Mathematical Methods in Engineering and Science
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Module – VI
Ordinary Differential Equations
Lecture – 02
Linear Second Order ODEs

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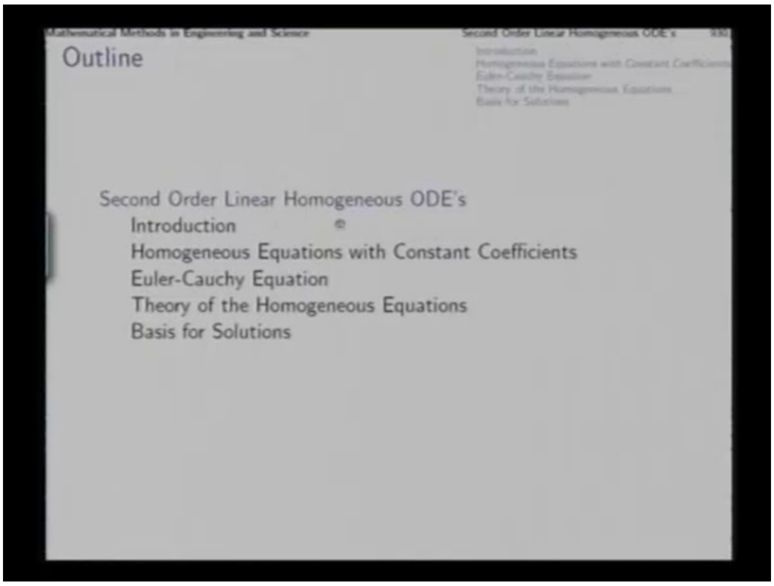

Good morning. So, in this lecture we start with second order linear differential equation for the homogeneous cases and then we will go to the non homogeneous cases.

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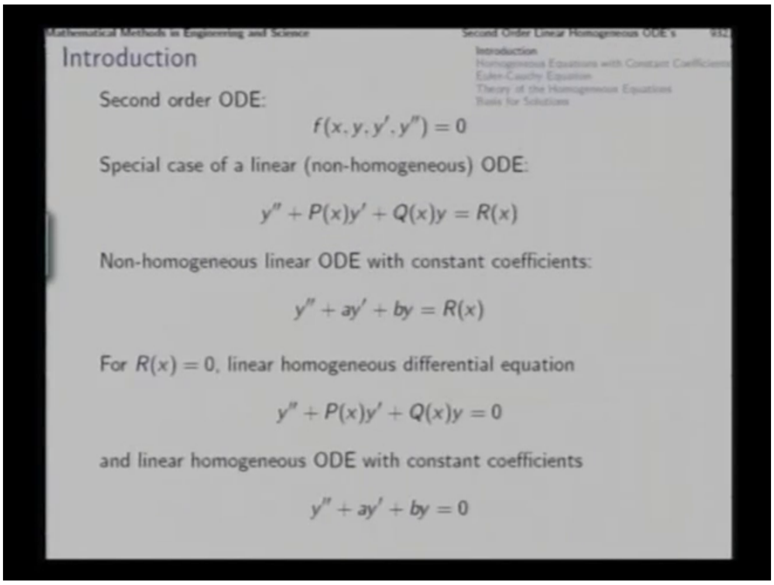

As outline in the previous lecture, we start with the simplest case which is this homogeneous differential equation with constant coefficient and then we will consider the homogeneous differential equation with variable coefficient and then one by one we will consider these mode difficult cases. So, first the simplest differential equation of second order.

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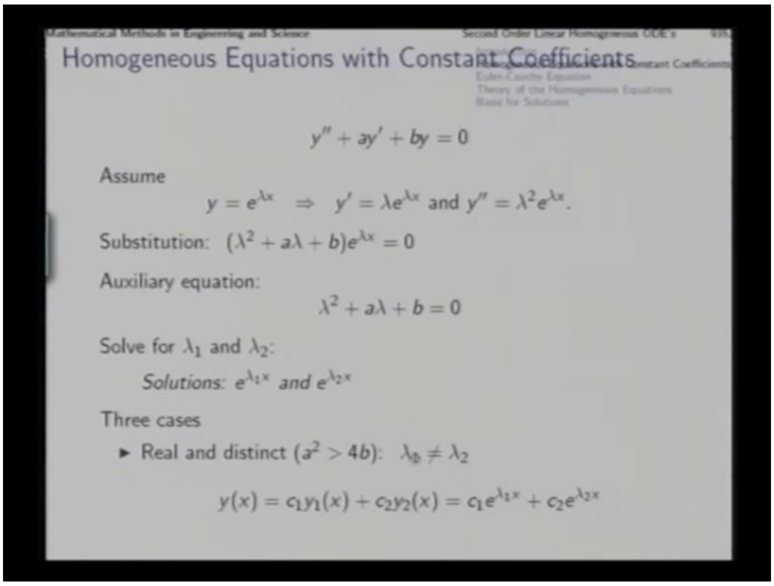

So, for this differential equation, we try to find the solution and the search is for a function y which up on differentiation produces such functions, which in an additive manner can cancel one another. So, we look for the type of function which up on differentiation will produce its own kind so that a sum of such functions can vanish together.

So, what kind of a function we are talking about? Exponential function fits the description, because the derivative of e to the power x is e to the power x itself. Derivative of e to the power k x will be k in to that is constant in to e to the power k itself. So, then the function itself and its derivatives with certain coefficients here can be added together to produce the same kind of function and as the total if the total coefficient can be made to vanish then as a sum we can get 0. So, we know what kind of solution we will expect. So, we assume y equal to e to the power lambda x and then simply differentiate it twice. So, the first derivative we get as lambda in to e to the power lambda x, and the second derivative as lambda square in to e to the power lambda x

Now, these 3 expression if we insert in this, then we get this equations and now we say that since this part cannot be 0. So, for this equation to be satisfied we must have this equal to 0 and this equation is called the auxiliary equation of the differential equation. So, from here directly we can say that if we are looking for this coefficient lambda in the exponent, then from here we write lambda square from here we write a lambda plus b and that equal to 0 gives us the auxiliary equation for this differential equation and since this is a quadratic equation. So, we will expect 2 routes from here that is 2 solutions of this quadratic equation and that is very easy.

So, we solve for the 2 solutions of this quadratic equation let us call them lambda 1 and lambda 2 and then putting those values in turn here we will get 2 solutions and we expected that because it is a second order differential equation. So, e to the power lambda 1 x and e to the power lambda 2 x are the 2 solutions that will satisfy this and now if e to the power lambda 1 x satisfies then any multiply will also satisfy similarly for this. However, a quadratic equation can yield 3 types of solution and according to that there will be certain variations in the form of the solution that we will get. First case is the real and distinct solutions that is when lambda 1 and lambda 2 are both real, but they are unequal that is the case when the discriminant of this quadratic equation is positive that is when a square is greater than 4 b.

Then will have lambda 1 and lambda 2, 2 distinct real solutions and in that case these 2 will be linearly independent and these 2 in a linear combination will provide us the complete solution that is this. Now if the 2 routes of this quadratic polynomial turns out to be same that is real and equal then e to the power lambda x, e to the power lambda x they will be the same solution repeated and therefore, they will not be 2 linearly independent solutions it will be the same solution. So, therefore, we cannot linearly combine like this to get the general solution or the complete solution.

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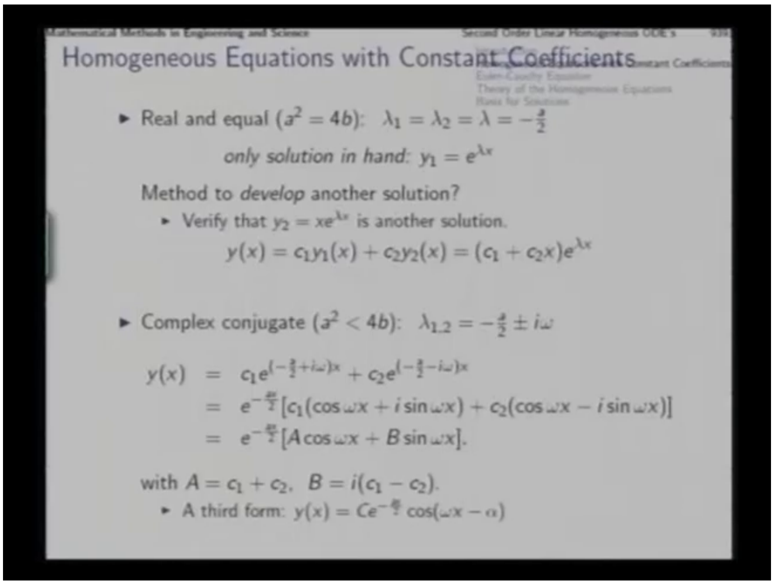

So, what to do with that, that is in this case if a square is equal to 4 b that is a discriminate is 0 then we get both routes same that is simply if here a square is equal to 4 b, then what do we have here is lambda square plus root over 4 b lambda plus b; that means, we will get the common value of lambda as minus a by 2 right. So, in that case will have both the values of same as minus a by 2 and the only solution in hand is this. So, we need a method to develop another solution. Towards the later part of this lesson of this lecture we will see a methodical way to find that second solution when we have one solution in hand, but currently let us simply verify that x into e to the power lambda x is another solution of the same differential equation.

So, let us try to insert y 2 that is x into e to the power lambda x into this equation, while lambda is minus a by 2 and a square is equal to 4 b in this particular case.

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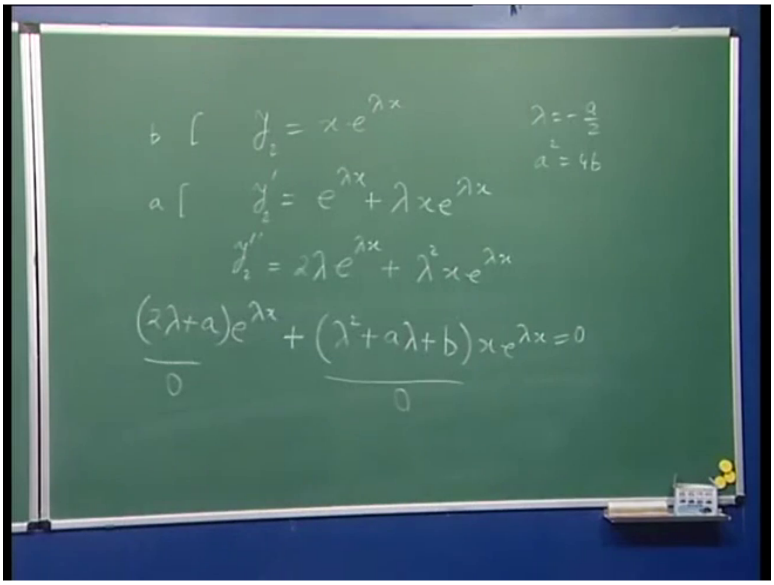

So, we try to insert this solution into the differential equation and see whereas, x satisfies the differential equation. This and then the second derivative to be the derivative of this, will get lambda from here and another lambda from here right. So, here the derivative of this has been included and the derivative of this the first part of it in which this is differentiated is included and the second one in which this part is differentiated that will give us this. Now, we will multiply this with a and this with b and add up right. So, e to the power lambda x terms will get from these 2 phases and x into e to the power lambda x terms will get from here here and here. So, let us put all of them together. So, this plus a times this plus now x, e to the power lambda x with that we will get lambda square from here a times this and b x time b times this right.

Now, we have already seen that lambda is a solution of this; that means, lambda square plus a lambda plus b is 0. So, this is 0 and in this particular case lambda term happens to be minus a by 2. So, twice lambda plus a this is also 0 so; that means, these 2 terms independently vanish and therefore, the sum is 0 that means, this sum which we have constructed the left hand side of the differential equation that is satisfied. So, here we just verify that this is another solution and obviously, this solution x into e to the power lambda x is linearly independent from e to the power lambda x because the ratio of the 2 solutions y 1 and y 2 is not constant it is x. So, then like this we can construct 2 linearly independent solutions and then we can get the general solution as a linear combination of these 2 linearly independent solutions this is this.

The third possible case of the solutions of that quadratic equation is when the discriminant is negative and a square is less than 4 b, then we get the 2 root as complex conjugate like this minus a by 2 plus minus i omega right and in that case these are certainly the corresponding solutions e to the power minus a by 2 plus i omega x, and e to the power minus i omega a by 2 minus i omega x they are certainly linearly independent; however, this is not a very nice form of writing this solution.

So, we reorganize the solution a little bit; e to the power minus a by 2 x we take outside and then inside we will have c 1 into e to the power i omega x which is this and on this we will have the c 2 into e to the power minus i omega x which is this and then we club together cosine term which we get as c 1 plus c 2 let us call it a and we club together the sin terms which will be i into c 1 minus c 2 let us call that b then this becomes the more nice looking form more elegant form of the same solution in terms of 2 new constants A and B. There is a third form also which is quiet useful in many situations that is in terms of the phase. So, what we can say is that rather than having a here and b here if we say that let us call a square plus b square under root as c then we can make the sin cos substitution here and call A equal to c cos alpha and B equal to c sin alpha then getting c outside we can put together cos omega x cos alpha plus sin omega x sin alpha which is a. So, this is a third form of the same solution which is also quiet often found useful. So, in this in all the 3 cases we can find the complete solution for this differential equation.

Now, apart from this differential equation this one, there is another differential equation linear, but not with constant coefficients currently we are discussing the solutions of homogenous equations with constant coefficient. But then there is a particular kind of differential equation which is not with constant coefficients, but in many aspect it resembles this differential equation and that is the famous Euler Cauchy differential equation.

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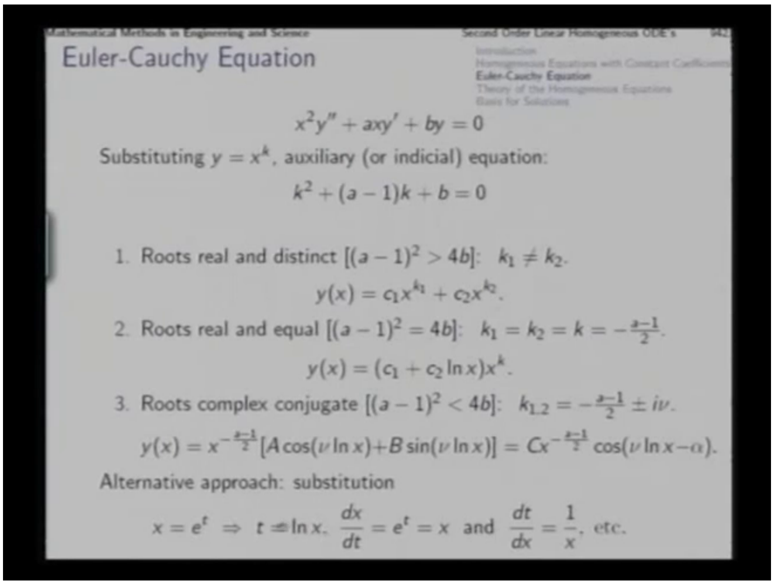

In which the coefficient of y double prime is x square, coefficient of y prime is x and coefficient of y is 1 that is apart from constant coefficient a and b. So, constant coefficients can come there and apart from that the variable part of the coefficient has whatever is the power of that for the coefficient of y there is one more power in y prime and another more power in y double prime and so on. So, in this particular case in order to have a sum of these terms in such a manner, that together they can vanish that is one can compensate for the other we will need all of them to be same type of functions and that means, that we should have a function here with upon differentiation, gives another function which is one degree less and that such that when that multiplied with x will produce something which will be similar to y. So, that it can be together added similarly here.

Another differentiation should involve another degree less. So, such that another multiplication with it will again produce similar thing. We know that f to the power k is a function like that which as many time as you differentiate it degree will go on reducing. So, if we substitute a function of this kind in the case of Euler Cauchy equation, then very quickly a similar auxiliary equation we can develop, in that other case it was lambda square plus a lambda plus b. But in this case it will be k square plus k minus 1, k plus b that can be verified easily by just 2 differentiation and substitutions in the same manner as we handle the previous equation. Any way then we get an auxiliary equation or initial equation and we know that those values of k which satisfy this equation if inserted here, and then x to the power k is taken then it will satisfy this differential equation.

So, our first job is to solve this quadratic equation and again similar to the last case we get 3 cases. If the roots are real and distinct then we get 2 different values of k 1 k 2 both real immediately and we put x k x to the power k 1 and x to the power k 2 and this gives the general solution. On the other hand if roots are real and equal then this will be this will give us one solution as x to the power k, but the other solution other linearly independent solution similarly we can develop a logarithm of x into x to the power k th that will give the second linearly independent solution. I would advice that this particular case we should verify the way we verified the case in the previous example previous differential equation.

In the case of root being complex conjugate you have k 1 and k 2 which are like this and again in this case also rather than having the solutions in terms of the complex number, we can club together at the cosine terms and the sin terms and develop the solution in this manner. The real part of this k is taken out side and the imaginary part is put inside. so that it can be clubbed together in terms of cosines and sins and again through sin cos substitution another form can be obtained in the with the help of the frame. So, this entire set of solutions can also be obtained very easily by making a substitution of the independent variable rather than x being here, if we insert x equal to e to the power c that is a very elegant approach because if we insert x equal to e to the power t, in which case t is log of x and then d x by d t turns out to be x itself and d t by d x is 1 by x and all this substitutions with further derivatives, if we insert here then we will find that this differential equation which is in terms of x can get reduced to a differential equation in terms of t.

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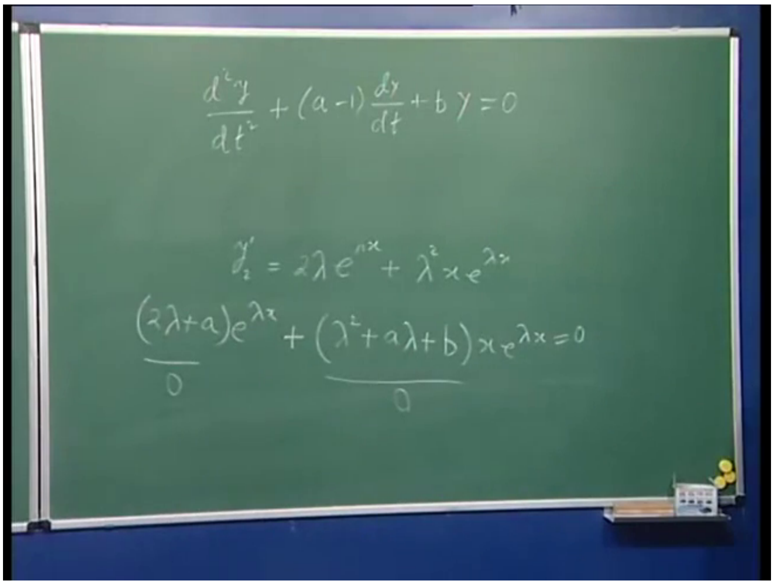

And that differential equation will be this and therefore, this particular differential equation Euler Cauchy equation has a very close relationship with the differential equation is constant coefficient which we have studied earlier ok.

Now, this much for the case of homogenous differential equations with constant coefficients and its immediate cosine which is the Euler Cauchy equation; now let us go back and see what we plan to do after that for this the entire complete solution is quiet simple. Now our next issue is to solve a similar differential equation with variable coefficients that is where coefficient, coefficient part functions of x. Now note that all through this discussions we will consider these coefficient function P x and Q x and in this case R x also as function which are continuous and bounded and therefore, we will get the advantage of the existence and uniqueness theorems, which we discuss in an earlier lecture.

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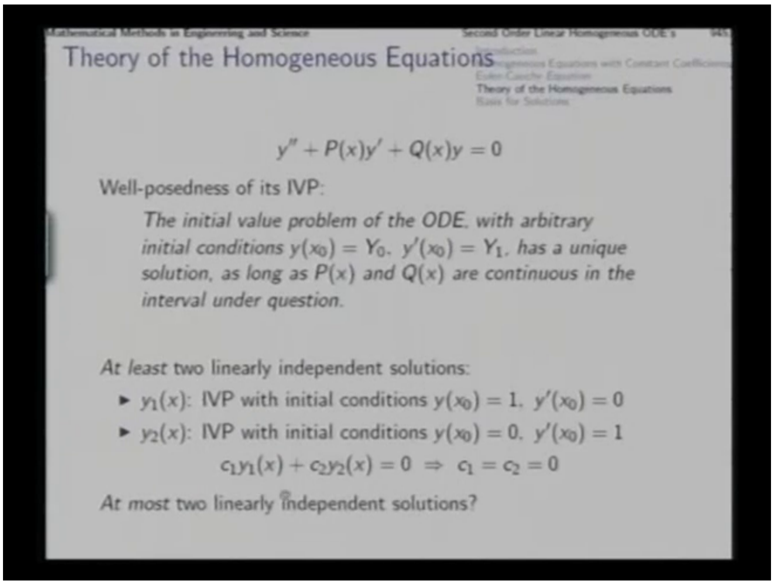

So, now we consider the fundamental theory of homogenous equations, that is the second order homogenous equation we take like this and first we use the Well-posedness of its IVP. Now we already establish that a linear differential equation with coefficient function which are continuous and bounded is well posed with an arbitrary set of initial conditions. So, the initial value problem of this ordinary differential equation with arbitrary initial condition, any position and any field any y and any y prime at the initial point x 0 has a unique solution, and it depends continuously on the initial condition as long as P x and Q x are continuous and bounded also in the interval in which the solution is being studied. So, this result this particular set we will use in establishing some of the results in a quiet state forward manner.

Now, one issue can be very easily noticed that at least 2 linearly independent solutions we can see very clearly. In one case we consider this initial condition that is at x 0 y is one and y prime is 0 and let us call that solution as y 1 that is that solution of this differential equation which has value one at x 0 and rate 0 at x 0. Another solution we can consider as that solution of this differential equation with initial condition y at x equa x 0 is 0 and y prime is 1. So, these are certainly 2 linearly independent solution because if you consider a linear combination of this to vanish that is this plus this equal to 0, then just simply by putting one of the value x 0 we find that y 2 of x 0 x 0. So, this goes to b this goes to 0 and y 1 of x 0 is 1. So, here we get only c 1 the right side left side is c 1 which is 0.

Similarly if we differentiate it and then insert the initial condition x 0 then we find that the c 2 coefficient is also 0; that means, a linear combination of these 2 solutions being 0 necessarily means that 2 contributions are independently 0 and that is the definition of linear independency. So, if you can think of 2 solutions of this differential equation which have these initial conditions, we can see very easily that the 2 solutions are linearly independent, that shows that at least 2 linearly independent solutions of this we can find. Now, we said at least 2 linearly independent solutions which we can see very clearly. Can we say that there will be at most 2 linearly independent solutions also that is other than these 2 we can find no other solution which will be linearly independent to both of them answer is yes, we can say also this that is 2 and exactly 2 linearly independent solutions.

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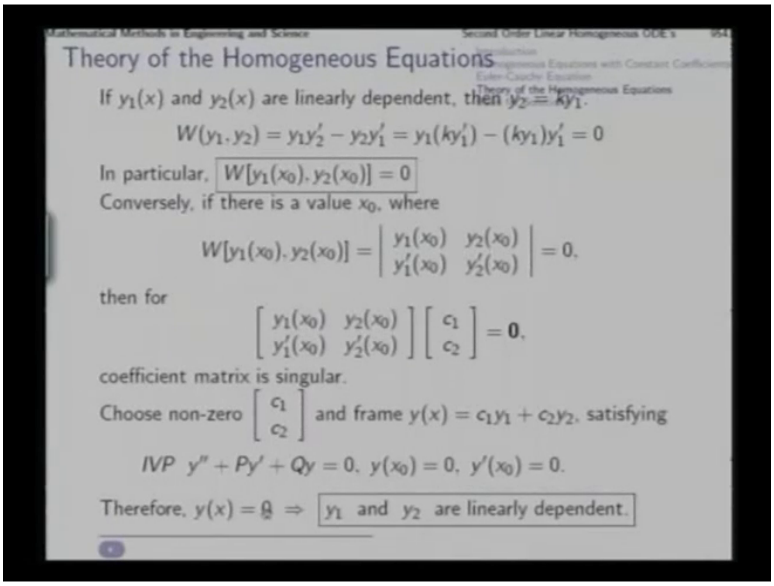

We can find out for this differential equation and to establish that let us consider one or 2 important point first is the definition of Wronskian. Wronskian function of 2 solutions y 1 and y 2 two functions is defined in this manner the determinant of this 2 by 2 matrix y 1 y 2, y 1 prime y 2 prime that will be y 1 y 2 prime minus y 2 y 1 prime this is defined as the Wronskian of these 2 solutions.

Now, the important result is that 2 solutions y 1 and y 2 are linearly dependent if and only if, there is some value x 0 where the Wronskian vanishing. Now if we want to establish this result then we need to establish 2 point. One is that if the Wronskian vanishes at some point then they are nearly dependent and other is that if they are nearly dependent then the Wronskian will vanish. So, let us first consider that if the 2 solutions are linearly dependent, then there is some point where the Wronskian will vanish. So, in order to establish that we take this y 1 and y 2 and consider them to be linearly dependent 2 functions linearly dependent means that one will be k times the other that is the 2 should be proportional if we take that then y 2 prime will be k y prime.

So, in the expression for the Wronskian we simply insert y 2 as k y 1 and y 2 prime as k y 1 prime and we find that k will be common and we will have y 1 y 1 prime minus y 1 y one prime which is 0. So, that shows that the Wronskian is 0 everywhere, in particular at some point x 0 it will be 0. So, forward part of the result we have found very easily that is if y 1 and y 2 are linearly dependent then the Wronskian vanishes everywhere not only at x 0. So, in particular at x 0 it vanishes now we want to show the converse that is if at some value x 0 the Wronskian vanishes, then the 2 solutions are linearly dependent that is why if there is a value x 0 where the Wronskian vanishes like this right.

Till now we have not shown that it vanishes everywhere, we have just in the converse proof that is we have taken some value x 0 where the Wronskian is vanishes that is the premise. Now, if this is 0 this determinant is 0; that means, the corresponding matrix is singular that is this matrix now if this matrix is singular then it will have a null space, which means that there can be non zero vector c 1 c 2 which will be in the null space of this matrix that is which will be the solution of this linear system homogenous linear system of equations.

Now, we choose such a non zero vector c 1 c 2 which is a solution of this that is we choose a vector in a null space of this matrix. Since this matrix is singular we will always have a null space. So, we choose a vector in that null space a non zero vector in the null space and construct this function with these c 1 c 2 values. Now, we claim that this function so constructed with these values c 1 and c 2 satisfies this initial value problem. The same differential equation and 0 initial condition, you can see very easily that this is true because y 1 is a differ is a solution of differential equation. So, c 1 y 1 is also a solution similarly c 2 y 2 and when insert this you will have c 1 into y 1 double prime plus P y 1 prime plus Q y 1 which will vanish, thus c 2 into y 2 double prime plus c y 2 prime plus Q y 2 which will vanish. So, this differ this function satisfies the differential equation.

So, far as satisfying the initial condition is concerned, you can see that you evaluate this at x 0 and you get c 1 y 1 at x 0 plus c 2 y 2 at x 0 that is 0 that is immediately satisfied because c 1 c 2 is a solution of this system of linear equations. So, c 1 y 1 at x 0 plus c 2 y 2 at x 0 equal to 0 that is the first row of this vector equation; similarly the second row of this vector equation. So, c 1 y 1 prime plus c 2 y 2 prime at x 0 is 0 tells us that this function also satisfies this second condition. Now, you will see that this function satisfies the differential equation and both the initial conditions; that means, that this is a solution of this initial value problem. Till now we are saying this is a solution of this initial value problem.

But remember that in a previous lecture we established here that a linear differential equation with continuous and bounded coefficient function has unique solution with any arbitrary set of initial condition right; that means IVP of this differential equation part any initial condition any set of initial conditions is unique so; that means, that this is a solution of this IVP means this is the unique solution of this IVP, but then we can also see that y equal to 0 is certainly a solution of this y equal to 0 satisfies this differential equation trivially this initial condition trivially and derivative equal to 0 trivially. So, y equal to 0 satisfies this initial condition a initial value problem trivially and just now we have seen that this is the unique solution, how can that be that can be possible only in one way in which this solution is nothing, but 0 because this is the unique solution of this IVP and we can see that y equal to 0 is certainly a solution of this.

So, that this can be the unique solution if it is that same solution y equal to 0 so; that means, this solution that we have. So, constructed is actually equal to 0 that shows what that shows that we can find 2 numbers c 1 and c 2 not both 0, such that this vector is a non zero vector such that this sum vanishes without c 1 and c 2 being both 0; that means, that the 2 functions y 1 and y 2 are linearly dependent. Now we have shown the converse also; that means, if there is a value at x ze a value x 0 when the Wronskian vanishes, then that will imply that the 2 solutions y 1 and y 2 are linearly dependent, and that will mean from here you see that the Wronskian vanishes everywhere. In a circular manner that will mean that if the Wronskian vanishes at vanishes at some point then it will vanish everywhere.

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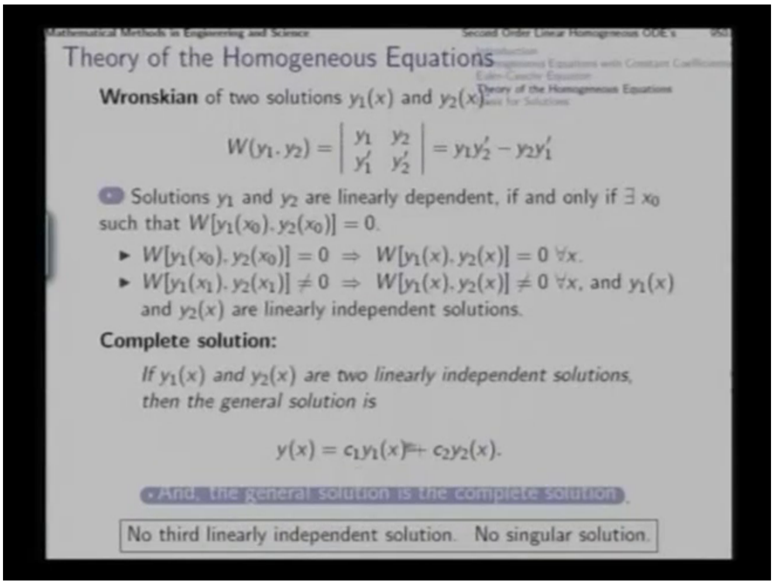

So, as a consequence of this we find that if the Wronskian is 0, then not only it means that the 2 solutions are linearly independent linearly dependent it also means that Wronskian vanish everywhere for all x.

And now, if the Wronskian is found to be non zero at some point for a situation then that also will mean that it will be non zero everywhere because if somewhere else it is 0 that will imply that it is 0 everywhere which will contradict with this; that means, at one point if you find a Wronskian to be non zero the from there you can directly claim that it will be non zero always and therefore, the Wronskian function will never change sin. If it is positive then it will remain positive for all values of x if it is negative then it will remain negative everywhere because due to continuity from for going from positive to negative or vice versa it has to cross 0 which it cannot do ok.

So, we find that if the Wronskian can be shown to be non zero at one point immediately we can conclude that it is non zero everywhere and y 1 and y 2 are linearly independent solution and now the general solution you will get in that case as a linear combination of the 2, and we can say that this general solution is also the complete solution what does it mean? It means that no third solution is possible which is linearly independent to both of this and; that means, is the complete solution that is any solution called the differential equation that you can find can be certainly put in the form of a linear combination of these 2, which means that there is no singular solution for the linear ODE.

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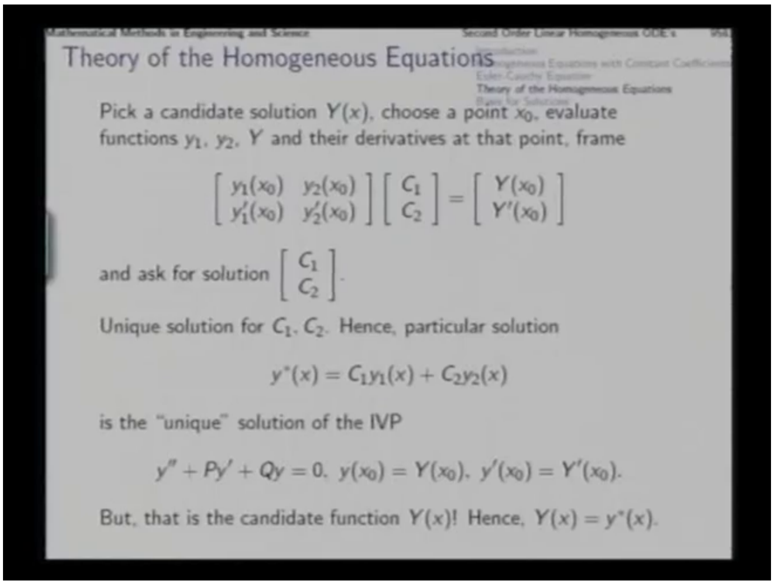

If we want to show that then what we will do? We will pick a solution candidate solution suppose capital Y is a solution of the differential equation and then we will try to put it in the form a linear combination of y 1 and y 2 which are 2 linearly independent solutions. If we succeed; that means, that for any solution of the differential equation we can always put it in this manner. So, what we do is that for this y of x we choose a point x 0 and evaluate the 2 basis members 2 solutions y 1 and y 2 which we found earlier and this new solution also. For all these 3 solutions we find a value at x 0 and the values of their derivative also at that point and then the values of y 1 y 2 we put here values of that derivative we put here and the y and y prime values at the same point we put here and then we construct this linear system of equations and ask for the values of c 1 and c 2.

Now, see y 1 and y 2 are 2 linearly independent solutions so; that means, there Wronskian is non zero, which means that the determinant of this is non zero which means this is a non singular matrix if this is a non singular matrix. Then when we ask for values of c 1 c 2 satisfying this we get a unique solution that is unique values of c 1 and c 2 we get now as we get that unique values of c 1 and c 2 and then we construct with the help of this c 1 and c 2 this particular solution y 1 and y 2 are linearly independent solutions of this differential equation.

So, now with the help of these coefficients which we found from the solution of this we develop these particular solutions y star, and then we know that this y star satisfies this differential equation and this y star must satisfy these 2 initial conditions also why so? Because c 1 into y 1 c 1 y 1 plus c 2 y 2 at x 0 that is the first equation in this system, c 1 y 1 plus c 2 y 2 at x 0 that is equal to this the solution of this system of equations is c 1 c 2. So, this one is evaluated this function when evaluated at x 0 gives us the left side of this first equation which is equal to this. So, it satisfies this, similarly from the second second line second row of this equation we find that this condition also is satisfied; that means, this function y star that we have constructed out of the solution of this linear system of equations, that satisfies this initial value problem that satisfies this differential equation and that satisfies this initial conditions; that means, it is a solution of the initial value problem.

Now, again based on the uniqueness theorem we know that if this is a solution of this initial value problem then it is the unique solution of the initial value problem, but then the way we define capital Y and its derivative, capital Y itself the original candidate solution itself is also a solution of this; that means, this solution that we have constructed with the help of this c 1 and c 2 happens to be exactly same as capital Y, because of the uniqueness of this solution. So, that is a candidate function itself so; that means, that candidate function which we picked up some where we started turns out to be expressible as a linear combination of the 2 solutions y 1 and y 2 which we started with and that shows that there is no third solution possible which is linearly independent of y 1 and y 2.

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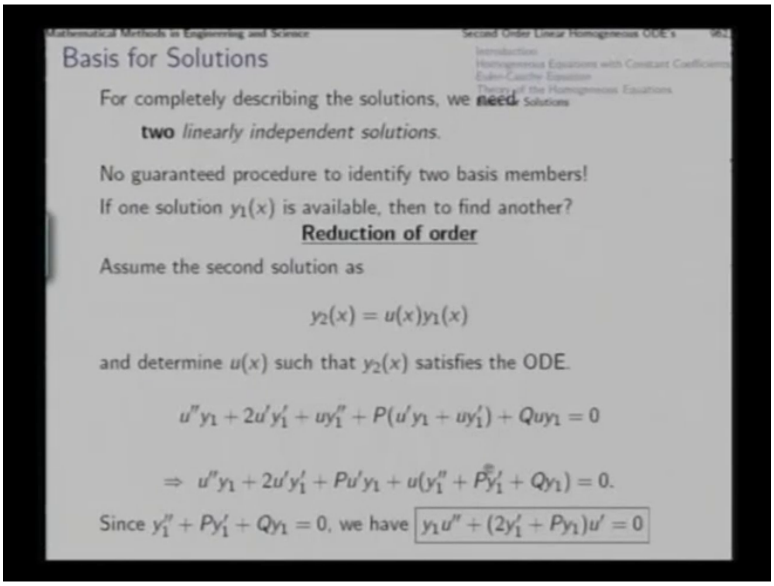

So, that is why we say that for completely describing the solutions of the second order differential equation, we need 2 and 2 only linearly independent solutions of it, and that gives us the complete solution. However, based on the differential equation itself if we want to find out 2 linearly independent solution then there is no guaranteed procedure to identify 2 such solutions analytically in general, and that is a something block. There is however, a way to find a second solution if we have already in hand one solution. That is if we have one solution in hand that is if y 1 is available which is already known to satisfy the differential equation and we want to find another solution say y 2 which is which is linearly independent of y 1 then there is a method to do that and that is called reduction of order.

That means if we want to solve a general second order differential equation homogenous differential equation, then in the most difficult case we may not be able to identify analytically 2 linearly independent solutions; however, if some somewhere through some consideration we can identify one solution then onwards we can completely solve the problem; that means, we can find out a second solution which is linearly independent to this and we can combine the 2 in a linear manner c 1 y 1 plus c 2 y 2 and get the complete solution. The way we do that is the method called reduction of order; now suppose y 1 is a solution of the differential equation y double prime plus P y prime plus Q y equal to 0 and we want to find second solution which is linearly independent of the (Refer Time: 36:50).

So, we assume the second solution as u x into y 1. Now as long as u x is variable depends on x this will turn out to be linearly independent of y 1. Now then taking u y u y 1 as a second solution we force this to satisfy the differential equation. So, as we insert its second derivative first derivative and the function itself in the differential equation. So, the second derivative of this is u double prime y 1 plus twice u prime y 1 plus u y 1 double prime this is y 2 double prime plus p y prime y prime y 2 prime. So, u prime y 1 plus u y 1 prime. So, this is first derivative plus Q into y 2. So, this is the result of substitution of this y 2 into the differential equation.

Now, here we club together this term, this term and this term, in all these 3 terms you find u is common. So, take this term here then this term here and this term here u is taken outside rather 3 terms that is this is here this is here and from here this term is here. Now y 1 is already available as a solution of the differential equation. So, this is 0 and therefore, we have this equal to 0, there is this part turns out to be 0 and now you see that here in this differential equation in terms of u, u double prime is appearing u prime is appearing u itself is not appearing because all the terms containing u have together vanished then what we can do is that this u prime we can call as something else let us call it as capital U and then we find that here we have y 1 into capital U prime plus this thing into capital U.

Now, if you divide it with y 1, then here you will get capital U prime plus this expression divided by y 1 into capital U.

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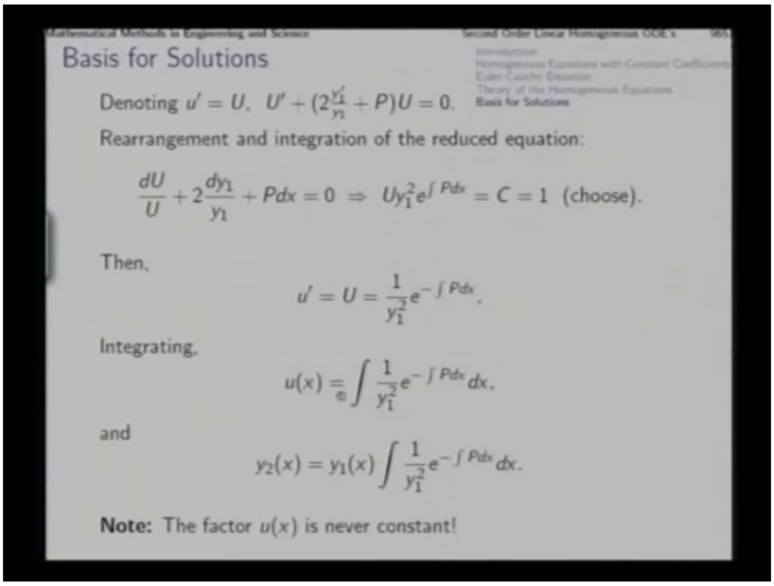

So, you get this, this is the differential equation that we get in capital U which is actually small u prime derivative of small u. Now this is the reduction of order now this is a first order differential equation and as we rearrange this that is as we call this u prime as d u by d x then multiply over all with d x divided over all with u then we get d u by u here which is this and plus d 2 2 plus twice d y 1 by d x. So, multiplication with d x will take away that d x. So, they have d y 1 by y 1 plus P into d x this u has gone here now you get this.

Now, we can see that as we integrate this differential this will be l and u this will be twice l and y 1 plus this will be simply integral PD x equal to constant or taking exponential all over this sum will get converted to product and therefore, will get U into y 1 square into e to the power integral p x P d x. So, this whole thing will be constant whatever constant we choose we reach the same final result. So, we can choose as one and as we do that we can expression we can get an expression of U in terms of everything else that is we take all this things and divided here. So, this capital U turns out to be one by y 1 square into e to the power this negative because it is going on the other side. So, this is capital U which is actually u prime. So, we simply integrate it.

So, the integral gives us u x right the coefficient function which needs to be multiplied with u one to get the second solution y 2. So, this is the way u into y 1 that gives us the second independent solution second linearly independent solution of the same differential equation and this is linearly independent, because u x is not constant and see that u x the factor u x can never be constant because for it to be constant this integral must be 0 and that will not be possible because the way the solutions we are constructing y 1 will be continuous and bounded.

So, this cannot be 0 similarly this p is continuous and bounded. So, its integral will not be minus infinity or something like that. So, infinite infinity and minus infinity this will not be. So, therefore, this cannot be 0 and similarly y 1 cannot be infinite. So, this factor also cannot be 0. So, product cannot be 0 and therefore, its integral cannot be constant. So, therefore, u will be always variable and therefore, y 1 into u will be linearly independent of y 1. So, this way if we have one solution in hand from there we can work out another solution which is linearly independent.

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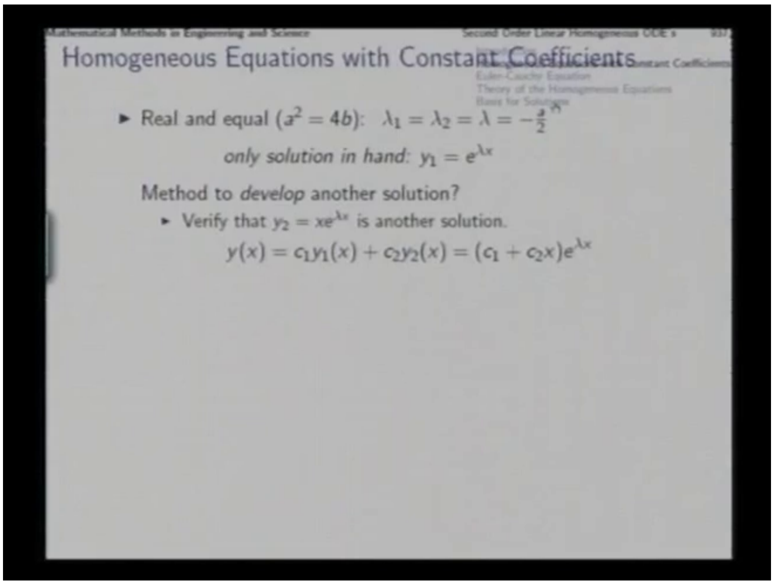

Now, you will remember that in this particular case I told you that currently for the time being let us just verify this. If we wanted to find the second linearly independent solution with this y 1 known, we could not done that with the use of this reduction of order and in that case after the necessary steps, we would get u is equal to x and that would show that x into e to the power lambda x is another solution which is linearly independent of the first solution. So, now, that we have got this.

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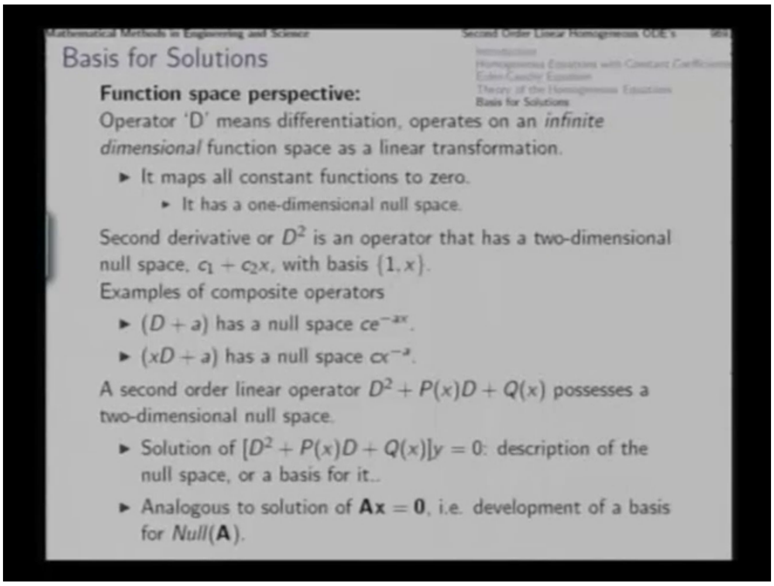

So, let us summarize the findings till this point and see a particular function space perspective of this solution of differential equation. Operator D that is d by d x means differentiation and in the context of function space, it operates on an infinite dimensional function space as a linear transformation. That means, it maps all functions in the infinite dimensional function space to other functions in particular it maps all constant functions to 0 and that one dimensional sub space of the function space is it null space.

The second derivative or D square is another operator that has a 2 dimensional null space that is c 1 plus c 2 x with a basis which is this. So, all linear combinations of 1 and x 1 operator upon by this second order operator, that vanishes; that means, this has a 2 dimensional null space. You can think of composite operators like this D plus a that is D by D x operated over something plus a into that something a is a constant number this again will have a null space like this, x D plus a is another first order operator which will have a null space which is this. Similarly a second order linear operator which is this now this operated over y equal to 0 gives us the second order differential equation which we have been studying till now.

Now, this itself apart from the y part of it is a second order linear operator and it possesses a 2 dimensional null space. So, the differential equation which we have been discussing till now, that the solution of that basically turns out to be the finding of the null space of this second order linear operator or in particular we try to find a basis for it; that means, basis means 2 linearly independent members of that null space and this issue is very analogous to the solution of this homogenous system of linear equations and that is the development of a basis for the null space of a. Now as we have seen that this solution y is a member of the infinite dimensional vector space of continuous functions, here this x the solution is a member of a finite dimensional vector space and apart from that many things are common.

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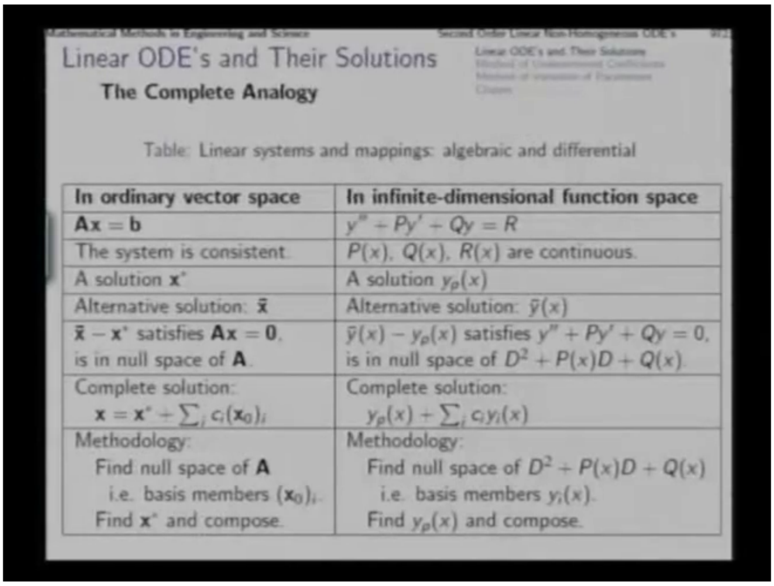

Now, to take the analogy further we consider the right hand side also and complete this analogy, in ordinary vector space a finite dimensions and in the function space of infinite dimensions. This A x equal to b a system of non homogenous equation, the corresponding differential equation is this here the unknown is x which is a vector of finite dimensions, here the unknown is a function y which is a vector of infinite dimension. Now for this saying that system is consistent corresponding statement here is

P x Q x and R x are continuous and bounded functions, now a solution x star here is analogous to a solution a particular solution y p of this differential equation.

For this linear system of equations apart from x star if there is an alternative solution x bar then similarly in this case. So, consider another alternative solution y bar, now the way in the case of ordinary linear systems of equations 2 different particular solutions give a difference which satisfy A x equal to 0 the corresponding homogenous system of equations, similarly here y bar minus y p will be another function which will satisfy this differential equation. The same left hand side, but in the right hand side it is 0 and here it will mean that the corresponding difference is in the null space of the coefficient matrix a and here it will mean that the corresponding difference is in a null space of the linear operator D 2 D square plus c D plus Q.

Now, here also we found the complete solution by adding a complete basis member of the null space to a particular solution same thing we will do here, when we need to solve this non homogenous equation that is one particular solution added to a general member of the null space will give the complete solution of this non homogenous equation. Till now we were solving the homogenous equation, now we get into the non homogenous equation also what is the methodology? To solve this and find the homogenous solutions we find the null space of A that is basis members of the null space and then one particular solution of the differ this system of equations you find and compose in this manner.

Similarly, here we will first find the null space of this, which is the solution of the corresponding complete solution of the corresponding homogenous equation, which we were doing just till now this equal to 0 and then find out one particular solution of this which is y p and compose in this manner. One particular solution of the non homogenous equation added to a general member a general solution of the corresponding homogenous equation gives us a general solution or the complete solution of this non homogenous equation. So, this is the way we will now next take up the problem of solving this differential equation which is non homogenous.

First we will start with the case of constant coefficient that is in this manner we will consider the first case with constant coefficients and see one simple method.

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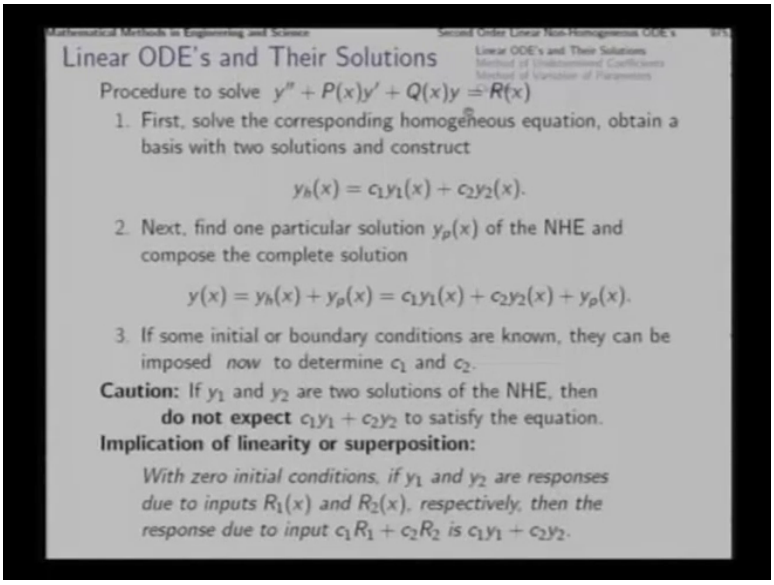

So, what we will do? We will first solve the corresponding homogenous equation in any case that is this equal to 0 and then obtained a basis with 2 solutions and construct this. Now note that this is not a solution of this differential equation, this is the solution of the corresponding homogenous equation and it is shown as y a, this is sometimes called the complimentary function. Next we will find one solution of this differential equation no non homogenous equation and then construct the complete solution of this and finally, if there are some initial or boundary conditions known then those conditions can be now imposed to determine the particular solution with the determination of the values of c 1 and c 2.

Now, as I just now noted that y 1 and y 2 and therefore, this is not a solution of this equation, but they are solutions of the corresponding homogenous equation and therefore, and in the another manner if you find 2 solutions which are solutions of this you do not expect the linear combination of that to satisfy this differential equation as it did in the case of the homogenous equation. Here the impact the implication of linearity of the differential equation is in the sense of super position, that is with 0 initial conditions if y 1 and y 2 are responses for 2 different functions R 1 and R 2 then the response or the solution corresponding to c 1 R 1 plus c 2 R 2 will be c 1 y 1 plus c 2 y 2 in that sense linearity operate on this non homogenous differential equation.

Now, we first consider the case with constant coefficients in which a particular very simple method work and that is the method of undetermined coefficient set.

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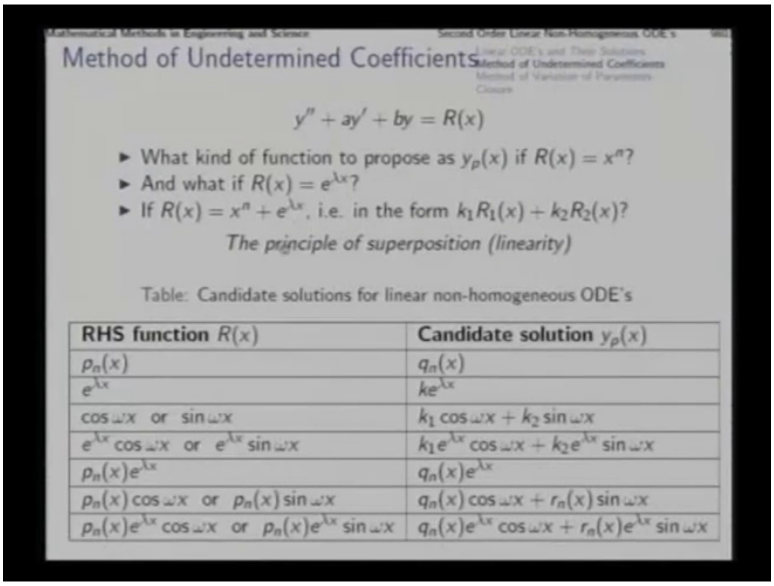

Let us take this non homogenous equation with constant coefficients. Now if R x is a function of if you particular kind and if the coefficients are constant as shown here, then this method of undetermined coefficients will work very easily in that what we do? We choose certain type of functions for y and then substitute to find out the undetermined coefficients. So, for example, let us consider if R x is x to the power n then what kind of y p we should choose which will satisfy this. Now understand that before we take up this question we have already found the complete solution of the corresponding homogenous equation y double prime plus a y prime plus b y equal to 0 for that we have got y 1 and y 2 and therefore, c 1 y 1 plus c 2 y 2 if the complimentary function that is y h solution complete solution of the corresponding homogenous equation that we have already got.

Now, we are looking for a particular solution of this non homogenous equation. If R f is x to the power n then we ask what kind of y p we are looking for now in order to satisfy this x n x to the power n part, something here should give a similar term which will cancel with this in order to satisfy this equation. So, if y y p we take as x n then that can be managed with this part, but then its derivative will have x to the power n minus 1 there is nothing on this side to cancel that. So, what we have to do is that we have to cancel that x n minus 1 and that x n minus 2 that will come here with the help of these

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Now, that will mean that one by one we will end up including x to the power n minus 2 then n minus 3 then n minus 4 till we find that we are at the end x to the power 1 x to the power 0 which is constant and that is it so; that means, that for positive integer value n here x to the power n, whatever is n choosing y p to be a function of k function of type k into x to the power n will not suffice. But an entire n s degree polynomial we have to include and apart from x to the power n if in R x many other terms are also present x to the n minus 2 n minus 4 n minus 3 n minus 1 for all of that you will have the same thing.

Now, similarly if r x is an exponential function, then the similar exponential function e to the power lambda x we can include and that through derivative can cancel this part this type. So, what we will do? For this we will choose k into e to the power lambda x and try to fit the function here and then equate both sides to determine the value of k. Now if we have a sum like this appearing here then we use the super position that is k 1 into R 1plus k 2 into R 2 if that is here, then we will choose the function in the form k 1 into this plus k 2 into this fine. Now here we use the principle of super position; to summarize if the right hand side function is the polynomial of degree n then the candidate solution also should be chosen as a polynomial of degree n. Now note that some of the terms missing here will not help us in reducing the term here, we as long as the term x to the power n is present here we had to start with a complete polynomial of an actually and coefficients we have to determine it may be a full co polynomial as a result with as all the term.

If we have the RHS function as e to the power lambda x then we choose this as the candidate function candidate solution and try to find out the value of k. If the right hand side function is cosine or sin or a linear combination of that 2, then in all these cases we have to choose this and determine k 1 and k 2 2 substitution.

Now, even if the right hand side function has only cosine or only sin still here in the candidate solution we must include both; because through differentiations sin and cosine will produce the other kind also. Now if there is a product of these 2 e to the power lambda x cosine omega x or e to the power lambda x sin omega x, now you will notice that e to the power lambda x cos or sin or this all of this are actually same kind of functions because through a quick reference to complex algebra you can convert these into these itself. So, therefore, here also the same will apply if this is involved in R x or this or a combination of both, then we need to include a term like this and further polynomial into e to the power lambda x also will give the same kind of candidate solutions.

Now, all these we can do in the case constant coefficient cases and this particular few candidate RHS functions right hand side functions here also.

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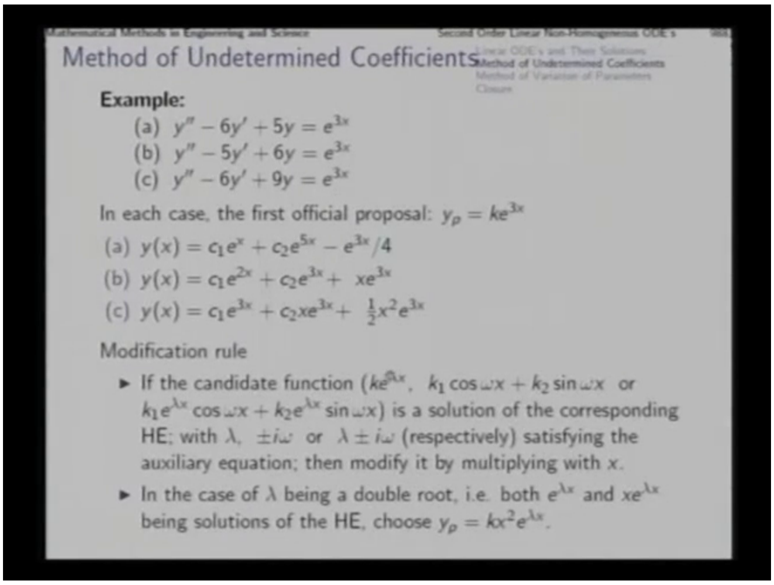

There is a situation in which remain it to modify the rule for example, in this case with the homogenous equation giving solutions e to the power x and e to the power 5 x in the first case, we can choose the candidate function as k into e to the power 3 x and that will give us the solution which is this. On the other hand in this second case there is a problem in the second case the homogenous equation itself have a solution which is e to the power 3 x and therefore, using k into e to the power 3 x as the candidate solution for the non homogenous part will not succeed, because this is already included here and it is evaluated on the left side as 0 there is this fellow when inserted here evaluates to 0.

So, there is no way to satisfy this and in this case the candidate is not k into e to the power 3 x, but k x e to the power 3 x and the value of k turns out to be 1. In this case the homogenous equation itself has not only e to the power 3 x, but x to into e to the power 3 x as well and in that case choosing even this will not help and in this case we try to choose k x square e to the power 3 x and that suffices and the coefficient turns out to be half. So, this is a particular modification rule that is if the candidate function this or this or this is already a solution of the corresponding homogenous equation with this, this or this satisfying the auxiliary equation then we need to modify the candidate solution by multiplying with x. Now if that multiplied with x version along with the original version both are setting as y 1 and y 2 here, then we had to further modify it and take k into x square into e to the power lambda x. Now, this method succeeds with only constant coefficients and with only a selected few right hand side functions R x.

In the next lecture we consider the general method which is the method of variation of parameters which you succeed in all cases. And after that in the next lecture we will continue to generalize the findings still now the discussions still now for second order differential equations in the case of the higher order differential equation without any limit to the order of the differential equation. So, these 2 things we will do in the next lecture.

Thank you.