**Mathematical Methods in Engineering and Science**
**Prof. Bhaskar Dasgupta**
**Department of Mechanical Engineering**
**Indian Institute of Technology, Kanpur**

**Module – V**
**Selected Topics in Numerical Analysis**
**Lecture - 05**
**Stiff Differential Equations, Existence and Uniqueness Theory**

Welcome friends, in this lecture we will make the transition from our module of numerical analysis to the module of ordinary differential equation general theory. In the last lecture we were discussing the merits of implicit method in comparison to explicit method. Now, the particular advantage of implicit methods come in the context of stiff differential equation today we will first see that with the help of an example.

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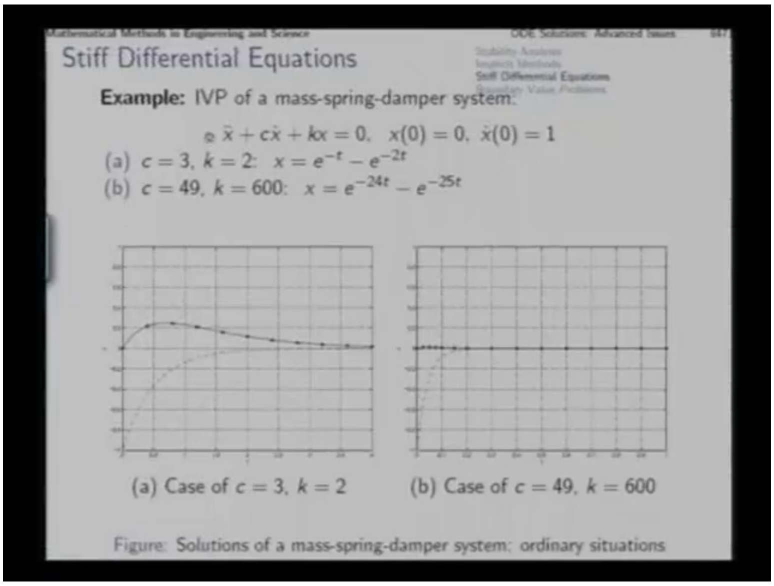


Consider this initial value problem of a mass spring damper system here c is the damping coefficient and k is the stiffness and masses unity unit mass, and these are the initial conditions initial position is 0 and initial speed is 1, now this system we will consider for 3 cases of the coefficient c and k. First we take c equal to 3 and k equal to 2, so far as expressing this differential equation in step space etcetera is concerned that is fairly trivial.

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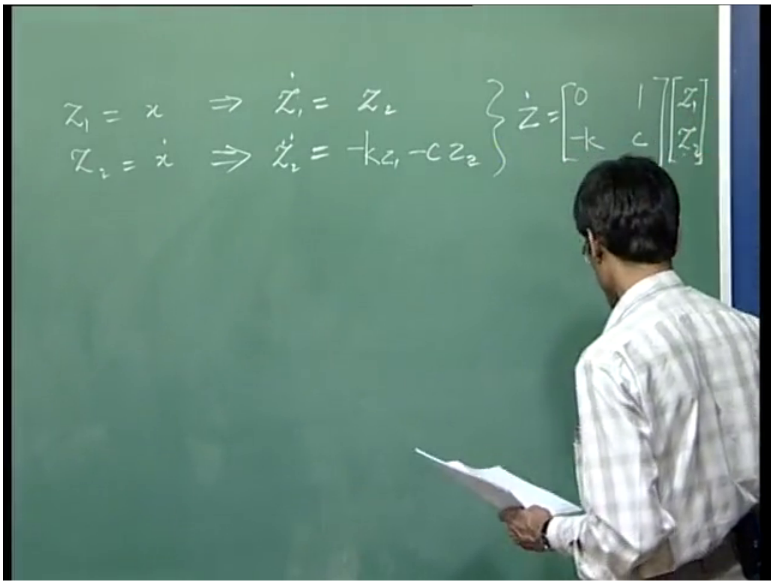


So, we quickly go through it we take the state variables as z 1 and z 2 which will give us z 1 dot as x dot which is z 2 and z 2 dot which will be x double dot and that we will get from this equation that is minus k minus c s dot; that means, minus k x minus c x dot. These two equations together will give us the state specific equations of the system. So, this z 1 z 2 with coefficients of that we will get from here 0 1 minus k c and this is nothing but z.

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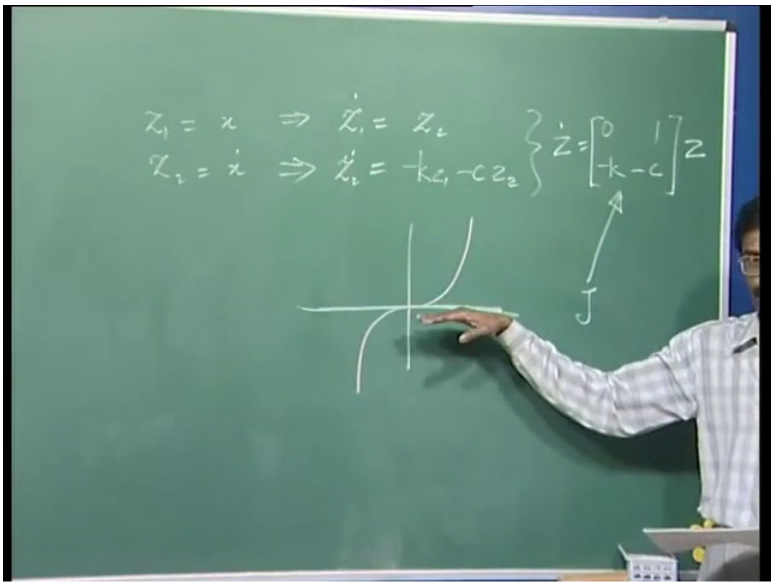

So, z dot is this matrix in to z and this matrix turns out to be the Jacobian of this entire function, the slope function f of t z that you get. So, that there the Jacobian of that the derivative matrix of that will be this J which enter in to our discussion with respect to which we carried out an eigenvalue analysis. So, now, with the value of c as 3 and the value of k as 2 if we put in this matrix sorry this is minus, if we put this 2 values of c and k in this matrix and try to find eigenvalues we get the eigenvalues as minus 1 and minus 2 and accordingly the solution transfer to be this.

Now, we will make note that both this minus 1 and minus 2, this two eigenvalues of this matrix are of reasonable magnitudes.

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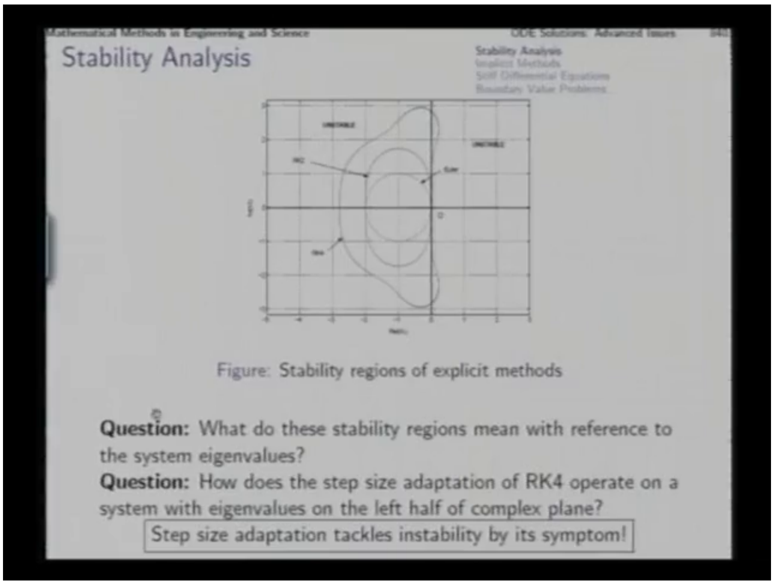


And they fall here I mean with sufficiently small h which is not too small the 2 values h lambda h lambda 1 h lambda 2 will fall in this circle and therefore, the or within this it will certainly fall within the RK4 boundary, stability boundary. So, then RK4 will be stable for this particular situation. So, we find that with the help of Runge Kutta 4th order method as we try to solve this then we will find that we get the solution these stars give out the solution points with adaptive RK4.

So, in this particular case you will find that there was no requirement of adaptation below 0.4. So, 0.4 is the step size here at from 0 to 0.4 and 0.8 then 1.2 and so on the integration of the differential equation went on mostly without any trouble. You see here the solution x equal to E to the power minus c minus E to the power minus 2 t that you will find at the solution satisfying these initial conditions also and E to the power minus t is this part and E to the power minus 2 t is this part. According to the 2 eigenvalues of the matrix j here you will find 2 components solutions on its E to the power minus t the other is minus E to the power minus 2 t. So, they are these two solutions and the sum of these two solutions is this which is the complete solution.

Now if we consider this pair 49 and 600 for c and k and put here and carry out the eigenvalue analysis then we will find at 2 eigenvalues are minus 25 and minus 24 these are the minus 24 and minus 25 are the 2 eigenvalues. And accordingly the two solutions will have explanations with that kind of exponent values exponent coefficient and you will find that the solution is E to the power minus 24 t minus E to the power minus 25 t. Here the 2 component solutions are one is this and one is this dash, one is this dotted curve and the other is this dash curve. And the resulting total solution this x is this made up of the solid line with stars, stars are the points where the adaptive Runge Kutta method gave the outputs.

Now you see here adaptation was not required much that is point 1 we are in step of point 1 you have got points appearing here, but here the error was going high and therefore, the adaptation sub divided the intervals and your here you have got so of course, points. Now, here so far as stiffness is concerned it has become stiff the system mass in the mass spring damper system the spring has got become much much stiffer, but at the same time the damping coefficient has increased sufficiently well, so that the resulting result of this increase in stiffness does not lead to what is called the stiffness of the differential equation

What has happened is that here minus 1 and minus 2, 1 and 2 were comparable in order here also the both the eigenvalues are comparable in order one is 24 the other is 25 minus 24 minus 25. So, their magnitudes are comparable. So, you do not see the real damage that the stiffness of the differential equation may cause that we will see in the third case.

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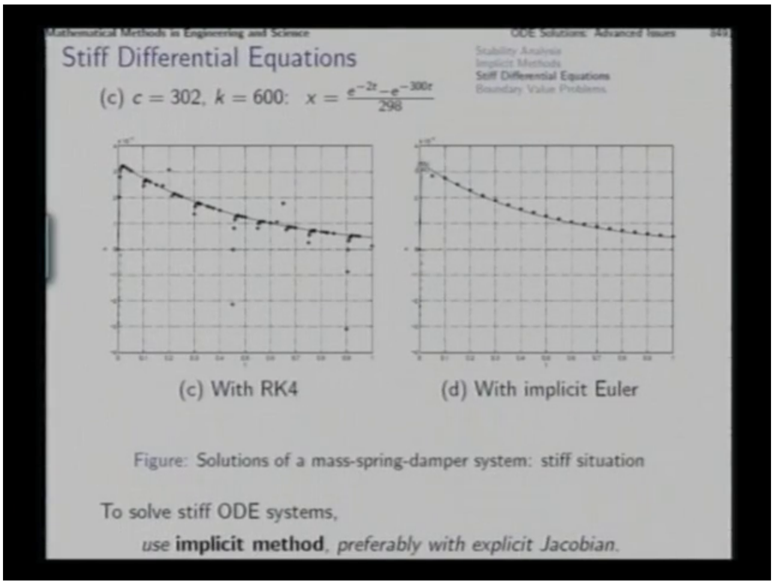


Where we take c as 302 and k as 600 in this case the 2 eigenvalues are minus 2 and minus 300 and with the RK4 the further the actual solution if this solid line the solution shoots up fast and then decays like this, this is the solution. Now here you will notice that if you try to work out the component solutions E to the power minus 2 t by 298, that is this slowly decoying decaying component that is its start form here and goes down.

On the other hand this term minus 300 minus E to the power minus 300 c by 298, that start from here immediately dies down to 0 and then goes 0, so that means, after a very little time after a very small amount of time say this is 0.1, this will be 0.03 after 0.03 units of time it is actually this part which will have a nonzero component and this part is has gone down to 0, but then the part which goes down to 0 that has such a large negative eigenvalue that this part of the solution will require further and further sub division of the interval in the adaptive Runge Kutta method and therefore, in order to predict this solution, the sub division of intervals goes on and all these points in (Refer Time: 08:46) manner are generated in the Runge Kutta 4th ordered method.

See how much is the error huge error because the Runge Kutta adaptive method does not have a an inbuilt mechanism to check the error when it is building up, after the error has built up quite a bit then its tries to sub divide the interval and then the quire is no longer possible. On the other hand first ordered implicit method, implicit Euler method that gives you these solutions, the implicitness of the method gives you this nice solutions here. Of course, there is no statement made about accuracy, accuracy has suffered a little bit the stars, the points are little away from the actual solution curve, but the method is stable the error never shoot out too much like this. So, this is the first order implicit Euler method giving this result.

So, therefore, when you try to solve ODE systems which are stiff in nature; that means, in which in the solution if you have components of hugely different scales then you must use an implicit method of differential equation solution and in that you will required the Jacobian of this function. In this case this is constant, but in other cases where this is variable you may need to find the Jacobian of that. In this case Jacobian is this constant value, sometimes you this functions will be. So, complicated that it is Jacobian will not be constant you may have to evaluate those Jacobians. Now when you need to evaluate those Jacobians which are not constant then you can develop those Jacobians by the process of numerical differentiation or if you can give explicit expressions for Jacobian then it will be better because the errors which can built up build up will be less to begin with.

Therefore, to solve stiff ODE systems it is always better to use an implicit method and in that implicit method when the Jacobian is needed this Jacobian matrix is needed for updating the Newton Raphson iterations then explicit Jacobian in terms of expressions are numbers will be a better use compared to Jacobian developed through numerical differentiation. These are some of the questions which we have addressed regarding the numerical solution of ODEs in initial value problems. And now we will raise some further questions regarding the fundamental nature of physics of the problems in which we encounter ODE systems and through those fundamental questions of existence and uniqueness we will build up some of the basic case for the theory of ordinary differential equations that we continue from the next lecture.

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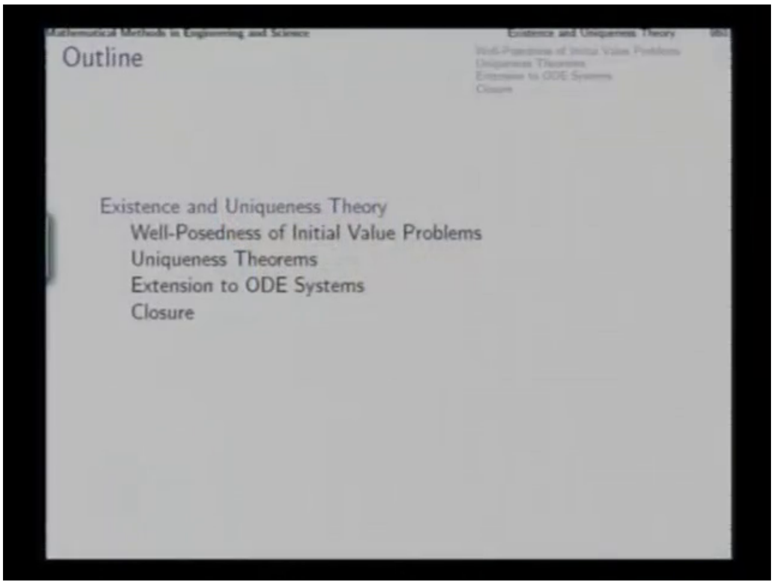


So, here now we discuss the existence and uniqueness questions of solutions of ordinary differential equation.

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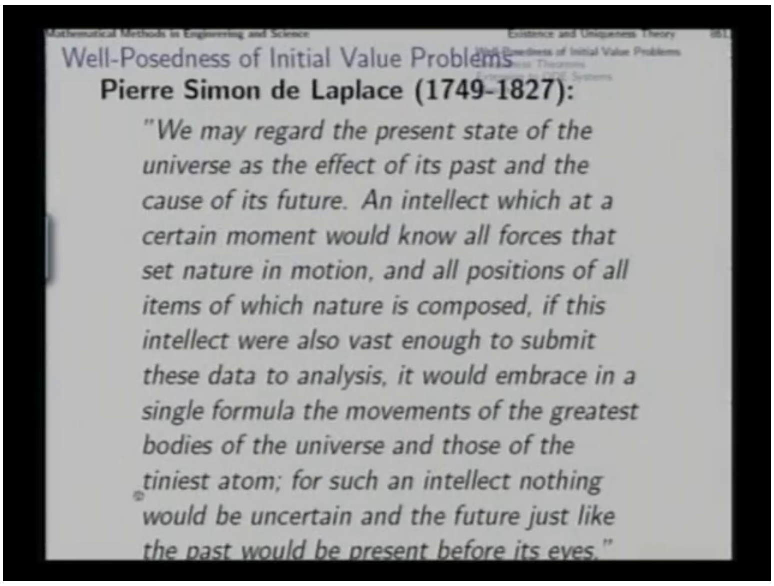


A few hundreds of years back when Newtonian mechanics was at the height of its success in predicting the nature and building up devices and gadgets and systems which were about to make revolutionary changes in the world at that time there was a great enthusiasm in the minds of scientist that they felt, they felt as if the entire mystery of nature is almost solved that is the background the theoretical background of the solution of the mystery of nature has been already honored and all that is now required is extensive amount of computation because with the initial value problems where we have methods which can solve the initial value problem they in a way can predict the future, they in a way can honored the path in principle. And the enthusiasm of that knowing everything or at least being in principle able to know everything was at such height that the one of the leading scientist of that time Laplace makes this statement.

“We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed. If this intellect were also vast enough to submit these data to analysis it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.”

What is the essence of this type of a statement? Considered the entire world of made up of elements components for which we know all the laws of nature governing their motion and then consider that initial condition of all these bodies all this elements will measure and all these laws of nature we keep together as a set of differential equations and the all the measurements that you have made all that we keep together as the complete initial conditions vast. Then what do we need? We need a large intelligence an intellect which is in of enormous capacity in our time we need not talk of intellect we can talk of a very powerful computer which has enormous disk space and which has extremely high computational capacity. Now, to that computer if we can submit this entire differential equation system with the entire initial condition that we have measured, then we ask this computer program to solve the initial value problem. As a result of this computation shall we not get the positions and velocities all the state variables at any future time it looks as if we should get it and that is the result that is the call of this enormous enthusiasm.

However, there are a few fundamental questions raised by mathematicians which kills this enthusiastic advantage and at the beginning come the questions on existence and using uniqueness and continuous dependence on initial conditions. That is the questions basically translate to will a system of differential equations have a solution always, is it necessary that there is a solution then if there is a solution is it necessary that the solution is unique whether the future is definable clearly and then also if that can be defined then does thus that definition have any meaning practically. These are the questions which are raised by mathematicians to this understanding of the natural laws and as a result we have got this existence and uniqueness theory.

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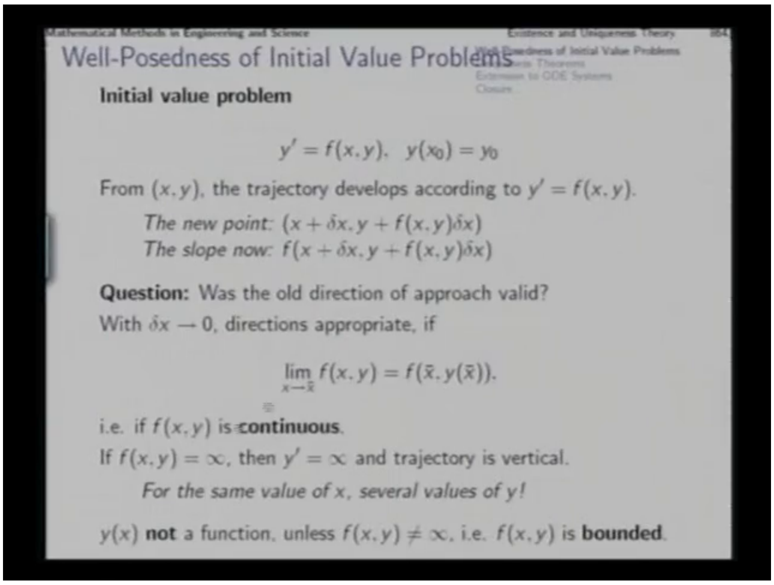


To start with to understand the first question we consider this initial value problem y prime is f of x y and this is the initial condition. Now from x y the trajectory develops according to this rules that is as x changes y will change according to this y prime rate of change. So, from a given point x y with delta x change in x the change in y will be delta y which will be this y prime in to delta x that will be this. So, that will be the new point.

At this new point if we try to develop the slope then we will get f evaluated at this new point f evaluated at this new point. Now we ask this question that we approach this new point through this slope f of x y, but then after coming to this point when we evaluated the slope we found the slope is not exactly this, but it is f of x plus delta x y plus f x y delta x that is this slope value will be in general different from the slope value with which we approach this point. Then we asked a question should we not have approached along this slope at this point at the new point, so that is why we can ask this question was the old direction of approach valid. So, we should have arrived at this new point by along a line which is with the new slope here so we ask this question was the old direction of approach valid.

Now, if we take the step very small then this old direction and new direction look like not changing too much, and then they will actually not change too much if this function is continuous that is if the function is continuous. Then with small enough step we will always find that the difference between this old slope and the new slope will be small. That means, that for whatever small change small difference we allow between the old slope and the new slope for this direction to be called valid we can get a suitably small size of the step delta x that is not be. So, if the function here is not continuous if the function is not continuous then whatever small step you take across one step there will be gross change in the function value of f. And therefore, this o direction of approach is valid with sufficiently small x if this function is continuous. So, this is the first point; that means, the continuity is required for this step by step solution of the initial value problems.

Next, if there is a point x y at which the function value of f can become extremely large there is if it can go towards infinity then at that point y prime is infinity and y prime is infinity means at in the x y plane the curve will go vertically up, if the curve goes vertically up the trajectory is vertical that will mean that along that vertical trajectory for the same value of x there will be large number of values of y which will be valid along that vertical line and that will mean that as a function y of x will not make sense. So, y of x will not be a function because for a single values of x you will be able to define lots of values of y which will all be valid along that vertical trajectory and that is why y of x is cannot be called a function and the meaning of the solution of this differential equation and developing the function y of x will break down. And therefore, you say that this solution will not be a function unless this function f of x y is bounded, that is if it is never allowed to become infinite then only you can talk of the solution as a function. So, therefore, from this you get the picture that continuity and boundedness of the slope function is essential for desistance of the initial value problem. And that is precisely what is formally stated in Peano’s theorem.

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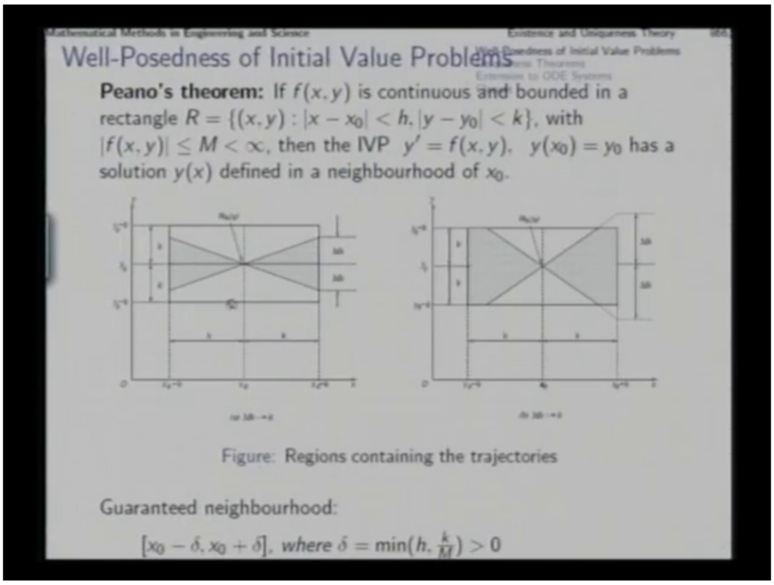


Peano's theorem says if f of x y the slope function is continuous and bounded in a rectangle around x 0 y 0 rectangle of size 2 h by 2 k this rectangle with the slope function bounded by M then this initial value problem has a solution defined in the neighborhood of x 0 say this. So, this is the point x 0 y 0 around which we construct this rectangle from x 0 minus h to x 0 plus h in the x direction and y 0 minus k to y 0 plus k in the y direction.

So, in this rectangle, within this rectangle if the function is continuous and bounded say bounded M if the slope function is bound bounded by with the value M then if with the M slope we will draw a line like this and with minus M slope we draw a line like that then any solution starting from here and continuing along the trajectory will have a slope which will be less than this slope and more than this slope right, whichever direction it goes it will remain within this shaded region. So, it will never go unbounded. So, you can say that up to the value x equal to x 0 plus h you can say that you guarantee the solution of the differential equation solution of the initial value problems in this case up to h you will give the guarantee in this case up to h you cannot give the guarantee because before that it goes out of the box out of the rectangle in the y direction. So, in that case you will give the guarantee up to x equal to x 0 plus this value and this value in that case will be k by M.

So, around x 0 y 0 there will be a rectangular there will be an interval x 0 minus delta to x 0 plus delta in which the solution is guaranteed this delta in this case will be h and in this case it the delta will be k by M, whichever is smaller right. So, within that neighborhood the solution is guaranteed this is Peano's theorem for existence. Roughly you can consider that if the slope function f is continuous and bounded then solution of the initial value problem is possible.

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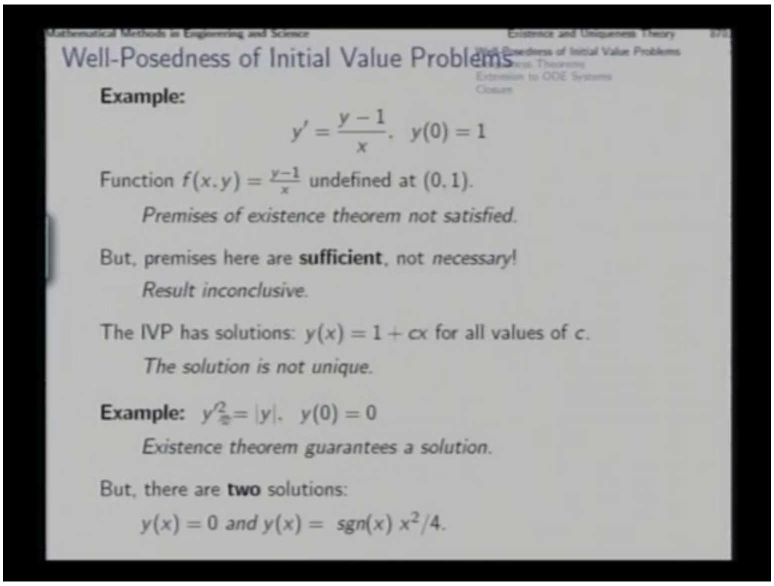


Now, we take an example again, suppose this is the differential equation this function is f of x y and this is the initial condition from this initial point x equal to 0 1 y equal to 1 we start, as we start we find that x equal to 0 y equal to 1 y prime takes a form which is 0 by 0; that means, at the initial point itself the slope function is not defined. So, if the slope function is not defined then premise of the existence theorem not satisfied so that means, existence theorem will not guarantee existence of the solution of this initial value problem. But note that the existence theorem is a one sided theorem it gives only sufficient condition not necessary; that means, that the existence theorem not guaranteeing the solution does not mean that the solution does not exist. In this particular case solution does exist.

So, premises here at sufficient, but not necessary, so premise not being satisfied means that the conclusion is not falsified only this is that no conclusion can be drawn, so result is inconclusive. In this particular case the IVP has a solution not only h solution, but it has infinite solution. For example, consider this solution y of y equal to 1 plus c f. If you put this y here then see what do you get 1 plus c f minus 1; that means, c f, c f by x is c, so y prime is c. And initial value is satisfied at x equal to 0 you have y equal to 1. So, if we satisfied this differential equation as well as initial condition is satisfied by this function, this is a solution. Now you find that this initial value problem has not only one solution, but infinite solution there is a solution is not unique, for all slopes all lines passing through the 0.01 will be solutions of this initial value problems.

Now, take another example definition value problem starting from origin 0 0 and this is the solution. Now in this case existence theorem guarantees existence a solution because if you take square root of this then you will get square root of mod y will be the slope function. Existence theorem will guarantee a solution that is not a problem because the slope function will be continuous. But then there are at least 2 solutions y go to 0 satisfies this because y equals to 0 will give y prime 0 y as well vary also 0 and this is anyway guaranteed. So, the constant value y equal to 0 is the solution of the initial value problem.

This also is a solution of the initial value problem this is the two solutions that you will get from here, one solution is this green line y equal to 0, another solution this is red curve, on this side it is x square by 4, on this side it is minus x square by 4 and together you can write them as this plus or minus x square by 4 depending whether plus or minus depends up on the sign of it. So, both of this the horizontal line at that is 0 suggestion trivial suggestion and this solution both are solutions of this. Now in this case existence theorem guarantees a solution solid existence of the solution is guaranteed, but whether the solution will be 1 or more than 1 that is not given.

So, for that we now again consider these from a practical stand point from a physical system suppose we develop a mathematical model that is in terms of differential equations and initial conditions and so on. After finding that mathematical model we try to find out the solution of that initial value problems and that mathematical solution we get in terms of numbers and from that we try to make interpretations about the physical system.

Now, if the mathematical model in terms of expressions or in terms of numerical values gives out more than one solutions then what does it mean what is the meaning of the non uniqueness of a solution.

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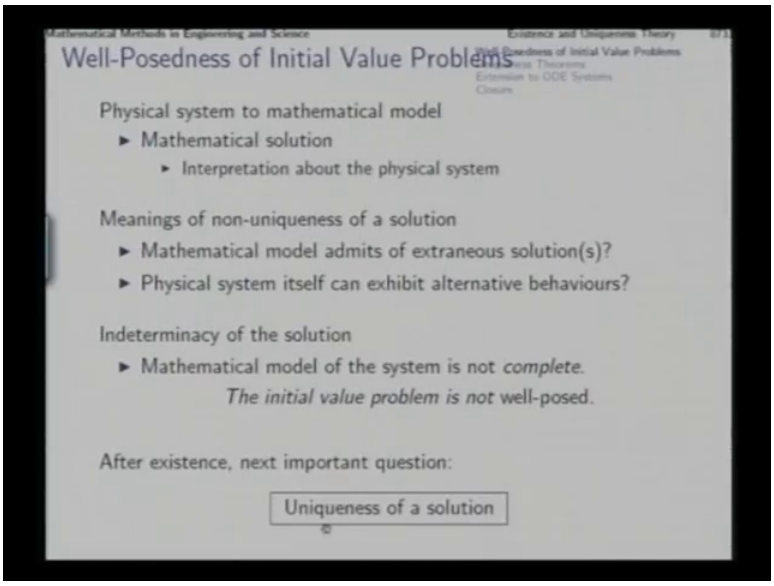


There can be two possible meaning one is that the physical solution will follow, one future will follow one trajectory the physical system will respond in a particular manner the mathematical model of the system apart from predicting that solution predict yet another solution or many other solutions that is the meaning of nonuniqueness one meaning. There is mathematical model admits of extraneous solutions which are not valid for the original physical system, but the mathematical modeling of the physical system has introduced these additional solutions which are invalid in the original physical context this is one possible meaning.

Another possible meaning which is even more dangerous is that the physical system itself can exhibit alternative behaviors it may be at the physical system itself is capable of exhibiting more than one possible behaviors and in that case you cannot model that system completely for all future. So, these are the 2 possible meanings of the nonuniqueness of a solution of the initial value problem.

Now, sometimes it happens that from the data the solution is in determinant. So, out of the nonuniqueness you cannot determine which solution is the correct solution. So, that also may mean the mathematical model of the system is not complete; that means, the mathematically posed question the find out the future behavior of this system that question has not been posed nicely because we cannot find a sensible answer to that. So, we say that thus initial value problem is not well posed. We will talk about when we say that an initial value problem is well posed, so that will require us to study another ratio. So, we have till now seen two important issues one is existence of a solution which is guaranteed by the premises of Peano's theorem that existence theorem and after resistance next important question is that of uniqueness of a solution there is a third question also, we appreciate that let us consider this equation.

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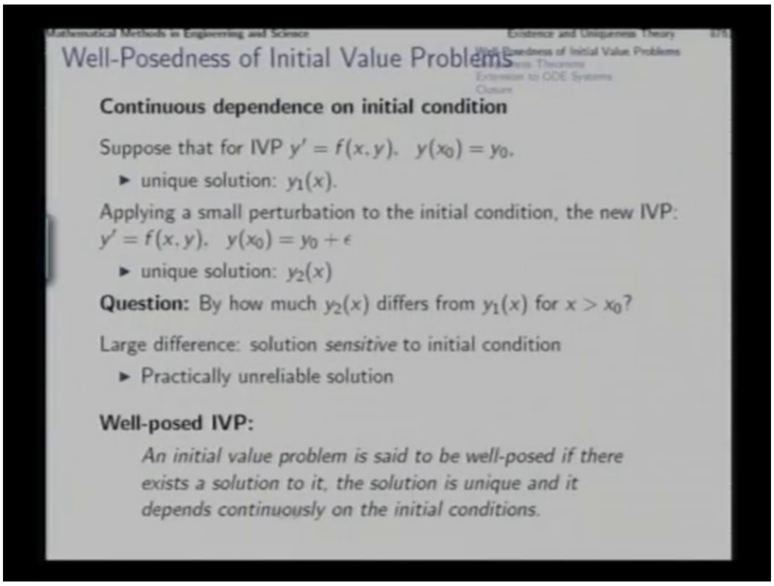


We have got a differential equation with an initial value and for which suppose we know that there is a function y 1 which is the unique solution of this. Now we say for the same differential equation we perturb the initial value solution and defined another initial value problem which is the differential equation this initial condition in place of y 0 we have y 0 plus epsilon. Suppose for that particular new case we have got another solution y 2 which satisfies this unique this is also a unique solution. Now, this is the unique solution of this IVP and this y 2 is the unique solution of the new IVP with a slight change in the initial value, now we asked the question how much difference is there between y 1 and y 2 for x greater than x 0. We know that for x equal to x 0 the differential is epsilon. Now for x greater than x 0 how much is the difference between y 1 and y 2. So, the issue is that if the difference is large that is with very little difference at the initial point if at feature points the difference is very large then we say that the solution is sensitive to initial conditions.

If the initial conditions are slightly perturbed then the perturbation in the resulting solution is large there is a that the solution is sensitive to initial conditions. This is a general way of telling the blunt way of telling the same thing is that the solution is practically unreliable it is a useless solution, why because this initial condition will be typically the result of measurements or results of other computations. Now both results of measurement and results of other computations are acceptable to small errors, now if the solution allows large change for future points future values for very small changes very small mistakes at the initial conditions then that will mean that the solution that we get are quite useless.

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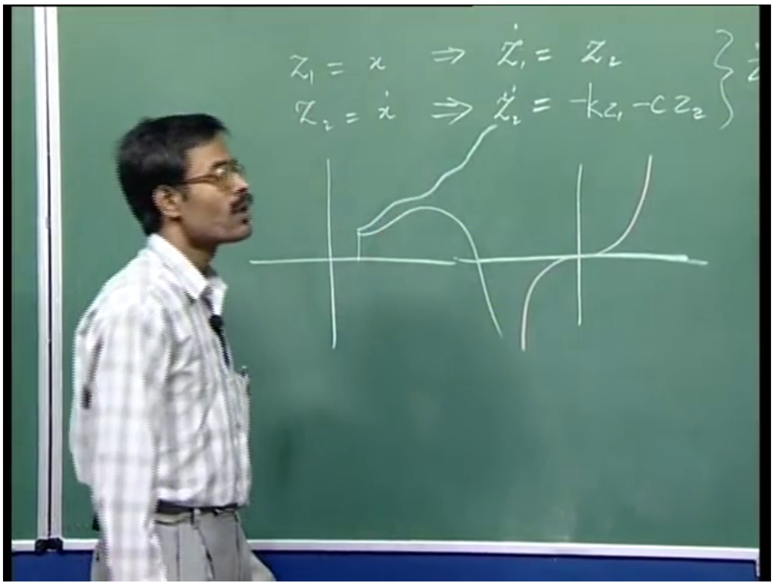


Say if the correct solution, if the correct initial condition at this point address to be this and the solution correct solution is this, on the other hand if because of a very small error at initial condition rather than this the initial condition was given like this and if the resulting error, resulting change, resulting difference in the solution is possible to be a very large and if it goes like this then you can see that the solution that we get is meaningless. So that means, that for the unique solution y 1 x or y 2 x to have sense, to have practical meaning it is also important that that not only the solution exists not only the solution is unique, but its error must have a sensible relationship to the error at the initial condition that is small error at the initial condition should have only small impact in the result; that means, the solution should depend up on the initial condition in a continuous manner.

So, that continuous dependence on initial condition is the third aspect which is required for an initial value problem to be well posed. So, you say an initial value problem is said to be well posed if there exists a solution it is unique and it has continuous dependence on the initial conditions. So, these 3 aspects full filled make an initial value problem well posed. There is an interesting condition which gives you the well posedness of an initial value problem, and that is called the Lipschitz condition.

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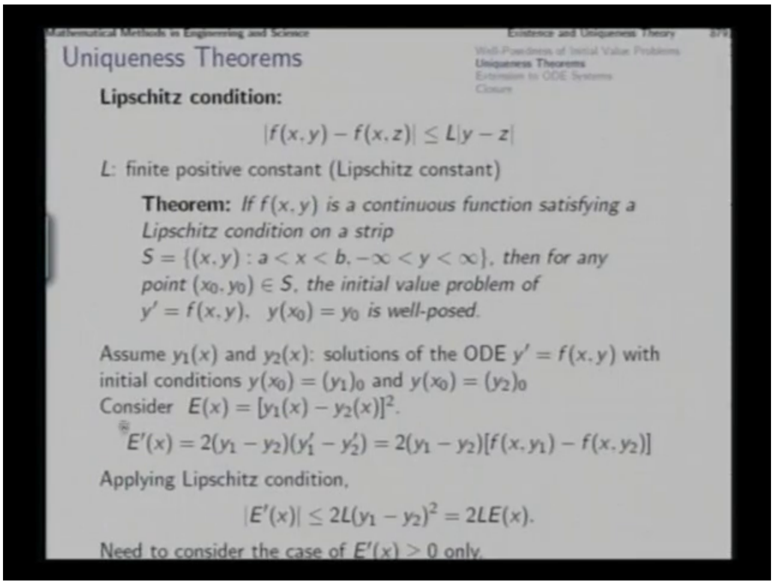


Since, we are discussing the effect of a difference in the initial value of y, so let us consider for the same x this point x y and this point x z that is at a different value of y.

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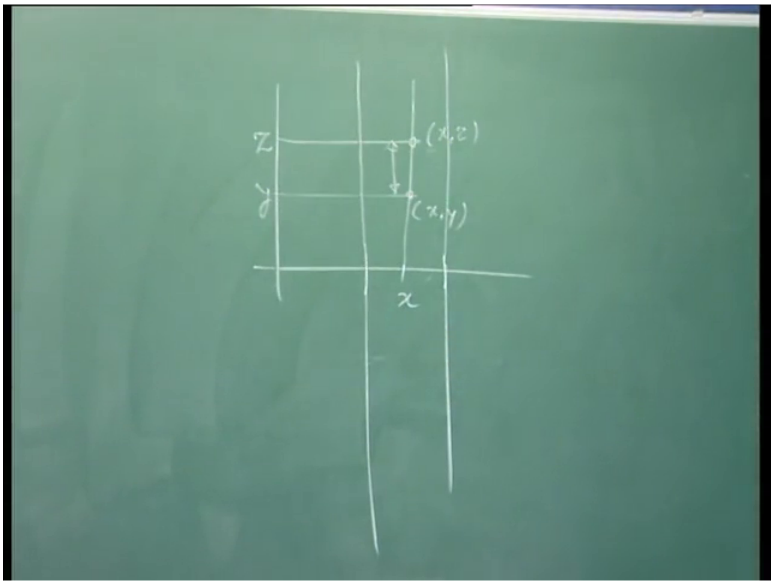


So, what will be the difference of the solutions if we start from here or here that is at the same value of x, but at different initial values in this case it is y in this case it is z. So, in that context if the slope function if the difference, in the slope function between this 2 points x y and x z if the difference in the slope functions here is less than equal to L in to difference of the values y and z where L is a positive constant. So, if this is if the difference of the slope if the difference of the slope function values is less than equal to is bounded by L in to the difference of the values y and z then we say that Lipschitz condition is satisfied this is the Lipschitz condition here L is a finite positive constant called the Lipschitz constant

Now, we have the important theorem which says that if f x y is a continuous function satisfying a Lipschitz condition on a strip x varying from this value to this value around this point and y is this, so infinite strip. So, in this strip if we have the point x y and then equivalently strip if the Lipschitz condition this condition is satisfied then for any point any starting point within this strip we can say that the initial value problem is well posed. To establish this result this is a powerful result to establish this result we consider two solutions y 1 and y 2 which are solutions for varying initial conditions for the same differential equation, same differential equation two different initial conditions giving two different solutions this start from y 1 0 and number and develop in to the full function y 1 similarly this starts from y 2 0 at x 0 and gets the another function y 2.

Now, the difference in this two y 1 and y 2 we consider and x square we will consider as the error function. If the error in the initial condition is this value minus this value then the resulting error in the solution is this. So, this is our measure of the error or difference. Now we consider the rate at which this error changes with x. So, that derivative of this is very simple twice y 1 minus y 2 in to y 1 prime minus y 2 prime the derivative of that. Now, if the Lipschitz condition is satisfied then y 1 prime minus y 2 prime will be f of x y 1 minus f of x y 2. So, this y 1 prime minus y 2 prime we put like this and now the Lipschitz condition tells us that this fellows absolute value is less than equal to L in to y 1 minus y 2 mod that is applying Lipschitz condition we will find that this mod will be less than equal to L in to y 1 minus y 2 mod.

So, both sides we take the mod. So, the actual value of the rate of change of E will be less than equal to twice y 1 minus y 2 in to L in to y 1 minus y 2 now we do not have to say mod in this side because we get square any way. So, this is the whole thing here is positive, so you do not have to separately say mod now here y 1 minus y 2 whole square is nothing, but E itself. So, we put that E back here and we find that E prime mod is always less than equal to this.

Now there could be two cases, one is where E prime is negative and the other is when E prime is positive if E prime is negative then we do not have to do anything because that will mean that as x increases the two solutions will come closer that is as x increases if E prime is negative that will mean that E will keep on decreasing; that means, the 2 solutions will come closer and closer. So, in that case we do not have to fear anything because they are different actually decays if E prime is positive that is as x increases if the difference turns out to be greater and greater then only we have a matter of concern if the difference can become too large. So, we consider only the positive case and if E prime is positive then E prime mod is same as E prime. So, we take that case and continue our analysis as we continue our analysis with this mod sign draw then we divide this with E then we get E prime by E is less than equal to twice error that is this, now this side is constant and this side is a function of x.

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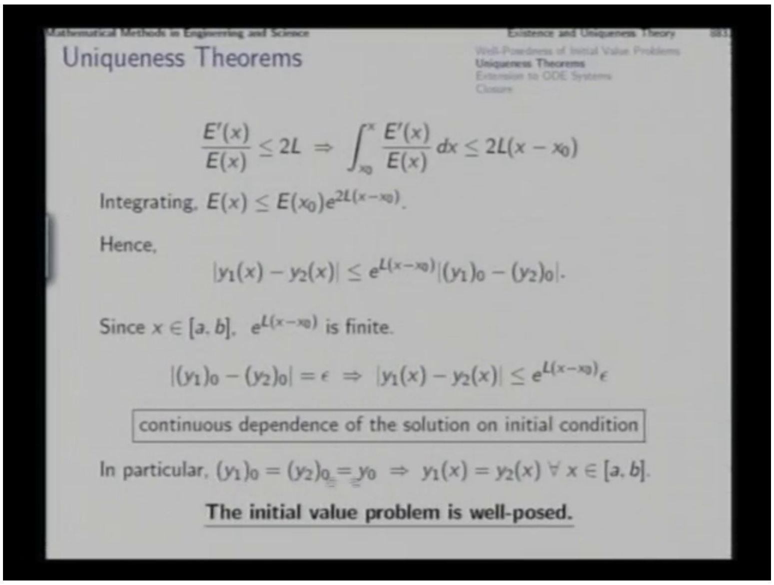


So, we can integrate, as we integrate we get integration from initial point x 0 to a general point x of this side will be less than equal to twice L in to integral of 1, integral of 1 from x 0 to L is this.

So, now this integral is less than equal to this relationship this function this expression. Now we can see here that this integral is very simple in the denominator we have got E x and in a numerator we have got it derivative. So, you know reorganizational integral easily this is log function the integral this derivative of log function log E x. So, its integral will be log of E x and then both sides we take the exponential. So, we get E x by E x 0 from here and on this side we get E to the power twice L x minus x 0. So, this inequality we get. Now see what we have got? We have got error at a later value of x future value of x is bounded by the initial error in to E to the power this and the square root of both sides will give us here the mod of y 1 minus y 2 from the definition of E, the definition of E was y 1 minus y 2 whole square. So, that square root of this is here and square root of the initial value of that is here and the square root of this get here with half power this 2 goes, the 2 is cancel cancelled.

Now, you consider if this is small then can it be large there is a question that you raise. So, in this interval since x is bounded within a to b in this finite strip. So, since x is bounded within this interval. So, x minus x 0 is also bounded and L is a finite number. So, this expression is finite this coefficient thing is finite. Now, if the initial value error is small say epsilon then this thing this error in the resulting later function values will be bounded by this thing in to epsilon.

And this is finite this is small; that means, small initial error can give only small error at later points. So, you find that the solution of the initial value problem continuously depend on the initial condition, if initial condition differs by small amount then the later error will differ by small amount only that is finite, it cannot grow indefinitely. In the case where there is a sensitive dependence you will find that despite of a very small difference in the initial value you might find that the later values may turn out to be quite different from each other.

Now, continuous once we have established continuous dependence on the solution of initial conditions solution on initial conditions then in particular if we consider both initial values to be same then immediately this epsilon will become 0, this will become 0, and that will mean that y 1 and y 2 will turn out to be the same solution and; that means, that the solution is unique. So, existence is guaranteed by the continuity of the slope function itself continuity and boundedness which we have already seen and now this shows that the Lipschitz condition satisfies all the requirements of well posedness solution exist it is unique and it depends continuously on the initial condition. So, we find that the Lipschitz condition is one good sufficient condition to check for the well posedness of an initial value problems and the initial value problem is well posed means that its solutions will have physical meaning.

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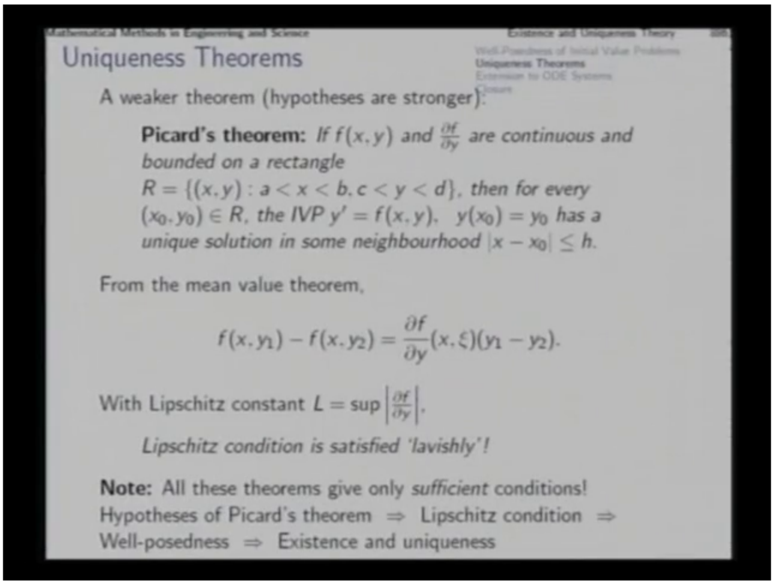


There is another theorem which is a little weaker theorem compared to the Lipschitz condition which is Picard's theorem it is weaker theorem in the sense that its hypothesis the premises are actually stronger you need more.

You need not only that this is satisfies, but you also need the partial derivative of f with respect to y to exist and to be continuous and bounded. So, with that additional requirements the theorem is weak, but the implications become more direct say if the function. So, function f and if partial derivative with respect to y are continuous and bounded on a rectangle a to b c to d then for every point in that rectangle the initial value problem is well posed. Now, so for this is extremely simple because if del f by del y is continuous then from the mean value theorem you will find certainly a point between y 1 and y 2 where point psi between y 1 and y 2 where the derivative is the mean value is gives the mean value and the difference in the function value is equal to this derivative at that point at that value in to y 1 minus y 2. And then since the derivative is bounded that means, in the entire rectangular whatever is the largest value of this derivative you take largest possible value or a little higher than that take the supremum of this and consider that as a lips Lipschitz constant. That is a finite number and put that Lipschitz L here and then you will find that mod of this will be less than equal to that L in to mod of this; that means, Lipschitz condition is satisfied very easily and with that satisfaction the entire well posedness follow as consequence. Now, in this all of this results theorems existence and uniqueness theorems you will find that they are all given in terms of sufficient conditions. So, hypothesis of Picard's theorem are more than enough more than enough to satisfy Lipschitz condition.

Similarly, Lipschitz condition is more than enough to satisfy is more than enough to imply well posedness which is with quite directly (Refer Time: 47:05) you the existence and uniqueness initial value problem and the unfortunate part is that all these are in terms of sufficient conditions. That is solution of Lipschitz condition will not imply Picard's theorems (Refer Time: 47:17) satisfied and so on. Well posedness does not have to imply Lipschitz condition satisfied is all one sided conditions, the one sided implications are there.

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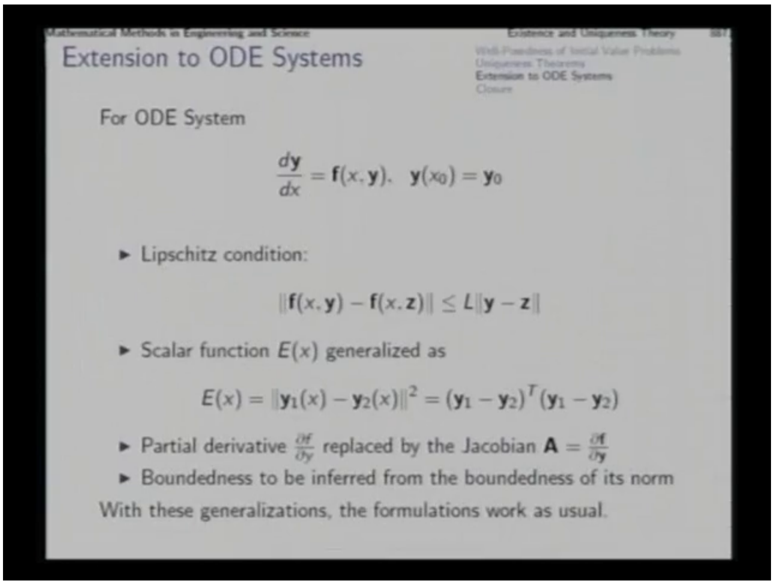


Now, this entire discussion can be directly extended to ODE systems also rather than a single ODE. Say for ODE system where y is a vector function of x and d y by d x is equal to this vector function f which is given and the entire initial condition is given that is y 1 y 2 y 3 y 4 all values at x 0 is given. Now the previous discussion that we had can be applied as it is with very little modifications in the meaning, rather than mod absolute value in the Lipschitz condition here you will have to put norm. So, f is affected function, so in place of f x y minus f x z mod you can make it an norm. So, now, in the (Refer Time: 48:15) space sense it will be the norm of this vector difference and that has to be less than equal to Lipschitz concept in to norm of y minus z. This scalar function which was earlier y minus z whole square, now becomes or y 1 minus y 2 whole square now will become y 1 x minus y 2 x norm square which is y 1 minus y 2 transpose y 1 minus y 2.

Partial derivative del f by del y will somewhat change because y is the vector now vector function and f is also a vector function. So, in place of del f by del y we will now have the Jacobian. So, which is the derivative of a vector function which is (Refer Time: 49:00) vector variable. So, Jacobian a will be del f by del y.

Now, we said in Picard's theorem that premise the premise of the Picard theorem need del f by del y to be bounded, now you will have to say that the this matrix should be bounded in what sense that is its norm should be bounded there is a maximum magnification that it can produce that size should be bounded with this realizations than if this condition and its power and its implications will remain as usual as the scalar case. So, these are the generalizations with which the formulation was exactly the same as in the scalar case. Now, let us take this much of background theory and apply that in two very important cases which will form as important necessary background for some of our theoretical developments in later lectures on the theory of differential equation.

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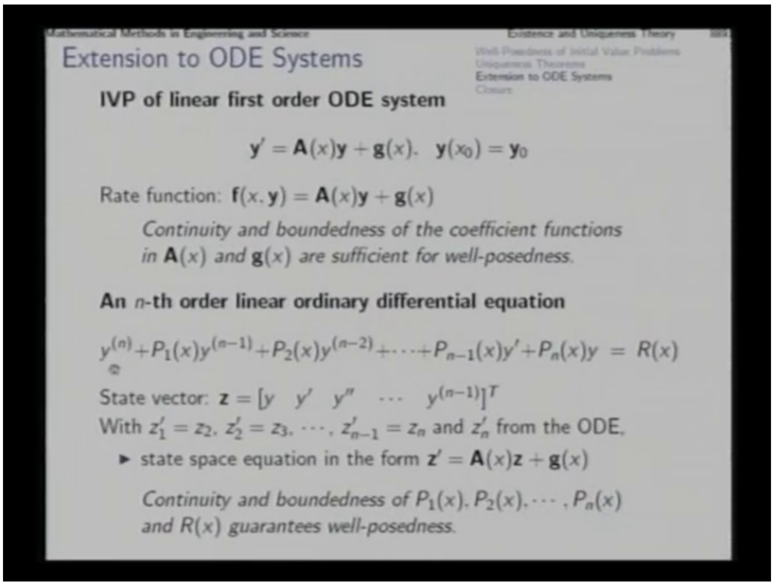


First is the case where we apply this theory on a system of linear first order ODE linear first order ordinary differential equation. So, here y prime which is f of x y that is A that will give us a linear system of ordinary differential equations like this where this A is function of x only does not depend on y that x with multiply with y plus g x. So, this gives us a linear system of first order ODE's. So, all this differential equation that we get are linear differential equations.

And the initial condition is given, now here you see that f of x y is this function and then del f by del y that will be the Jacobian of this function with respect to y that will be simply the matrix A right. Now that means, that the continuity of boundedness, continuity and boundedness of the coefficient functions that is a 11, a 12, a 13, a 21, a 22, a 23 etcetera this things and the continuity and boundedness of g 1 x g 2 x g. So, coefficient functions here and these functions g 1 x g 1 2 x here they will be enough to establish the continuity and boundedness of f as well as del f by del y which is simply A. So that means, therefore, a linear system of ODE's you can say that as long as the elements of this matrix function of x and elements of this vector function of x as long as these functions are continuous and bounded you know that both f and del f by del x del f by del y are continuous bounded and then you can say that any initial value problem with this differential equation; that means, with any initial condition will be well posed.

Similarly if you have an n-th order linear ordinary differential equation then first thing is that you try to take this n-th order linear differential equation consider this state vector y y prime y double prime up to the n minus 1-th derivative of y these are the state variables this constitute the n dimensional state vector. And then with z 1 prime being z 2 prime being z 3 up to z n minus 1 prime being z n and z n prime which is the derivative of this that is y n which you get from the differential equation you will get R minus this minus this minus this minus this all of this right and in this the y, y prime etcetera the derivatives they will be supplied in terms of z 1 z 2 z 3 etcetera. So, you find that this matrix A will be composed of 1s and 0s and these functions from here these functions from here and R right. So, that will all that will come in this coefficient functions in this matrix a which is going to be from the Jacobian and this right.

So, again here we will find that as long as the coefficients functions p 1 x p 2 x p 3 x etcetera and this right hand side function R x as long as these functions of x only not involving y as long as this are continuous and bounded we will find that guarantee for well posedness of the initial value problem no matter what are the initial conditions. So, for linear systems of differential equations and linear differential equations of higher order also which immediately can be converted in to systems of first ordered differential equations for these cases, as long as the differential equations are linear we find that it is only the coefficients function here a x and g x similarly here this p 1 x p 2 x etcetera and R x it is only the coefficient functions which are functions of x only the continuity and boundedness of which we need to examine and that will mean that with any initial conditions that we get in hand we will find that initial value problem is well posed. So, the initial condition according to this from one initial point to another initial point we do not have to check again and again it is the coefficient functions which are functions of x only which will decide the case for the well posedness.

So, these are important result with which we will be going applying again and again in the theory of differential equations that we take up in the next few lectures.

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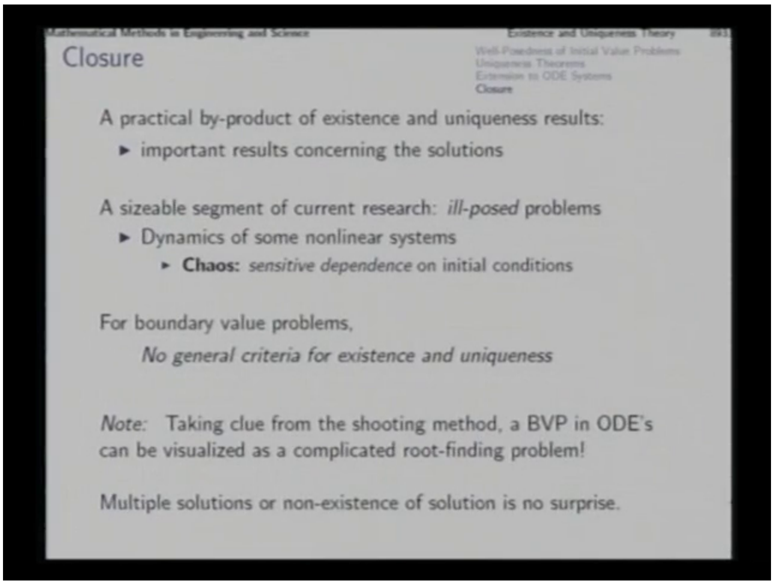


So, in closure let us quickly summarize the important issues a practical by product of the existence and uniqueness result which we just now established and which will be applying. In the case of the theory of linear differential equations is some important results concerning the solutions later when we study the properties or some of the solutions of linear differential equations these result of existence and uniqueness will help us in establishing some of the results very easily.

Now, one comment is that we have been talking about well posed problems a sizeable segment of current research on dynamic systems concerns ill posed systems and in which in some cases in study in the study of dynamics of some non-linear systems we find a situation a phenomenon called chaos which is actually characterized by a sensitive dependence on initial conditions. Now, apart from all this discussion on initial value problems if you ask this question that like Lipschitz condition, like Picard's theorem can we work out some general enough criteria for the well posedness of boundary value problems also for BVPs.

Now, the first issue is the point is that for boundary value problems there is no such general criteria for existence and uniqueness and that makes sense because if we take a clue from the shooting method then we find that a boundary value problem in ODE is can be considered as a very complicated root finding problem as we discussed in the previous lecture. That means, in this root finding problem we have to really solve a system of non-linear equation and as we know for a system of non-linear equations sol solutions are never guaranteed there is no a priory criteria for that one which we can take and similarly typically we express multiple solutions and therefore, for boundary value problems in such situations typically will be expect that there will be situations where there will be multiple solutions or there will be no solution at all. So, such situations may arise and that is not a surprise and therefore, for BVPs we cannot work out such general criteria like Picard's theorem or the Lipschitz condition and so on.

So, with this we close this issue of numerical solution of differential equation and their meanings as we come out with the well posedness of the initial value problems.

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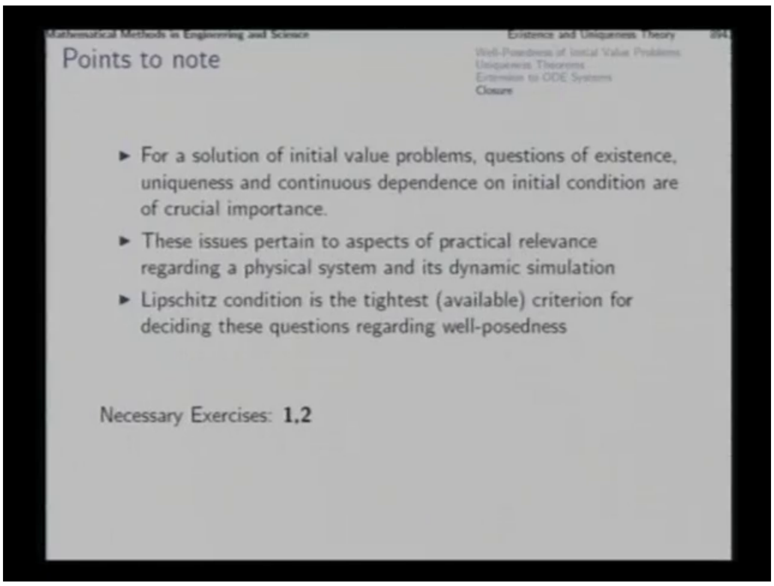

And in the next few lectures we will consider the solution of differential equation the theoretical results of that, the theoretical aspects of that starting from first order differential equations in the next lecture.

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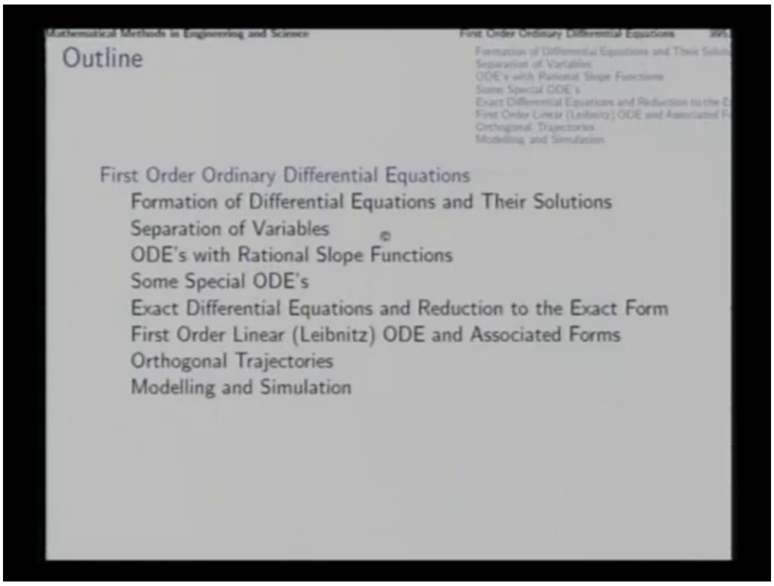


Thank you.