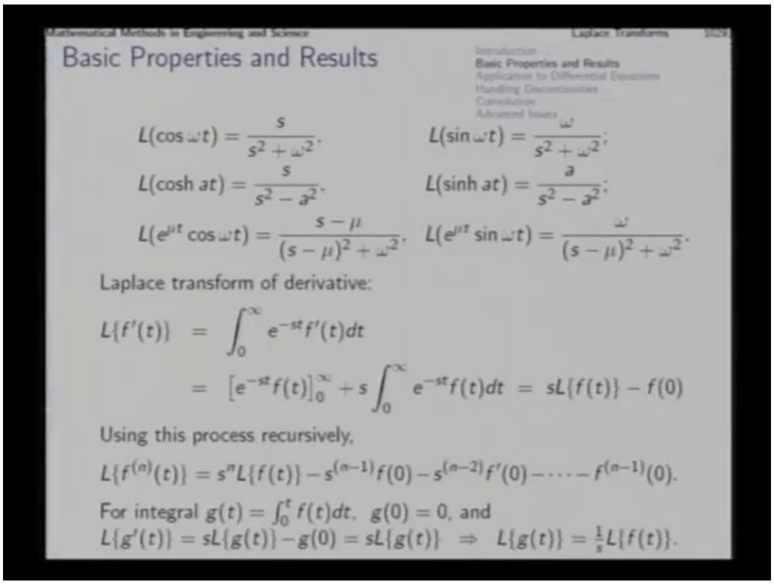

And going directly to the typical methodology of solution solving differential equations with the help of the Laplace transform method.

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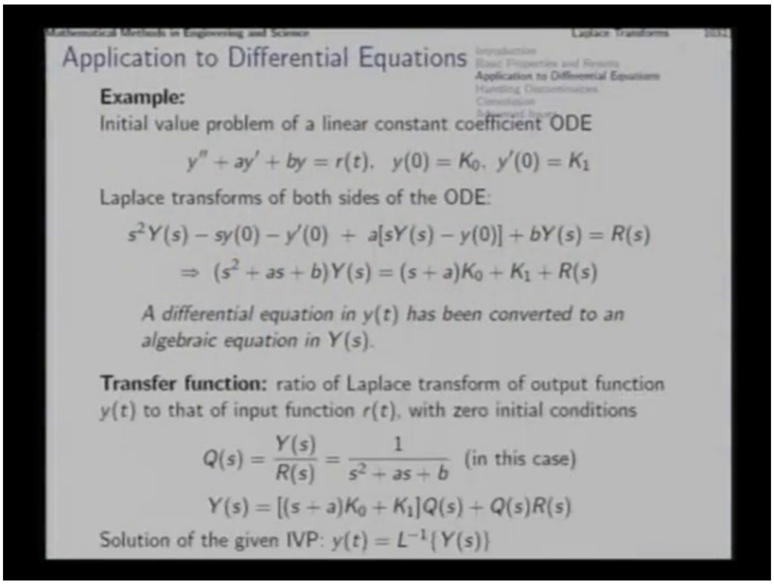

So, suppose this is the linear differential equation with constant coefficients and these are the initial conditions, second order differential equation the 2 initial conditions will be needed. Now if we take the Laplace transform of both sides of this using the rules for Laplace transforms of derivatives and in that how initial conditions appear etcetera etcetera we will get the Laplace transform of both sides of this equation in this manner. S square plus as plus b ys where ys is the Laplace transform of the unknown function y this is what we want to determine and on this side we will have s plus aK 0 plus K 1 plus Rs. Now, up to this point we can do except for this Rs even without knowledge of this right hand function Rt. The important point to note here is that this differential equation has been broken down to an algebraic equation and from here we can solve for ys in terms of the other quantity.

So, this whole thing divided by ys will be the solution that is rather than the solution it will be the Laplace transform of the solution. So, even without knowing the solution yt we know the Laplace transform of the Ys. So, then the next step will be after knowing Rs also if we know the Rs also, then this whole thing together that is the inverse of it this is Qs. So, that is why Qs has been put here. So, then as we solve for Ys from this algebraic equation of in capital Ys then we have the Laplace transform of Yt. So, its inverse Laplace transform turns out to be the solution.

So, in this the initial conditions have been involved from the very beginning and inversion of the Laplace transform is conducted at the end to resurrect the solution. Now in this framework of course, we can handle only limited number of plants with constant coefficients and so on easily, but then we can handle discontinuity of Rt through 2 important functions and their Laplace transforms.

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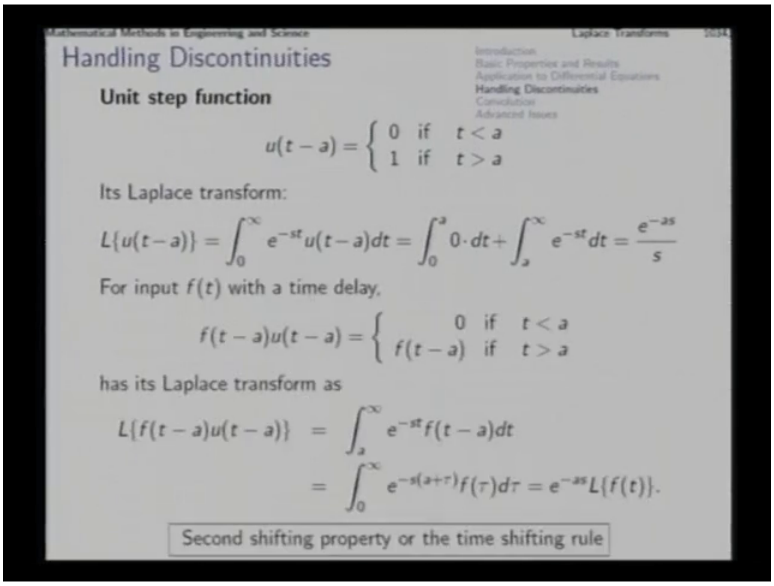

So, this is the unit step function if t is greater than a less than a the value is 0, if t is greater than a then its value is 1 and its Laplace transform has been determined and it is found to be this.

Now, if a function appears with a time delay after time a then this is its effect and for that this can be the detailed expression and for which we can work out the Laplace transform in terms of the Laplace transform of the original function, that is what is the shift in time that is the result of the shift in time. This is one important discontinuous function for which we have the Laplace transforms to be used whenever this kind of an input appears it is suddenly applied and then after application that value is known.

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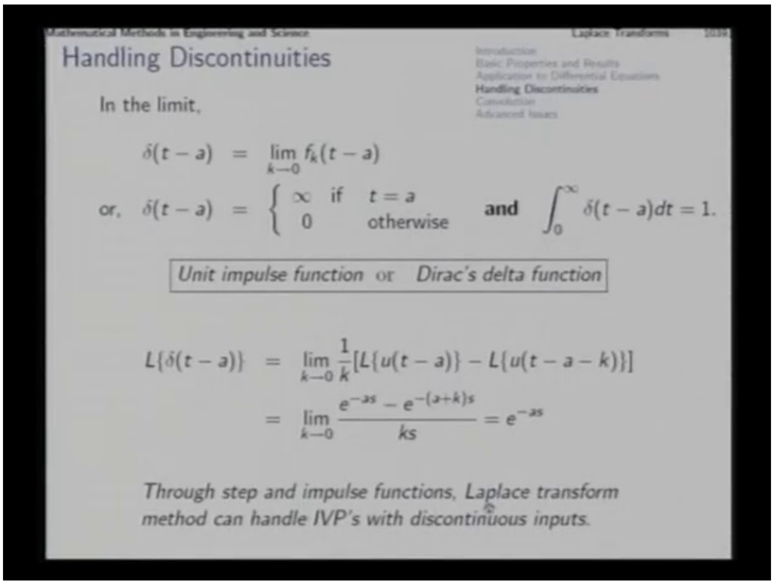

Another important discontinuous function is this Dirac's delta function.

Suddenly there is a huge jump in the value of r. So, and that value is suddenly at that particular moment the value is infinite very large every at other all other time it is 0 and with the special property that under that curve we have under that huge slash we have the area as unity. So, this is the Dirac's delta function or unit impulse function. So, for this also the Laplace transform has been determined and that turns out to be this. Now with the help of these 2 discontinuous functions and their Laplace transforms, we can handle the situations where the right hand side function of the differential equation is discontinuous.

So, through step functions and step function and impulse function Laplace transform method can handle initial value problems with discontinuous inputs as well. Now another important term is there which appears quite often in the discussion of Laplace transform method and that is convolution.

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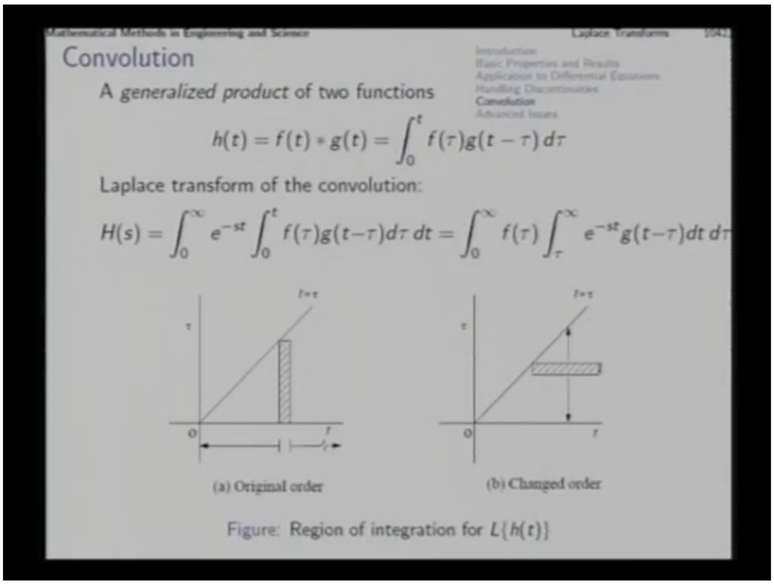

Convolution of 2 functions is actually a kind of a generalized product of 2 functions which is defined in this manner. Now you can show that the Laplace transform of the convolution is actually the product of the corresponding Laplace transforms that is this.

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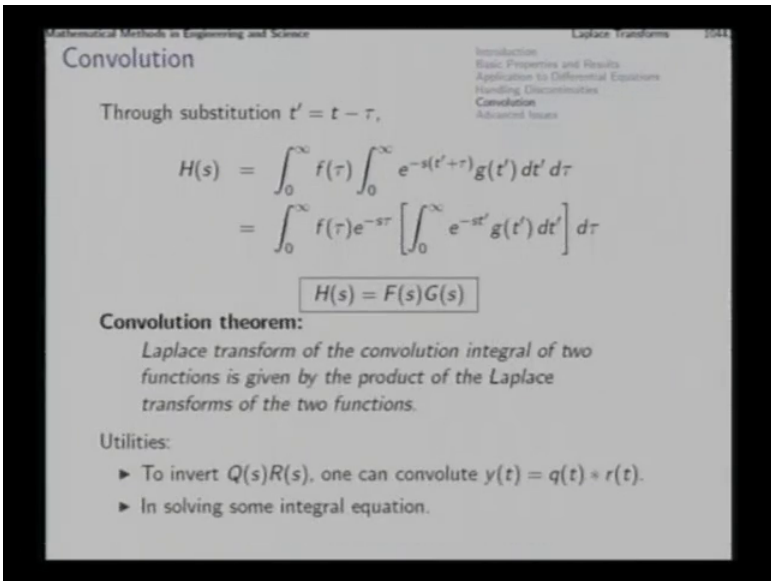

If you have 2 functions F and G and their convolution of is a then the Laplace transform of the convolution turns out to be the product of the 2 individual Laplace transforms this is called the convolution theorem. Laplace transform of the convolution integral of 2 functions it is given by the product of the Laplace transform of the 2 functions.

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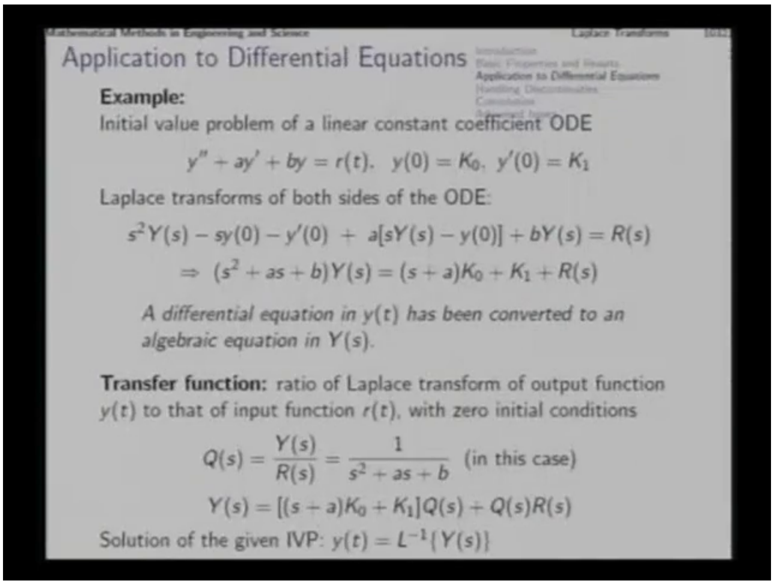

Now quite often what happens is that in this context we know QsRs and we know that the moment we say in this particular case, if the initial conditions were 0 K 0 and K 1 then these 2 terms will be 0 and this whole thing will go up and; that means, after this step with K 0 and K 1 as 0 we can invert this and find out Qs and that is that gives us ys as qsrs.

Now, rather than finding Rs there is after the input function Rt has been specified, then rather than finding its Laplace transform Rs multiplying that with Qs and then taking the inverse Laplace transform one could do well to directly find out the convolution integral of Q t original Qt and Rt, that is because the inverse Laplace transform of QsRs which we want will be the same as the convolution integral of the original functions in time QtRt. So, in that sense in many cases when the Rt changes then rather than waiting for Rt to appear and then compute Rs quite often, from Qs itself the transfer function itself this is called the transfer function.

From the transfer function itself we find out the corresponding Qt through its inverse Laplace transform and keep it, and then the moment a new input function Rt is supplied then we do not go into Laplace transform further. So, rather than going to Laplace transform to find Rs then multiplying this and then taking the inverse Laplace transform, we can simply take the new Rt and the originally earlier determined Qt and construct their convolution through the definition of the convolution integral.

So, this is the another important issue which appears in the analysis of Laplace transform method and this will again appear when we later study Fourier transform. So, with this little discussion on Laplace transform, we continue we will continue in the next lecture to the solution of ordinary differential equation systems. So, ODE systems we will take up in our next lecture and from there we will discuss the stability of dynamic systems which will be solving through these methods.

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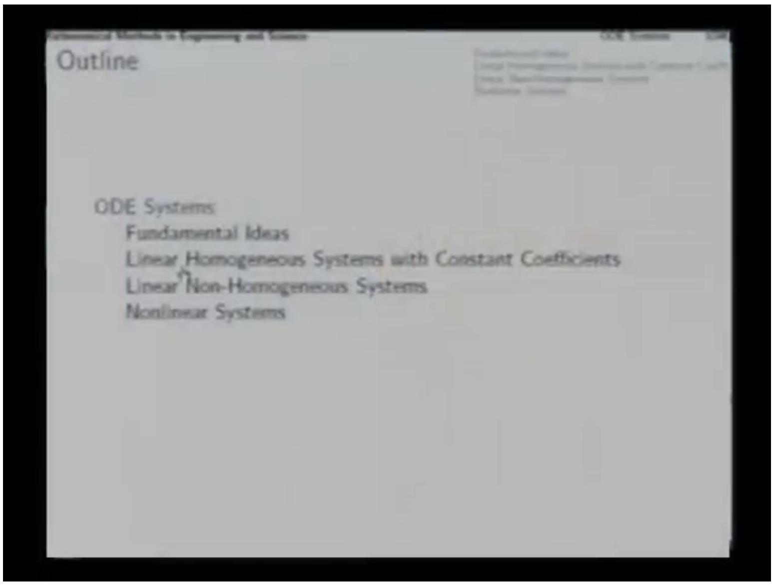

Thank you.