Mathematical Methods in Engineering and Science
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Module – VI
Ordinary Differential Equations
Lecture – 01
Theory of First Order ODE's

Good morning. With this lecture we start our module on the analytical method of solution of ordinary differential equation. Analytical method are more powerful in the sense that they can give you the solution of differential equation in the form of expressions without having to result to numerical values and that means, that groups of solutions you can represent together and the solution process is also much listed here

However, there is one good aspect and one limitation to the analytical method the limitation is that analytical method are applicable only on certain special classes of differential equations. Beyond that the huge possibilities of differential equations do not lend themselves to successful analytical treatment. However, the good part of it is that most of the differential equations arising in nature in the physics or in engineering can be to a good extend reduced or simplified to those classes of differential equations which can be analytically solved.

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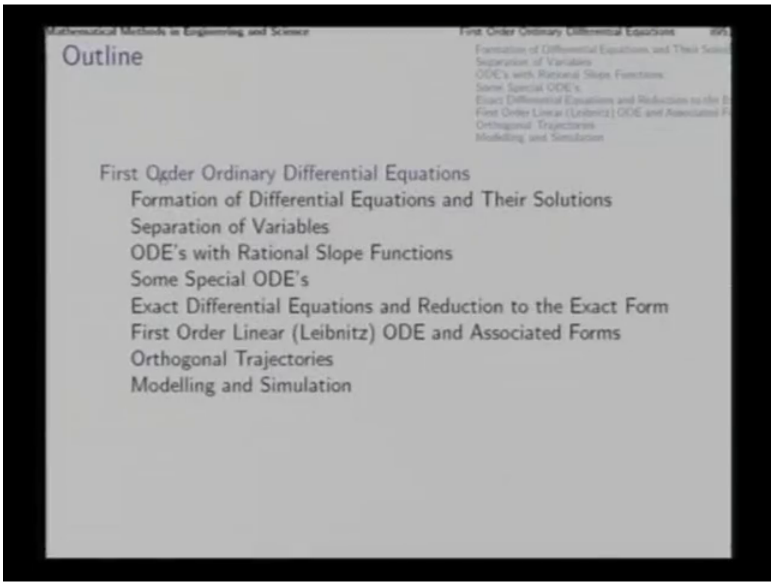

So, as we take up the analytical method of solution of differential equations we approach the problem slowly. First we consider first order ordinary differential equations and then we take up the problem of second order ODEs and beyond that after second order we will generalize whatever aspects of the second order methods can be generalized through higher orders.

So, first we take up the first order ordinary differential equation. Even before going in to the solution method let us see how typically a differential equation is formed and what it in general represents.

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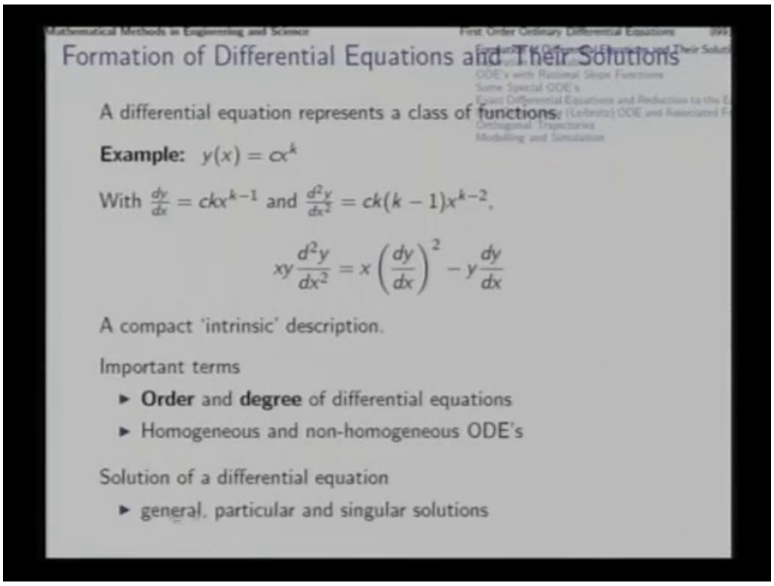

For example, let us take this equation which is an equation in x and y; that means, in x y plane this will represent a curve; however, it will represent a curve with a given value of c and the given value of k. Now, if we are interested in supplying in assigning all sorts of values for these 2 parameters c and k then we will get different curve. And a differential equation between x and y is capable of representing all these curves together independent of c and k and from that representation with the help of a differential equation we will get rid of c and k these parameters and the relationship will be then given by x y and its derivatives. So, that will be in some sense and intrinsic representation of all curve of this side.

So, for that what we do? We differentiate this and get the first derivative which is this and if you notice between this y and this dy by dx you can immediately see that the parameter c can be immediately removed. So, you have got 2 equations y equal to this and dy by dx equal to this. So, between 2 equations you can immediately eliminate one unknown say c. So, if you do that then we will have. So, for that what you will do you will divide dy by dx with y and then in the division on the right side c will get eliminated right.

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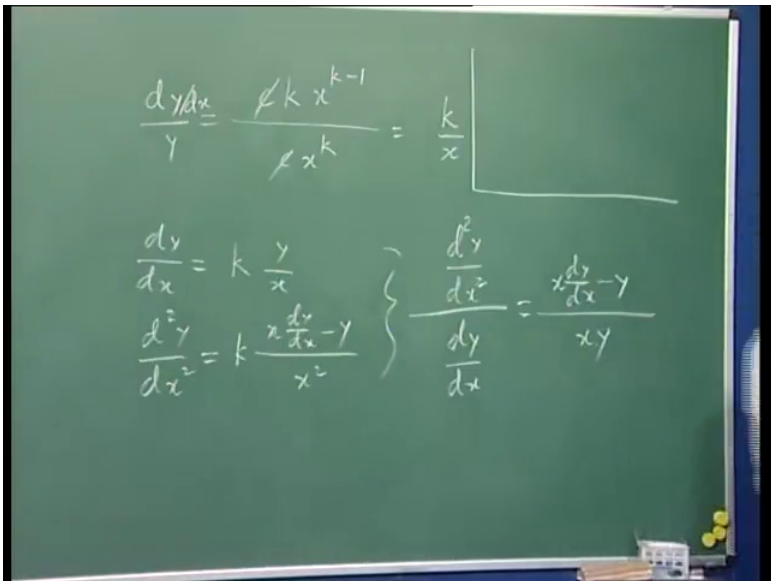

So, you will find dy by dy as dy by dx by y as this divided by this. So, c get cancelled and you have got k by x c gets cancelled and (Refer Time: 04:18) was the powers of x also gets cancelled and you get k by x.

So, from here you find that one of the parameters c has been eliminated, but the other parameter k remains. So, as many parameters you want to remove that many times you need to differentiate. So, before differentiating further what we can do? We can take y on that side and have it in this form. There was another possibility of differentiating it directly and then between these 3 equations we could eliminate 2 unknowns with this c and k which are c and k. Now, as you differentiate this you will find that you get this.

Now, again you will find that dy by dx has k in to something and d 2 y by d 2 x has k in to something else, now if you divide these 2 c has got eliminated already on dividing these 2 you will find that k also gets eliminated. So, from here you will note that dy by dx d 2 y by dx square transferred to be divided by dy by dx transferred to be this divided by this and that will mean this now you cross multiply and find x y in to this will be dy by dx in to this and you have got this differential equation.

So, this gives you a relationship between x y and its own derivative dy by dx and d 2 y by dx square, eliminated are or the variables parameter c and k. Now this is a compact intrinsic description of the curve of the relationship between y and x the other items the other quantities that enter in to this equation are derivatives of the same function y. That means, that this differential equation when you want to solve then basically you will get back this constant c and k. Sometime it happens that while solving you might find some other solution also which are not incorporated here.

Now, you have notice that every differentiation is accompanied with a laws of information. So, that laws of information is supplied at the time of solution of the differential equation with the help of initial or boundary condition. In the numerical solution or differential equation we have seen that with N initial conditions or N boundary conditions and n-th order equation differential equation can be solved completely in principle and that will be available in terms of numbers numerical values. However, if you do not give those initial or boundary conditions if you give only the differential equation that will mean that as many possible differential initial conditions are there that many different forms of solutions you can find. And that is why a solution of a differential equation without any conditions specify will be infinite and in that there will be constants of integration involved which are will be called as arbitrary constants they can be given any value and that will be a valid solutions.

So, we talk of different solutions of the same differential equation that is differential equation satisfying, different solutions satisfying different set of initial or boundary value, condition boundary values.

Now, in the classification of differential equation there are 2 important terms order and degree. Order of the differential equation is the order of the highest order derivative highest derivative that is appearing in the differential equation. For example, in this case the order of this differential equation is 2 because this is the highest order derivative that is involve. And if the differential equation can be organized in the form of a polynomial equation in y and x derivative as this equation certainly is then that degree in which the highest order derivative appears in that that is called the degree of the differential equations.

Now, in a differential equation there could be square root here for example, now square root of the right side appearing here. Also d a differential equation in that case the degree of this would be 2 because to free that square root sign, to free that radical sign you will need to square and both side and this term here the second derivative would appear in degree 2 in that case the degree would be 2. In this case this is the second order equation of degree 1. However, when we try to classify differential equations in the form in the classes of homogeneous and non homogenous ODEs there we count the degree as the total combined degree of y dy by dx d 2 y by dx square and, so, on that is the unknown function and its derivatives.

So, in that sense here you will find that this term has got degree 1 and degree 1 total degree 2, similarly this term has total degree 2, this term also has got total degree 2. The way x is appearing is not counted in that for example, x could have appeared even as exponential sin cosine logarithmic function and so on, the way x is appearing in the equation is not counted in this count of the degree. Now, when we try to classify the equation the differential equation as homogenous and non-homogenous for that we count the total degree of every term. So, in this case total degree of each term transferred to be 2 and therefore, we call this a homogenous equation if there are some terms which are of degree 2

If there are some term which are of degree 3 or 1 then that would be called a non-homogenous equation. In particular a differential equation having the terms with only degree 1 and 0 is called the linear ODEs. So, that is a particular case which will be of great interest to us.

Now, when you try to solve the differential equation there are 3 kinds of solutions that you talk about one is that general solutions. So, general solution of this differential equation for example, would be this that is in which there will be 2 arbitrary constant y 2 because the order of this differential equation is 2; that means, its solution process in one way or the other within was 2 integration and in that there will be 2 constants of integrations involve and they will be the arbitrary constant. So, the general solution is in terms of 2 arbitrary constants in this case. So, the number of arbitrary constants will be same as the order of the ODE.

When we supply when we assign particular values to those arbitrary constants they may get a particular solutions which we were getting during the numerical solution. There are some situations where there is a solution of a differential equation, but it cannot be obtained by giving particular values to the arbitrary constants nevertheless it transfer to be a solution in the sense that the function and its derivatives together satisfy this that is a single solution, one solution without involving any arbitrary constant which satisfy this and which cannot be found from the general solution by assigning special values. That kind of a solution is called a singular solution which is possible only in the case of some non-linear equation non-linear differential equations and that is not possible in linear equations as we will see further.

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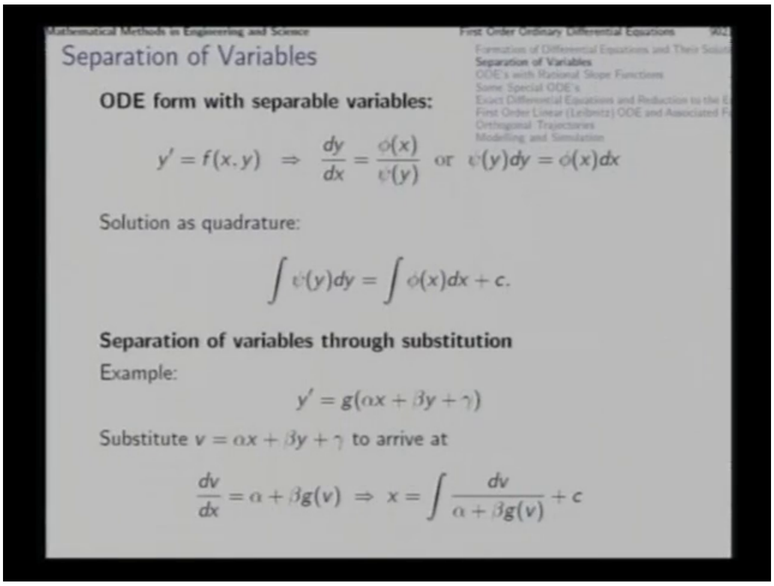

Now, let us go in to the solution processes of firs order ordinary differential equations the simplest possibility is when the differential equation is given in such a manner that you can split it in this manner and the dx taken on this side and psi y taken on this side results in to this situation where the variables are completely separated. The left side has only y and its differential in the right side has only x and its differential in that case independently we can integrate the 2 and we provide for the situation that the 2 integrals are allowed to differ by a constant. So, then just it is a state forward quadrature or integration. So, integration of the left side is equal to integration of the right side plus some constant. So, this is the arbitrary constant which will be involved in the solution of the differential equation.

Now, the moment you have got this expression after integration which will be an expression in x y and this arbitrary constant c, 3 of the derivatives you can say that you have solved this differential equation. So, these are simplest possible situation that is where the variables are separable

Sometimes the variables are not already separable the way the equation is given, but they can be separated with some suitable substitution. For example, in this case where the derivative y prime is given as a complicated function of alpha x plus beta y plus gamma then you can say that this entire expression if we substitute as v then what happens - we get dy by dx is equal to g of v right, but then dy by dx 2 variables x and y are involved now here v gets involved. So, what we do? We differentiate this and work out the relationship between v and x. So, as you differentiate this we find dv by dx is alpha plus beta in to dy by dx in the place of that beta dy by dx we put this g v. So, we get dv by dx is equal to alpha plus beta g v.

The moment we get this we find that as we take this entire right side and put it in the denominator on the left side and get dx here then we get dx equal to dv by this. So, with dx no element of v is involved and if dv this entire thing will depend on only v M constant no x. So, then immediately through a quadrature we get the solution.

Now, we say that we have solve the differential equation even though we have not found v as an explicit expression of x, but you have found x as an explicit expression of v, it is as good as finding the other. Not only that if you could find an implicit relationship between x v and c then also we would say that the differential equation has been solved. Now, after finding this relationship between x v and the constant c then 1 can put v equal to this and that will give us the relationship between x and y which we were looking for that is the solution of the differential equations.

Next complicated case is this where the slope function dy by dx is available in the form of a rational function.

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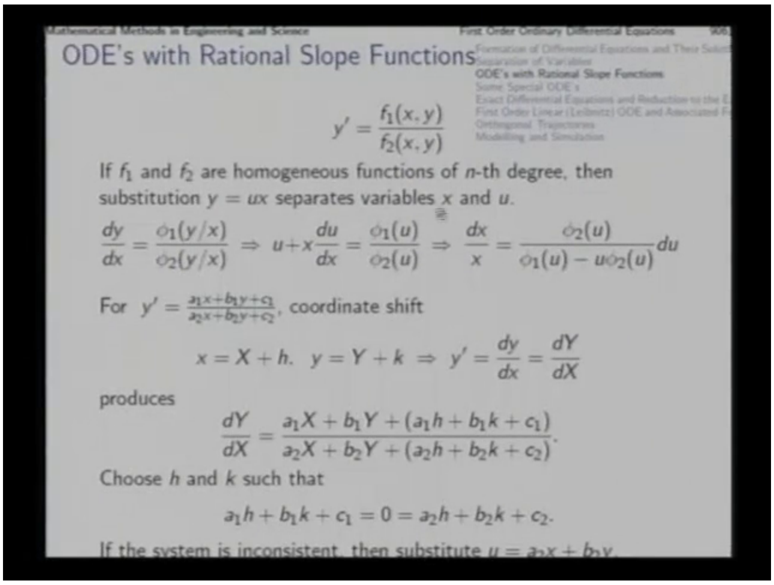

Some polynomial of x y divided by some other polynomial by x y of x y, that is a rational function of x y. So, in such a situation there are several cases that lend themselves to easy solution one is when f 1 and f 2 are homogenous functions of the n-th degree say this is a function and this is another function. So, if this is, if you divide the numerator as well as the denominator by x to the power something then if we can represent the 2 functions after division in this manner that is this is a function purely of y by x; that means, whenever y and x appear in this expression they appear together in the form of a ratio y by x.

And the same thing happens for this case also then the solution is very easy because in the place of y by x we can substitute u and as we substitute that we find that the dy by dx will turn at, turn out to be if you represent u as y by x, Then y will be u x and then the term dy by dx in the differential equation has to be replaced with d u by dx. So, you differentiate it and find out an expression for dy by dx that will turn out to be u in to 1 plus d u by dx in to x.

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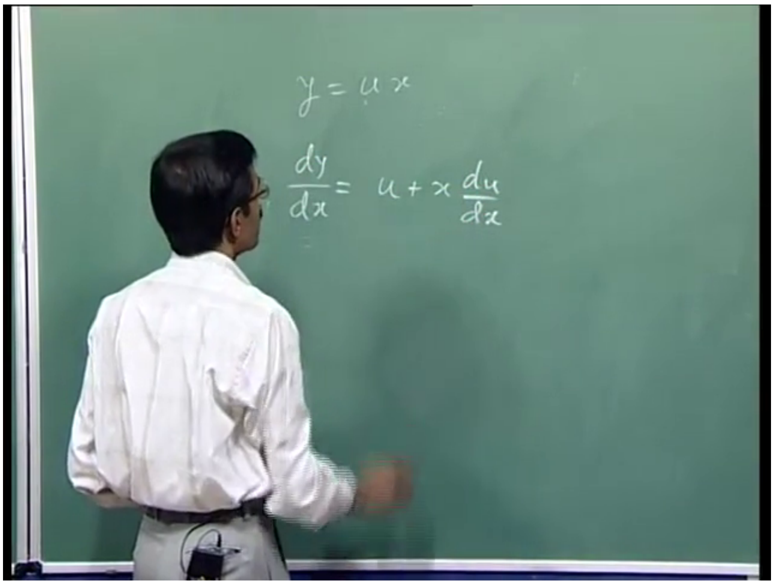

So, this expression for dy by dx we can insert here and that gives us this expression for dy by dx in terms of x u and d u by dx and on this side we have simply phi 1 by phi 2 which are both functions of u only. And now separating this separating variables from this equation is easy, we take u 1 this side this entire side is consisting of u only then and here we have only x. So, dx and x we take on the other side and the entire expression depending on u we take on this side. So, we get dx by x and here we will find d u with that phi 2 will join and whatever comes on the other side that will come here. So, that is phi 1 minus u phi 2 that u come in the denominator. Then we find that this can be separately integrated and this can be separately integrated.

Similarly, even if we cannot represent it in the term of in the form of y y x, but if both numerator and denominator turn out to be linear expressions of x and y, then also we can handle it almost in this manner. We observe that if c 1 and c 2 are missing then we could have done exactly this that is we could have divided both numerator and denominator with x then we would get here a 1 plus b 1 u here you we would get a 2 plus b 2 u. Now c 1 and c 2 are spoiling the game. Then we say we apply a coordinate shift x as capital X plus h and y as capital Y plus k and we choose h and k in such a manner that in terms of capital X and Y the c 1 and c 2 term vanish constant term vanish.

Now, since this are just coordinate shift only shift no scaling etcetera involved. So, therefore, y prime which is dy by dx from here you will get dy as d capital Y from here you will get dx as a d capital X. So, the derivative remains same. So, in the place of dy by dx we can directly insert this d capital Y by d capital X and as we represent x and y as with these 2 expressions we get this, a 1 in to x plus h; a 1 x and a 1 h here. Similarly b 1 in to y plus h, b 1 y here b 1 y plus k, so b 1 y here b 1 k here and c 1 remains. Now, we say that what h and k we should choose. So, that this entire constant term vanishes from both numerator and denominator.

So, as we do that we basically need to solve these 2, this system of 2 d 2 ordinary equations linear equations in 2 unknown h and k. Now if they have unique solution then we choose that h and k and through that substitution this particular case reduces to this situation. On the other hand if this pair of equations for h and k is in consistent then we can say that in that case a 1 by a 2 should be equal to b 1 by b 2 and in that case a choice of u as a 2 x plus b 2 y will make it separable. On the other hand the third possibility that these 2 equations having infinite solutions together will not arise because these 2 being consistent having infinite solutions would require a 1 by a 2 to be equal to b 1 by b 2 and equal to c 1 by c 2. In that case this entire expression and this entire expression will be just multiples of each other in that case the whole situation would not arise because even before that you would have divided and got a constant. So, that situation will not arise. So, if this pair of equation has unique solution then we choose that solution on the other hand if it is inconsistent pair then a 2 x plus b 2 y is something which we substitute as u and then a reduces to a separable form.

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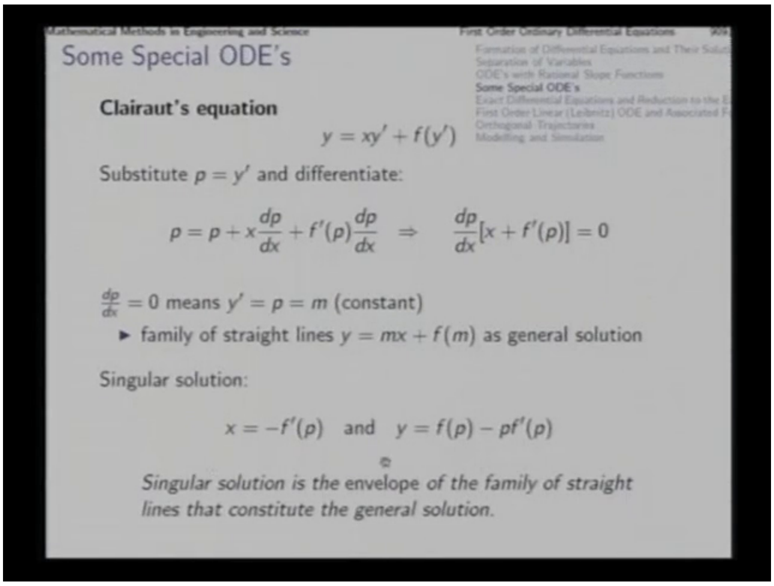

Now, we go to some more difficult situation and a particular kind of non-linear differential equation diff differential equations of the first order is called Clairaut's equation or it is called the Clairaut's form of first order ODEs that appears when we have y as x y prime plus some complicated function of y prime itself. So, this actually this term is making the differential equation non-linear.

Now, in this particular situation what we do we can call the derivative y prime as e and then insert it here x e plus f of e P turns out to be look like a parameter and then we differentiate this differential equation this entire equation both side with respect to f. On this side the derivative of y is P dy by dx here x in to P the derivative of this is P plus f d plus by dx, the derivative of this f P f of P with respect to x will be derivative with respect to P multiplied by dp by dx.

Now, when you organize this we find that these 2 ps cancel each other and we have left with this entire term equal to 0 and in that dp by dx is common. So, we organize it in this method, dp by dx is common we get x plus x prime p. Now, what is this? This is a product of 2 term which is equal to 0; that means, either this must be 0 or this must be 0 by making this equal to 0 we get 1 kind of solution by making this equal to 0 we get another kind of solution. So, first let us consider the simple 1 dp by dx equal to 0. If dp by dx equal to 0 that will mean that P is constant, but P is dy by dx. So, that will mean that y prime is constant m, then you say that if dy by dx is constant that will mean that y is a linear function of x and that you get as the family of straight line which is y equal to the integrate this you just integrate this. So, you find y is equal to M x plus something, but then you already know that from here M is y prime. So, you already know that y is equal to M x plus that something should be f of m. So, this gives you the general solution of this differential equation.

However one particular case rather one singular case that appears here is when this is equal to 0 and if this is equal to 0 then dp by dx does not have to be 0; that means, P does not have to be constant and the equation of the state of the curve need not be a straight line. So, when this is equal to 0 that is the singular solution. Then you get x equal to minus x prime P and y is equal to you already know that this is f P and from here you get x in to e and x is this. So, you get minus P f prime p.

Now, what is this? This is x equal to some function of P y equal to some other function of P this function is known because f of y prime that is a known function. So, this is a known function and this is also a known function. So, when x and y both are known functions of a parameter P that essentially gives you the parametric equation of the curve which is a relationship between x and y. So, P appears as a parameter the slope itself appears at parameter. So, and in this there is no arbitrary constant involved and this solution you cannot find from the general equation because this is the equation of a curve and not a straight line is in equation of a straight line.

So, with different values of M here you get different straight lines which are f family of straight lines which is a family of solutions of this differential equations. Now by giving the value of slope in this as P you do not get different solution, but you get different points on the same solution curve and that solution curve is called the singular solution and it turns out that it is a same slope that is appearing here and here and therefore, through every point that this curve passes a straight line like this passes this curve and this straight line having the same slope at that point that is with that value of M and that value of P here.

So that means, all these lines obtained from different values of M will be all tangent to this curve at their respective points of contact, you find that a singular solution turns out to be the envelope of family of straight lines which are the general solution which can be found from the general solution of the differential equation. So, this is one very typical case which can arise only in the case of non-linear equation.

Now, we slowly make a move towards second order differential equations and at this stage we consider only those second order differential condition equations which can be easily reduced to first order differential equations. In general second order differential equations particularly the non-linear ones are much more complicated to solve compare to first order differential equation analysis sorry.

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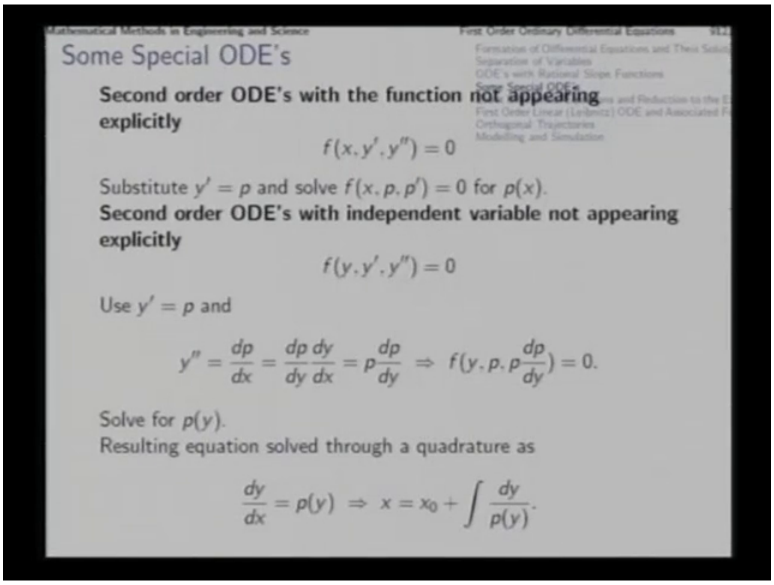

So, there are two distinct classes of second order ODEs which can very easily reduced to first order ODEs one of them is the one in which the function y the dependent variable y does not appear explicitly. It does appear as its in the form of its derivatives, but y itself does not clearly appear, in that kind of a situation it is very easy to reduce or break down this differential equation in to two first order differential equations in place of y prime if we write P then here we get P here we get P prime and the differential equation transferred to be this, this is a first order differential equation in p. So, we can solve it by the first order method.

After solution of that we can say now this expression is equal to dy by dx that will be another first order differential equation which will be then solved. Now, at every stage we will get on arbitrary constant. So, finally, there will be two arbitrary constant accumulated through the two stages. There is another class of second order ODEs in which y appears explicitly, but x itself does not appear y, y prime and y double prime in this case also we use y prime as p, but with a little difference what we do we take y double prime as dp by d x, but then x gets involved which was not here. So, we were happy. So, the now we need to get rid of this x from here. So, we say dp by dx is dp by dy in to dy by dx and dy by dx itself is p. So, we find y double prime as P dp by dy and when we insert this expression for y double prime here then we get this now this is a first order differential equation between P and y.

As you solve this we get the solution which is P of y and now which we can then solve. So, then dy by dx is equal to P of y will give us now P of y goes down dx comes up and we integrate it and it is variable separable. So, we get this as the final solution. When these two classes of second order ODEs you can apply a suitable substitution to break it down in to two first order ODEs which can be solved in turn in succession.

Now, the general theory of first ordered linear differential equation.

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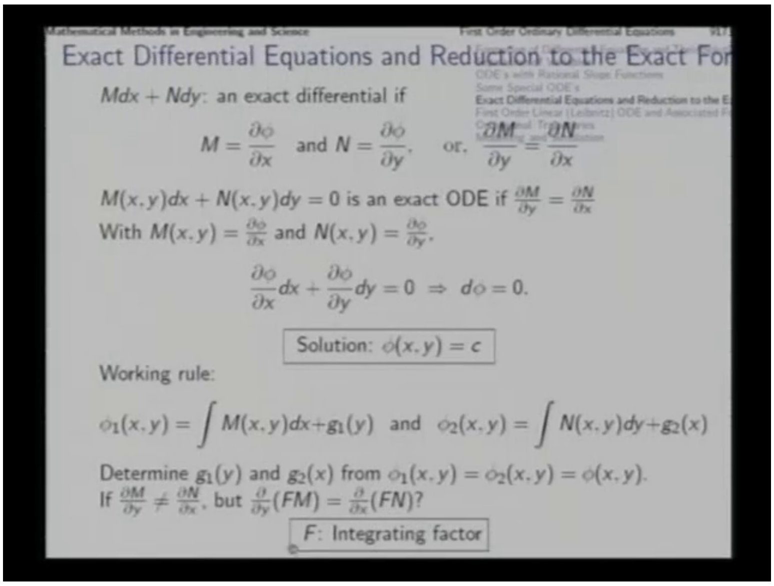

Before that we have a little discussion on exact differential equations.

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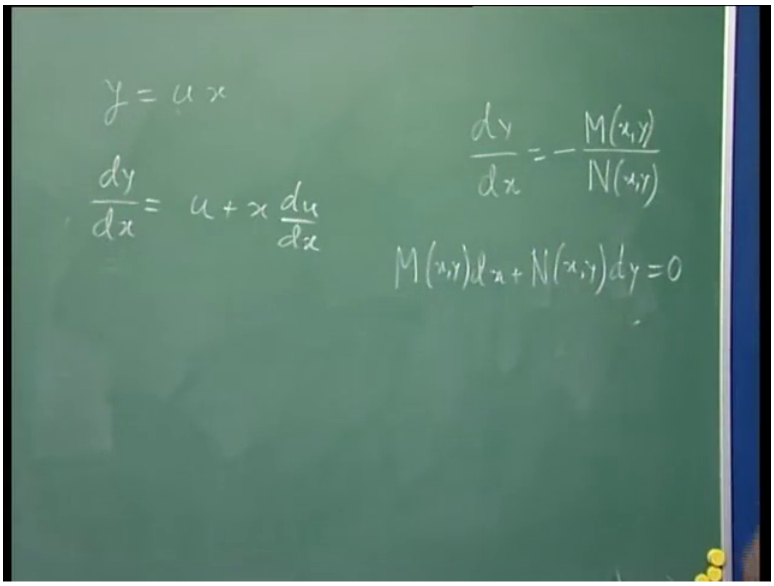

As you write a differential equation d y by dx is equal to f of x y, now if that f of x y can be organized in this manner. Now. here no simplification has been made x and y both are involved and then with it this on the dy side we take this on the dx side and then get everything on the left side then we will get M of x y in to dx plus N of x y in to dy equal to 0.

So, from here you will be able to see that dy by dx equal to any function of x y can be always organized in this manner and therefore, that differential equation and this equal to 0 the differential equation that you will get are actually equivalent. So, any first order differential equation can be organized in this manner that is this differential equal to 0.

Now, we already, we have already seen from our discussion on multivariate calculus that this is called an exact differential if there is some function of x y phi if there is some function phi of x y, the partial derivatives of which are M and N del phi by del x is M del phi by del y is N. In that case we call this as an exact differential because it is exactly the differential of the function phi because we find that if phi is a function of x y then we try to work out dy then that will run out to be del phi by del x in to d plus del phi by del y in to d y. So, in that case we call it an exact differential.

Now, whether they exist some function phi for which this is the x derivative and this is the y derivative that can be easily checked if we differentiate this with respect to y and this with respect to x; that means, del M by del y will be the second derivative del 2 phi by del x del y del x and del N by del x will be del 2 phi by del x del y. Out of continuity of the derivative these 2 will be same. So, this is the check del M by del y is equal to del N by del x. So, if we have an exact a differential like this and we want to check whether it is exact or not then we differentiate this with respect to y and this with respect to x if they turn out to be same then this is an exact differential.

And in that case the corresponding differential equation is called an exact ODE right which will immediately get solved when we integrate it both side that is when we write it openly then in place of M we write del phi by del x and in place of N we write del phi by del y this are exact differential and that is d phi equal to 0. The solution of that will be phi of x y equal to constant, we immediately get the solution.

Now, sometimes it happens that the differential equation as available in hand is not exact, but multiplication with something will make it exact now this is a little working rule for integrating this when you have M and x M separately. So, while integrating what you can do is that first round you integrate this partially with respect to x, considering y as constant for the time being and therefore, in the constant of integration you include an arbitrary function of y. Similarly here when you integrate this part with y with respect to y partially then you consider x as something like a constant and therefore, in the arbitrary constant you win good a function of x and them when you say that these two should mean the same thing; that means, we try to determine g 1 y and g 2 y such that this entire solution and this entire solution turn out to be same and that gives you the particular case of g 1 and g 2 which will make it same and that is the solution.

So, if these two are not same say if the condition is not satisfied and this is not an exact differential which means that different this differential equation is not an exact ODE then sometimes we look for a suitable function with which we can multiply the differential equation which will make it exact. So, that kind of a function f is called an integrating factor. And the succeeding theory of linear differential equations that we will we are going to discuss depends directly on the (Refer Time: 35:05) an integrating factor.

The general first order ordinary differential equation.

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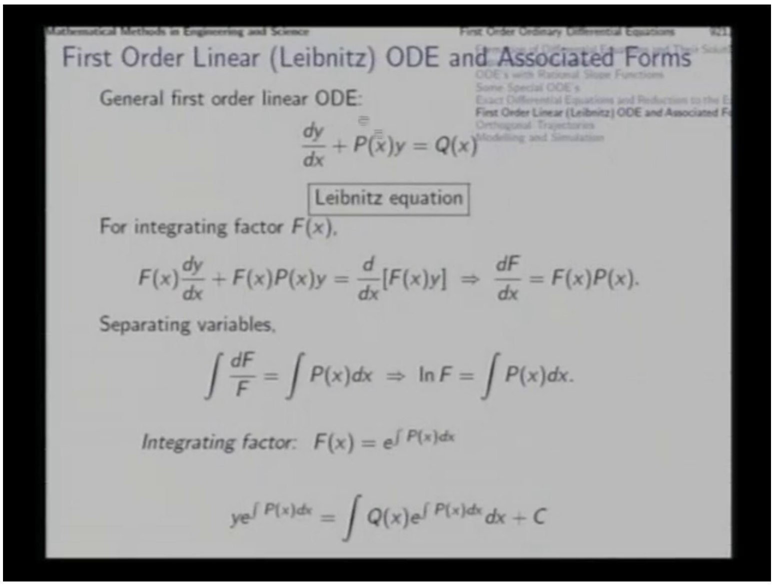

Even if there is some function of x appearing here we can always divide with that the entire equation to put it in this one and since it is linear. So, no term of y can appear here. So, in the first order linear ODE only 3 distinct terms we can get one is a term with dy by dx from which we have got rid of all the coefficient by dividing with that another term can be there which is the y term as coefficient of that any expression of x can appear and the other term which is making this equation non homogenous that can be any function of x including any other constants alpha beta etcetera similarly here for that (Refer Time: 36:05).

Now, x y and dy by dx are the 3 important items in which also y and dy by dx. So, this is the dy by dx term this is the y term and this is the term which is free of both this form is called the Leibnitz equation. And we try to find out the solution of this for the general first order linear ODE. So, we look for an integrating factor we thought to multiply this side. So, that this side turns out to be exactly the derivative of something. So, we say that suppose that something which we are looking for is f of f, we multiply it with f of f this left side and say f x dy by dx plus f x P x y this we want as the derivative of something. Now, it is clear that derivatives of derivative of whatever we are going to put that will be y in to f because the y can appear only in this manner otherwise this dy by dx term will not be, so simply in hand.

So, we find that f in to y gives us something the derivative of which will have one term which is f in to dy by dx the other term will be y in to d F by dx right. So, which is here y in to d F by dx that will require this coefficient f x P x to be d F of d F by dx. So, the integrating factor F x must satisfy this differentiation which is also a first ordered linear ordinary differential equation. The difference between this first order linear differential equation and this first order differential equation is that this is homogeneous this does not have a term free from F and d F by dx and solving this is extremely simple because this is already in the variables separable form. If we take f below then we find that here we have got d F by f and on this side we have got P x dx. Now, this is dependent only on f this is depending dependent x. So, integrate it the integral of this is one N f an integral of this can be whatever that is as long as that is free from f free from y we do not care how complicated expressions of x are appearing in that.

So, this is we keep this in the form of quadrature itself. If ln F is this integral then F is its exponential function e to the power this whole thing right. So, we have found the integrating factor. After we have found the integrating factor we know that if we multiply it this entire equation with this integrating factor then the left side is going to be exact d by dy, d by dx of f in to y that is this in to y (Refer Time: 39:05) to that.

So, the right side, left side after multiplication with this is exact derivative of y in to this term right. So, we say that then we integrate it because on this side we will have Q x in to this. So, Q x in to the integrating factor we integrate it and we get y in to this is equal to this integral whatever complicated that integral may be plus a constant. Now this gives us the complete solution of the differential equation of course, we can get the expression of y if we multiply both side with the inverse of this that is if we got minus integral of P dx with that if we multiply both side we get the expression of y in terms of x and the arbitrary constant c.

Now, notice that when we took this integrating factor F while solution of while solving this in this integral we did not take an arbitrary constant, even if we had taken we would get essentially the same solution back. So, in this case whether you take that arbitrary constant or not it will not matter. So, this is the way we solve a first order linear ODE that is in the hunt for it integrating factor we find a homogenous linear ODE the solution of which is trivial and that provides us the integration integrating factor which when plugged in to this by multiplication throughout gives us the left side left hand side of the equation as an exact differential coefficient.

Now, there are some non-linear equations which can be easily reduced to this Leibnitz form through certain substitution.

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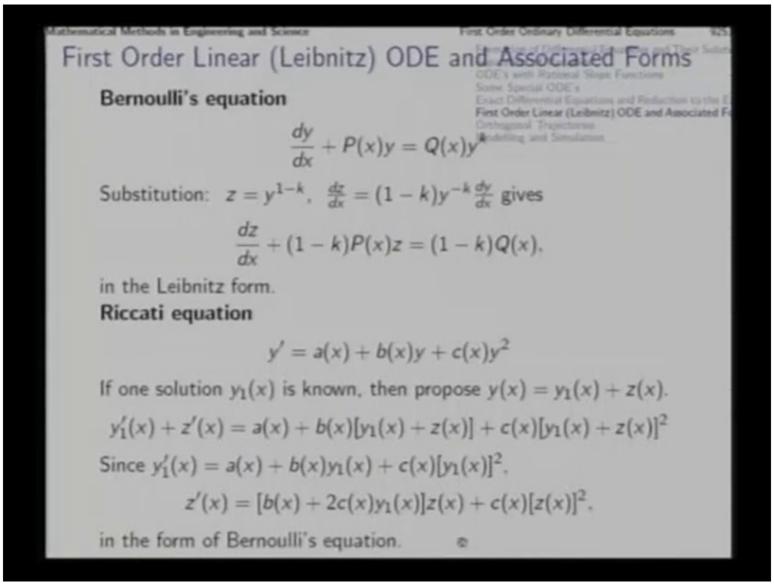

One of them is the Bernoulli's equation in which here apart from Q of x y to the power k term appears which makes it a non-linear equation; however, this y to the power k term can be easily removed by making this substitution when we substitute z equal to y to the power 1 minus k and then we differentiate this and try to find dz by dx we find that 1 minus k in to y to the power minus k in to dy by dx. And then we find that in case of dy by dx we now will try to put expressions in volume dz by dx and we will find that dz by dx turns out to be this does dz by dx plus this will substitute convert this equation in to this one. How? Say we multiply this equation with 1 minus k y to the power minus k. So, then this y to the power minus k multiplied here will get rid of 1 minus k will remain here also 1 minus k will remain and this y multiplied with 1 minus k and y to the power minus k will get z here right. So, 1 minus k P x will remain and z will come.

So, here we will get a linear expression of z and on this side the y k y to the power k will be removed through the multiplication of y to the power k and when we multiply when we put y to the power minus k in to 1 minus k in to dy by dx here this side is just dz by d x. So, now the resulting differential equation that we get in terms of z that is exactly in the form of Leibnitz equation where as coefficient of z we do not get P x, but we get some constant in to P x and in place of Q x we get 1 minus Q k in to Q x and this can be subsequently handled in the same way as we handled the previous equation.

Another equation is there which is Riccati's equation here the expression on the right side is only slightly complicated compare to a linear equation. If we had removed this equation this term from the equation it would be a Leibnitz equation now this quadratic term is causing the trouble; however, at when we try to solve the differential equation this actually poses more problem than Bernoulli's equation. However, if one solution of differential equation is known then there is a nice way to get the general solution.

In general without knowing even a single solution it is not possible to solve this analytically; however, somewhere from somewhere if you get one solution y 1 then getting the general solution is easy because then you say the latest cause a general solution as that known solution plus something which is unknown and then substitute this here. As you substitute that this entire thing as y then on this side we get y 1 prime plus that prime here this is a x simply b x in to y 1 plus z c x in to y 1 plus z whole square and as we substitute as we simplify this we already know that y 1 satisfies this. So, y 1 prime is already known to be this now I was just subtract this 2 equations y 1 prime goes up z prime remains here a x goes up this term goes up this term remains. So, b x z x remains.

From this whole square as we subtract this c y 1 square goes out the other 2 terms remain, one of them is a linear in z that gets put here the other is quadratic in z that is here now you note that this equation is in Bernoulli's form. z prime is here a linear expression of z is here with coefficient as x and known expressions of x and here in place of Q x y k you have got c x z square that is z to the power 2. So, then using this method we can reduce it to Leibnitz form and subsequently solve it.

Now, this is Riccati's equation. There are large number of physical systems and with the rough shoot as engineering systems where first order differential equation appear in a big way and they can be solved to predict the behavior of those physical system quite often in the description of curve and orthogonal trajectories first order differential equations are found quite useful.

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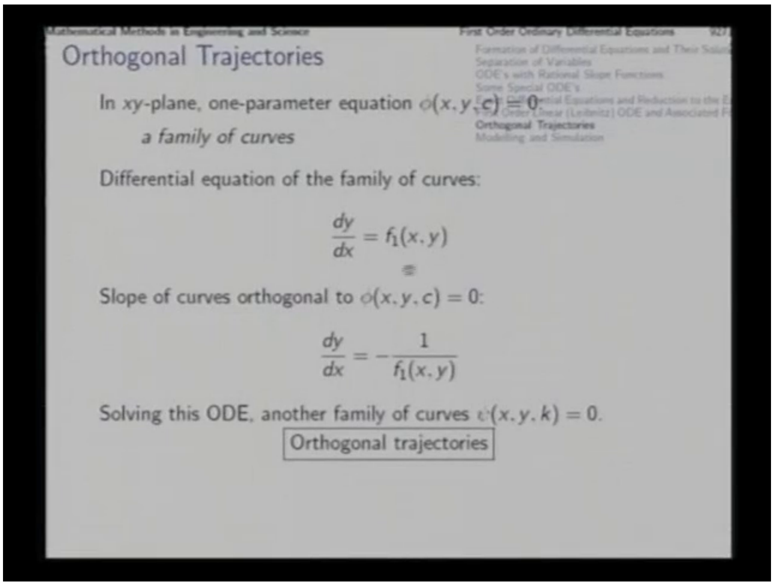

For example in x y plane a 1 parameter equation in x and y with c as a parameter represents a family of curve you know that now differential equation of the family of curves can be found by differentiating this with respect to x which will involve dy by dx and then eliminating the parameter c between this equation and the expression which we get by the differentiation. And as we eliminate that we get dy by dx as f function of x and y and gives us the same family of curve in its intrinsic equation.

Now, if we have got this family of curve in x y plane suppose in this manner and then in many application we want to know another family of curve which are all orthogonal to it.

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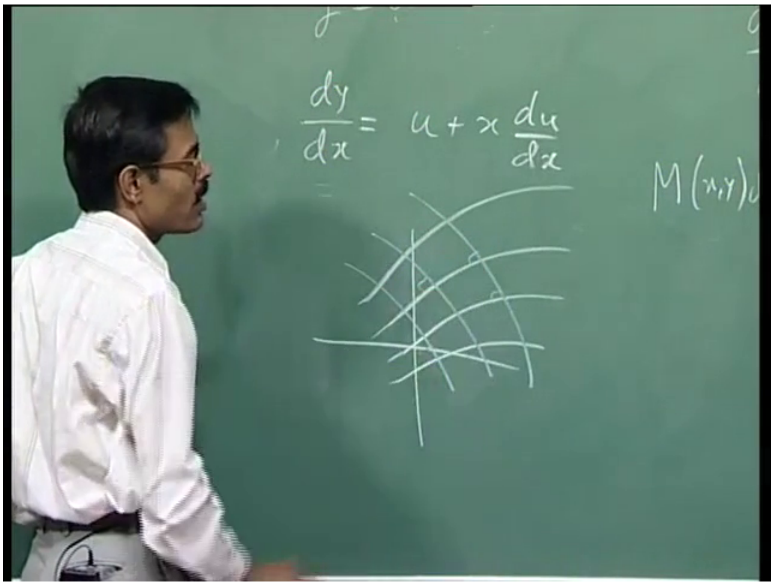

For example, if this family of curves represent potential curves or contours then the flow lines will be different curves another set of curves another family of curves which will be all orthogonal to these. So, this is an explicit an application which is involved in many different types of engineering problems.

So, in there what we do if the slope of the curves of this family is given by f 1 then the slopes of the curves of the family which is orthogonal to this wherever they intersect they make right angle they make right angle; that means, the slope of this white curve and the slope of the intersecting blue curve at every intersection point must be right angle. That means, this (Refer Time: 47:50) slope and this (Refer Time: 47:51) slope the 2 slopes their product should be minus 1.

So, when we use that fact from analytical geometry, analytic geometry, then we say that slope of the curve orthogonal to this will be dy by dx equal to minus 1 by f 1 and then this is another differential equation which we can solve and as I solve that we get another family of curve in the form of solution of this, this expression, this equation. So, this equation gives another family of curves and the family of curve given by this phi equal to 0 and the family of the curve given by this phi equal to 0 give us 2 orthogonal families of trajectories or curves that is 2 families of curves each curve from this family is orthogonal to each curve of the other family.

And as we give different values of c we get different curves of one family and as we supply different values of k we get the different solution curves of the second family, so these two together represent orthogonal trajectories. Now, if one of them is the velocity potential function of (Refer Time: 49:05) situation then the other one will be the giving the stream lines. Similarly, if one of them gives the electric potential controls the second family of curves will give us the directions of the electric fields and so on. In all potential problems and field problems we will find this pair of orthogonal trajectories in 2d it will be orthogonal trajectories in high dimension it will be more complicated.

So, in this lesson we find that these are the important point to make note of the different methods of solving first order differential equation.

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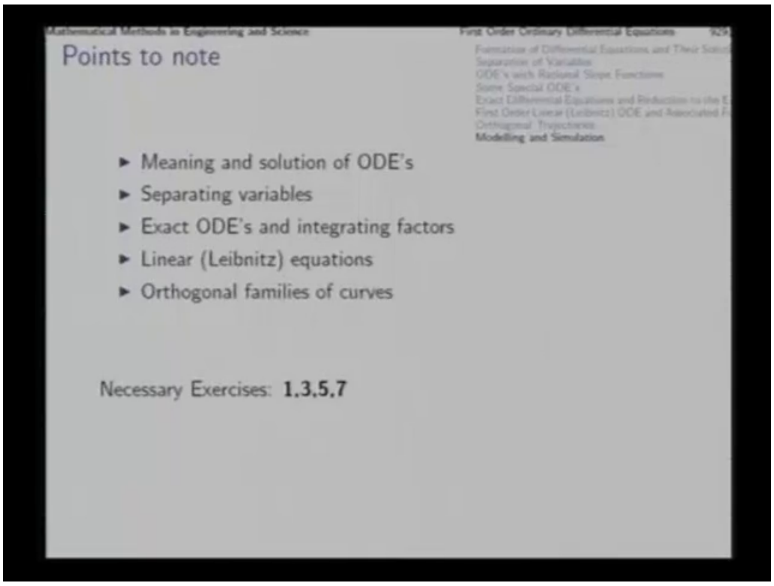

After this as we discussed earlier we make a move towards the second order differential equations.

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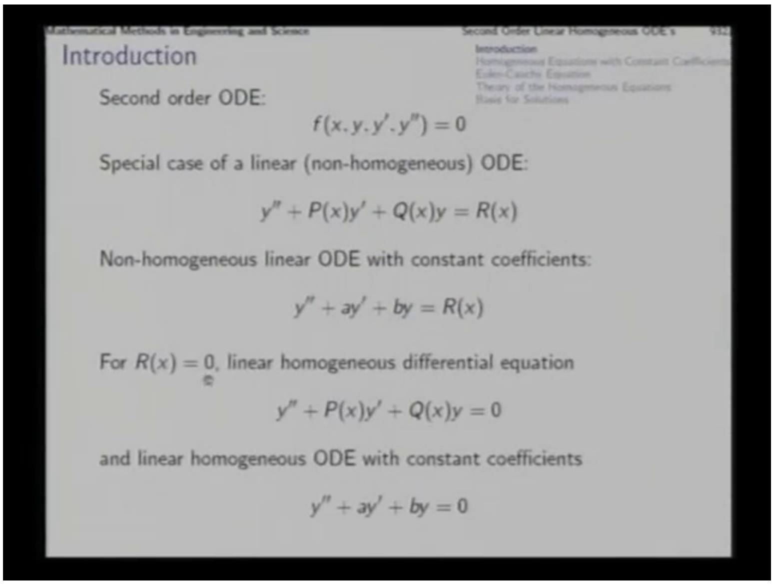

In general a second order differential equation would look like this in which x y and its 2 derivatives will appear. Now, in the particular case when we would be talking about linear ODEs second order linear differential equations, for that the special situation that will arise in that y prime square y prime in to y double prime, y in to y prime such terms will not be present. So, there what kind of terms can be present there will be 1 term with y double prime 1 term with y prime 1 term with y and there can be a term without y or its derivative that is x.

So, typically in many engineering situations or physical situations this side can represent the dynamic model of a plant of a system and this side can represent the input given to that system which drives the system and the response of the system is given in y which we need to solve in order to predict the behavior of that particular physical system.

Now, in this case which is a linear ordinary differential equation we have terms of degree 1, degree 1, degree 1, and degree 0. Now, if this side is not there then we will get a homogenous equation. If R x is 0 then we get a linear homogeneous differential equation which is this. Now in this, a particular case appears when the coefficients of y y prime y and y prime are constants. So, then we get this differential equations which is a special case of this and we call it the linear ordinary differential equation with constant coefficients.

Similarly, here if we have constant coefficients then we get a linear homogenous ordinary differential equation with constant coefficients which is this right. Now this can be considered a special case of this by making the coefficients constant or a special case of this by getting by putting the right hand side as 0. So, the linear second order ODE will have it general form here the solution of which we will consider, but before that will consider these special cases this one, this one and this one and finally, we will (Refer Time: 52:25) this general linear ordinary differential equation of the second degree.

So, in the next lecture we will start with this simplest case for which the solution will be found very easily.

Thank.