Mathematical Methods in Engineering and Science

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Module - V
Selected Topics in Numerical Analysis
Lecture - 01
Interpolation

Good morning. Starting from this lecture in a few lectures we will cover the classical topics of numerical analysis.

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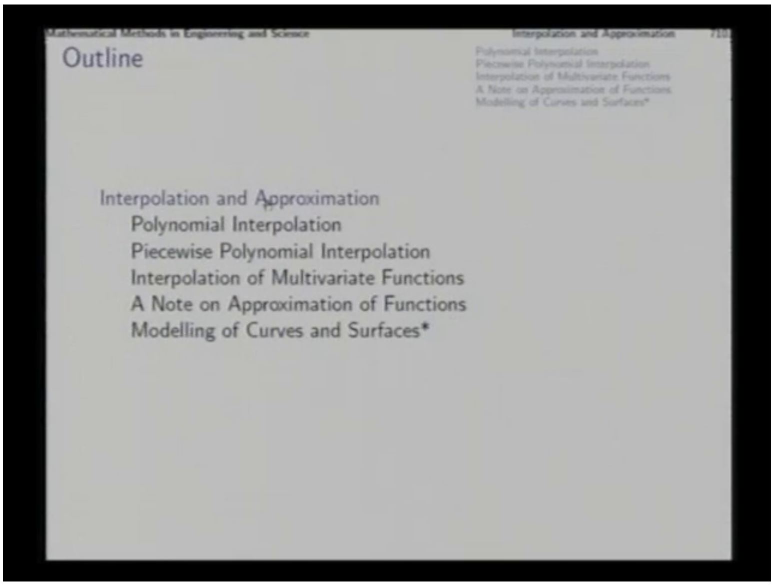

Function interpolation then numerical integration and numerical solution of ordinary differential equations first. In this lecture, we discus interpolation and function approximation.

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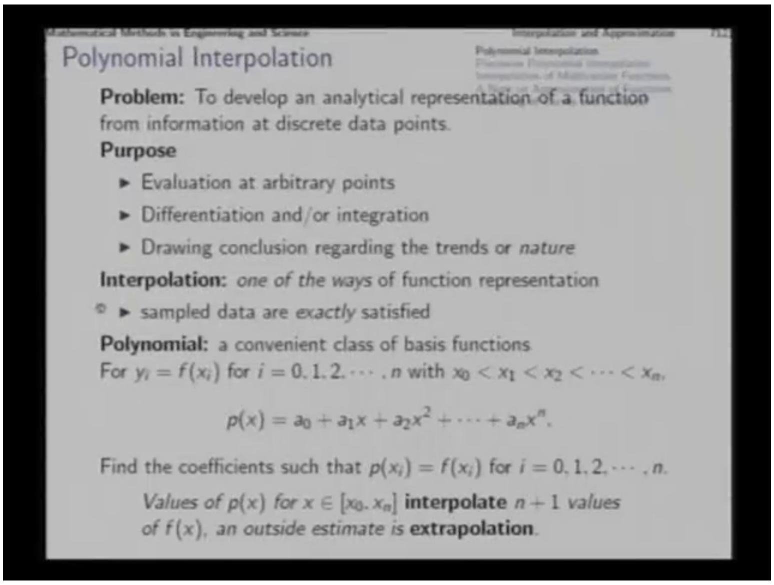

The why just used interpolation of functions is through polynomial. So, first we discuss polynomial interpolation the problem here is to find an analytical representation of a function from information at a finite number of data points for example, suppose you have got a function of x for which you have data regarding the function at several data points say here, here, here and here and from that you want to develop the function in the form and in the form of an expression. So that, with that expression you can develop the function continuously and evaluate the function at any point in between. So, that is called function interpolation.

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Now, the purpose is to evaluate at arbitrary point in which at which your data is not there and apart from that you if you have an analytical representation in the form of an expression, then you can differentiate or integrate the function as the need arises besides once you have an expression for the function you can draw certain conclusions regarding the trends of the function or nature of the function. So, with these purposes in mind we try to develop analytical expressions analytical representation of function from if information available at discrete data point.

Perhaps those data point those discrete information can be obtained from experimentation or through some other computation which cannot be conducted at infinite points. Now interpolation which we are going to considered; now which we are going to discuss now is one of the ways of function representation there are other ways also and in the interpolatary approximation of a function the sampled data the data which is given are exactly satisfied now polynomials offer a convenient class of basis functions as linear combination of which we can express the function quite easily.

So, in a particular problem suppose y i as f of x i is available at this n plus 1 points x 0, x 1, x 2, x 3 up to x n with the help of these n plus 1 pieces of data, we want to fit a polynomial of the n-th degree. Now 5 n x degree and n plus 1 points because a polynomial of n-th degree will have n plus 1 coefficients the determination of which will required n plus 1 pieces of data and that n plus 1 pieces of data you get from the function values at these n plus 1 points starting from a x 0 up to x n.

Now the task of formulating or finding this polynomial is basically the determination of these a 1, a 2, a 0, a 1, a 2 coefficients which will match the given n plus 1 pieces of data exactly that is interpolation and after that at any other value of x in this interval if we try to find out the value of p x from this polynomial that will in a way represent the function f at that point and this is what is called interpolation finding the value in a at an in between point if based on this same expression if we try to find out the function value or estimate the function value at a point outside this interval such an activity is called extrapolation.

Which for all situations is not considered safe and reliable enough now one question arises that for given values n plus 1 values of the function at n plus 1 points can we always find a polynomial of n-th degree like this and the other question that arises that if we can find, then will a single set of values for this coefficient will be found or further we can find multiple sets of values for this coefficients that is this polynomial fitting this data unique first of all whether it exists for all data of this curve and whether that is unique whether the polynomial exists which will interpolate this data and whether that polynomial is unique.

Now, if we just try to insert the values of x 0, x 1, etcetera in this and equate the corresponding expressions to y 0, y 1, etcetera, then we get n plus 1 equations in these coefficients.

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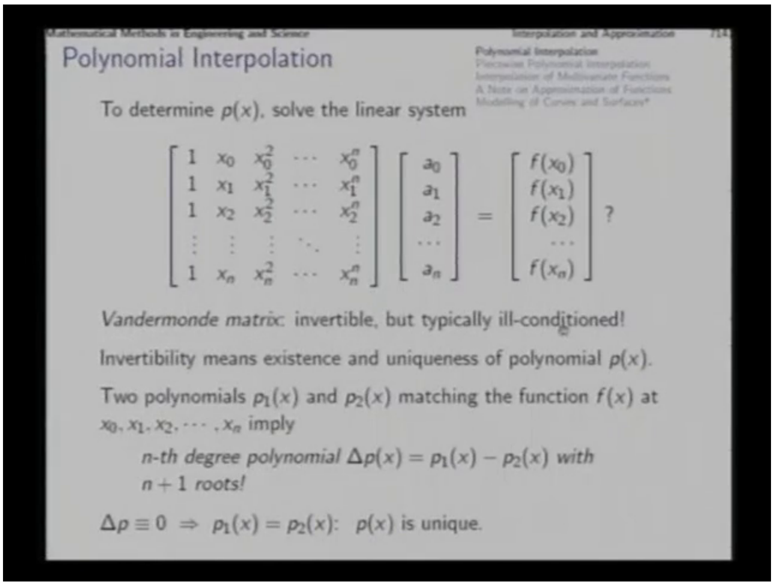

These equations are here a 0 plus a 1 x 0 etcetera up to a n-th 0 to the power n equal to function value at x 0 and so on. Now the values of a 0 a 1 up to a n will exist and there will be unique if this coefficient matrix is nonsingular now this particular matrix with different values of x 0, x 1, x 2, x 3, etcetera is a well-known matrix known as the vandermonde matrix and it is known to be invertible you can analytically show that it is invertible and, but then it is typically ill conditioned.

Now, you analytically show it is invertible, but here we will not take up that exercise because we are going to establish the inevitability of this matrix that is existence and uniqueness of these coefficients that is the polynomial through some other short cut means. So, therefore, we will not spend our time on formally showing that it is invertible from the existence and uniqueness of the polynomial coefficients the invertible will be established.

But keep also in mind this fact that is even though this matrix is invertible, but typically it is ill conditioned that is the solution process will quite often have numerical errors because of the ill conditioning. Now we try to see the uniqueness of this polynomial coefficient set suppose there are 2 polynomials; that means, 2 sets of values for this coefficients which match the function values at the given data points and those 2 polynomials are p 1 and p 2.

Now what can we say about p 1 minus p 2 that different polynomial. Now p 1 is an n-th degree polynomial p 1, p 2 is also an n-th degree polynomial. So, their difference will be at most and n-th degree polynomial. Now at n plus 1 data points p 1 matches it exactly; that means, at those n plus 1 points the p 1 polynomial; polynomial p 1 will have a exactly this values p 2 also. So, the difference of these 2 polynomials will have 0 values at n plus 1 points that will mean that delta p x is an n-th degree polynomial of x, but at n plus 1 values of x it has got 0 value.

Now, you know from the fundamental theorem of algebra that a n-th degree polynomial will have exactly n roots it cannot have n plus 1 roots right and; that means, that if delta p satisfied such a requirement that will mean that it is not only 0 at those n plus 1 values, but this the 0 polynomial; that means, the entire polynomial is 0; that means, all the coefficient are 0 which we will mean that delta p is 0 as a polynomial will mean that p 1 and p 2 actually the same polynomial that shows that p x is unique and p x is p x exists that we can argue based on the properties of linear system of equations, but let us go another round and existence also we established in a little shortcut argument.

How for that we do something which would have done otherwise also because we know that solution of this will require us to solve an ill conditioned system. So, typically may not like to do that for large value of n that is for high degree polynomials. So, what we can do the alternative is Lagrangian interpolation.

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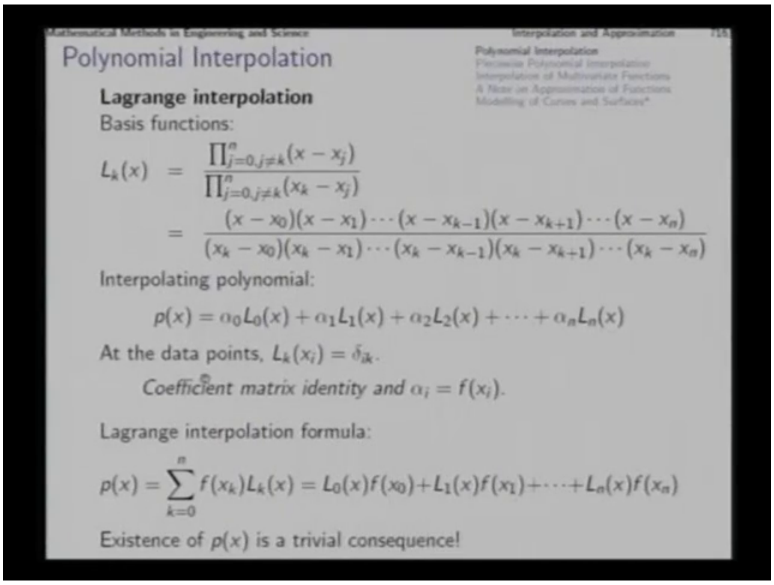

For that we considered the basis functions not as 1 x x square x cube the way we did last time in this function representation the basis functions where 1 x x square x cube etcetera and coefficients were a 0, a 1, a 2, etcetera.

So, we have noted that with this kind of a polynomial expression it will be very easy to differentiate and integrate etcetera, but then evaluation through the solution of this kind of system is going to be unnecessary time consuming. So, what we can do is that we can use basis functions not as 1 x x square etcetera, but this kind of basis functions examine carefully in this expression we have got in the denominator product of x k minus x j j running from 0 to one except j equal to k of course, because for j equal to k this will become 0.

So, that term we do not have all other terms we have now in the numerator in place of x k we have a x. So, the denominator is a constant number so; that means, for the second basis function k equal to 2 we have got here x 2 minus x 0 x 2 minus x 1 x 2 minus x 3 x 2 minus x 4 and so on up to x 2 minus x n and that is a number in the numerator we will have the corresponding x minus x 0 x minus x 1 x minus x 3 x minus x 4 etcetera up to x minus x n omitting x minus x 2 so; that means, that this numerator is an n-th degree polynomial in x denominator is a number. So, L to x is a is an n-th degree polynomial similarly, L 0, L 1, L 2, L 3, L 4 all of them will be n-th degree polynomial.

So, we have got n n plus 1 n-th degree polynomials. So, any linear combination of them also will be an n-th degree polynomials and this is the linear combination. Now here the basis functions are these functions L 0, L 1, etcetera and alpha 0 alpha 1 etcetera are the corresponding coefficients to construct the polynomial now these functions x 0 L 1 etcetera basis functions at x 0, then note that L 0 will be found to be 1 at x equal to x 0 because in that expressions for the expression of L 0 this x minus x 0 will be missing all others will be there and here x 0, x 0, x 0 will be there.

So, when we put x equal to x 0 this and this will cancel next cancel and so on all of them will cancel and you will get the value one at that same point x 0 if we try to evaluate L 1 then in the expression of L 1 this term will be missing this term will be there. So, at x 0 when you try to evaluate, then we will get x 0 minus x 0 which is 0 right and; that means, that the expression and here it will be x 1 minus x 0 right so; that means, that L 1 L 2 L 3 L 4 all of them will be 0 at x equal to x 0 so; that means, that suppose this is x 0 this is x 1 this is x 2 this is x 3 and this is x 4 so; that means, L 0 will be 1 here, 0 here, 0 here, 0 here, 0 here, L 1 will be 0 1 0 0 0, L 2 will be 0 0 1 0 0 at other places which are not the given data points all of this expressions will have some other values, but at the j-th data point L j will have value one all others will have value 0 and therefore, in this when we try to equate the function values at x 0, x 1, etcetera, then we will find that at x 0 when we try to put the value then we will have alpha 0 in to one plus alpha 1 into 0 plus alpha 2 into 0 and so on. So, we will get alpha 0 into one is equal to affect x 0.

So, immediately we will get alpha 0 as the function value itself at x 0 similarly alpha 1 will be the function value at x 1 and so on. So, at the data points we have this that is at the at as k is equal to i, we have one and k not equal to i we have 0 so; that means, that basically we are trying to solve a system of equations in which the coefficient matrix is identity this and; that means, alpha i is going to be jus f of x y y i so; that means, the function values are the coefficients here alpha 1 L alpha 0, alpha 1, alpha 2, etcetera. So, Lagrangian interpolation formula we get like this now note that in this development from here to here we have done nothing which will not be possible for arbitrary data arbitrary points and arbitrary function values; that means, for any set of points and for any function values this polynomial will always exist and this is an n-th degree polynomial and earlier we have seen that whatever energetic polynomial.

We can find to exactly satisfy n plus 1 data points that as to be unique so; that means, we basically get this polynomial which is a same as the polynomial that we would have got otherwise through the earlier process except that here we get the polynomials very easily without any difficulty, but then differentiation integration etcetera will be more cumbersome from this kind of expressions. So, here you will find that.

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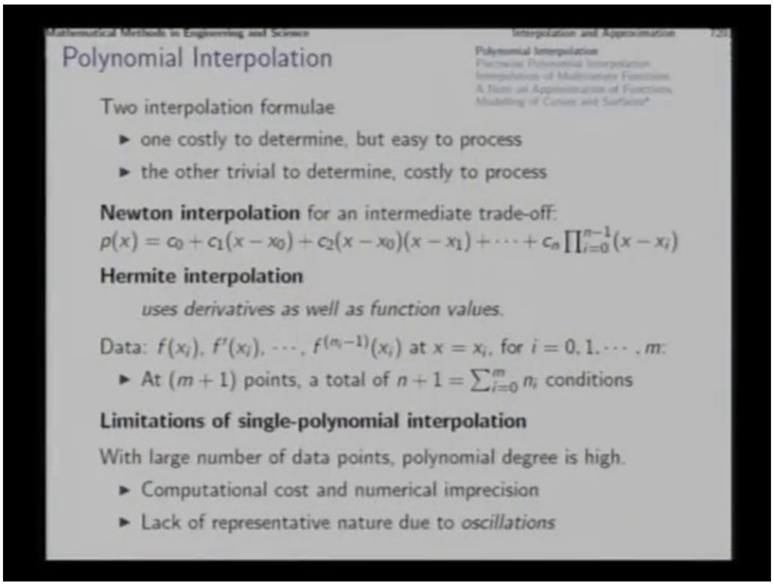

We have got 2 interpolation formulae one costly to determine with the basis functions 1 x x square etcetera, but easy to process afterwards the other trivial determine the Lagrange interpolation, but costly to process later differentiation integration etcetera depending upon your situation depending upon your need you will be in a position to frame the polynomial coefficients and the basis functions either this way or that way.

If you want a formula a methodology by which it will not be. So, costly to determine on the other side it will not be. So, costly to process you have an intermediate case also that is the newton interpolation which is an intermediate trade off where you have got the formula in this manner. So, in this case if you try to put x equal to x 0 you will find all these are 0s.

So, you will get c 0 is equal to f of x 0 then when you put x equal to x 1 you will have this and this and all others having a factor x minus x 1 will become 0. So, you will get another equation is c 0 and c one and. So, on. So, this will have a triangular coefficient matrix. So, it is easier to develop compare to the first case and more difficult to develop compare to the second case Lagrangian interpolation on the other hand the processing the differentiation integration etcetera also will have intermediate computational cost.

Now, another kind of interpolation is there which uses not only the function values, but also some of the derivatives such an interpolation is called Hermite interpolation it as a lot of applications because in many situations many problems you need to satisfy not only the function values, but the rates as well they are part of the problem that is you cannot have a have a polynomial which changes in a rate which is different from what is already known to be the rate of change of that function which you are trying model this is one important points the second important point is that some times with the help of interpolation which uses derivative information also at certain points or in the locality in the local neighborhood of certain points we can establish we can develop function approximations which have the required trends.

So, therefore, Hermite interpolation is quite well used kind of interpolation. So, in this what kind of data he will talk about you will still have a few points say x 0, x 1, x 2 put to x m a hand full of points at which we will use the data and at every point the data may be function value and some of its derivative it is not even necessary that at every point you have the same number of the derivatives to be used it is possible that at x 0 the function value and 2 derivatives are given at x 1 the function value and one derivative is given at x 2 only function value is given at x 3 the function value as and one derivative is given that is possible now whatever be the number of data items function and its derivative values at every point. So, if you add up those number of thus those conditions the total number of condition that is given in terms of function values and their derivatives that will equal one more then the degree of the polynomial that you can model.

Why because this many conditions will give you the number of that many equations in the unknown coefficients of the polynomial and if you have got n plus 1 coefficients in the polynomial; that means, you can go up to nth degree n plus 1 equations n plus 1 unknowns. So, the number of the conditions will be one larger than the degree of the polynomial by up to which you can correctly model the function. Now that way you can think of a function in which at this point 3 pieces of data are given at this point 2 at this point 4 at this point again 2 and this point one.

So, what is the total number of conditions that you have got 3 2 4 2 1; that means, dual conditions; that means, through this points satisfying all these functions values and derivatives values that are given you can make an eleventh degree polynomial which will fit this data exactly, but then there are very strong objections to high degree polynomials as function representations one is of course, the computational cost and numerical imprecision that is you will have to solve a 12 by 12 system of equations and that may be in condition.

So, that may lead to computational cost and numerical imprecision, but this is a minor objection the major objection is that the high degree polynomials sometimes fail in their prime duty to represent the function well let us see how.

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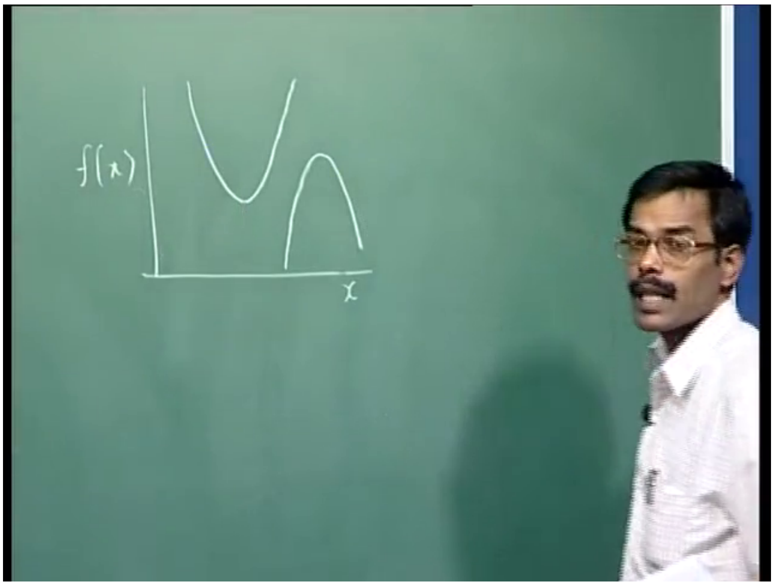

As you know a linear function of x either goes all the way up or goes all the way down or straight constraints in the particular exceptional case that is you if you remove this then you will say that typically a linear function goes all the way down or goes all the way up a typical quadratic function as this kind of a loop it goes down takes one turn and then goes up never takes another turn or it goes up takes a turn then goes down this will be the case if the coefficient of x square is positive this will be the case if the coefficient of x square is negative.

So, a quadratic function of degree 2 has 2 turns once going down and once going up right a cubic equation a cubic expression a cubic function that is degree 3 will have at most 3 such tends.

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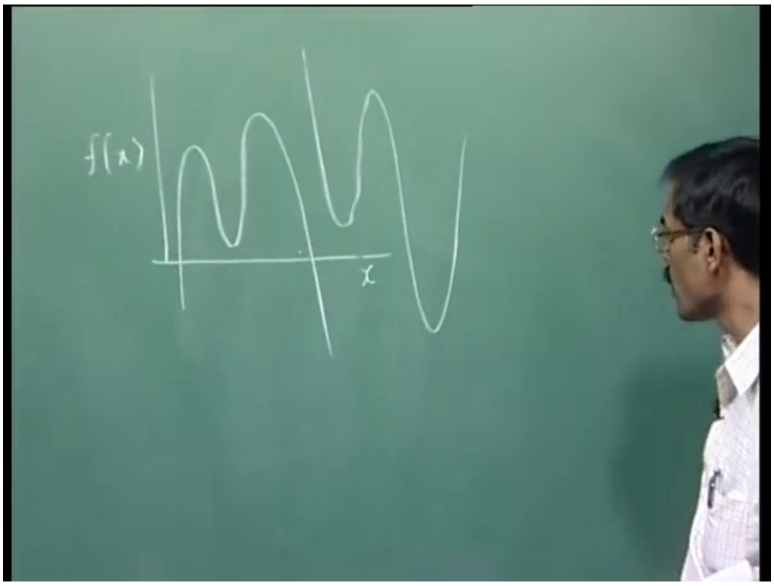

Going down then going up and then possibly going down again I said at most because a cubic could also go like this that is also possible, but then a cubic as the possibility of showing this king of a trend a 4 degree polynomial. Similarly will have the possibility of showing a trend of this kind or of this kind this is situation with 4 degree function.

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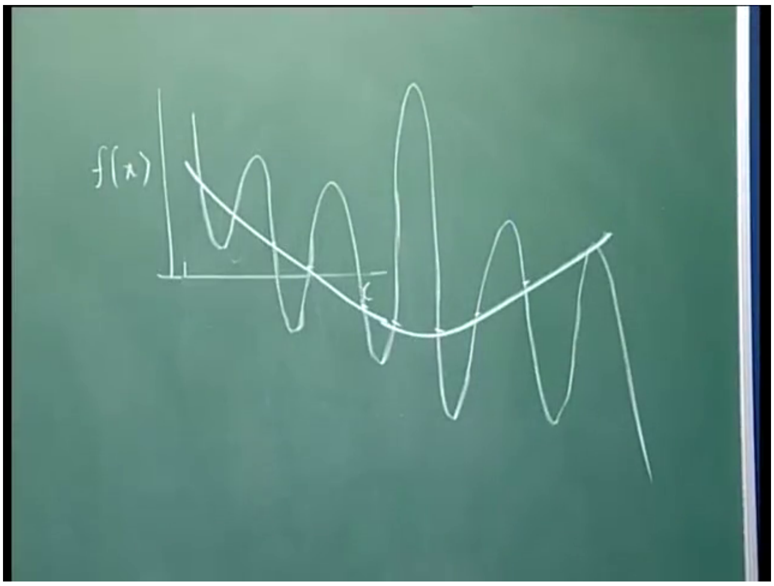

Now, if you talk of an eleventh review function that may have scope to 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 this kind of a profile it might show that up to that much it can oscillate; that means, that if your data are like this it is possible for an eleventh degree polynomial to give you this representation which is a very bad representation because through this data points the function that you were possibly approximating was this; it is a nice function, but because you allowed and eleventh be polynomial the polynomial representation turn out to be this which is matching the data exactly, but other than that it is join doing nothing sensible.

So, this is the prime objection to high degree polynomials representation of functions that it may be failing in the prime duty that to represent the function faithfully that itself may be failed in a problem of this sort. So, that is why single high degree polynomial interpolation is something which we usually do not do, then what do we do we make pieces wise polynomial interpolations ok.

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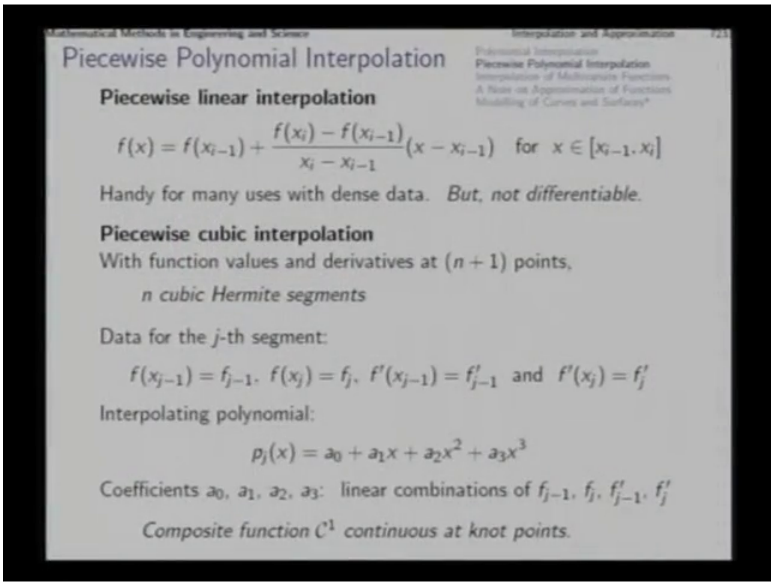

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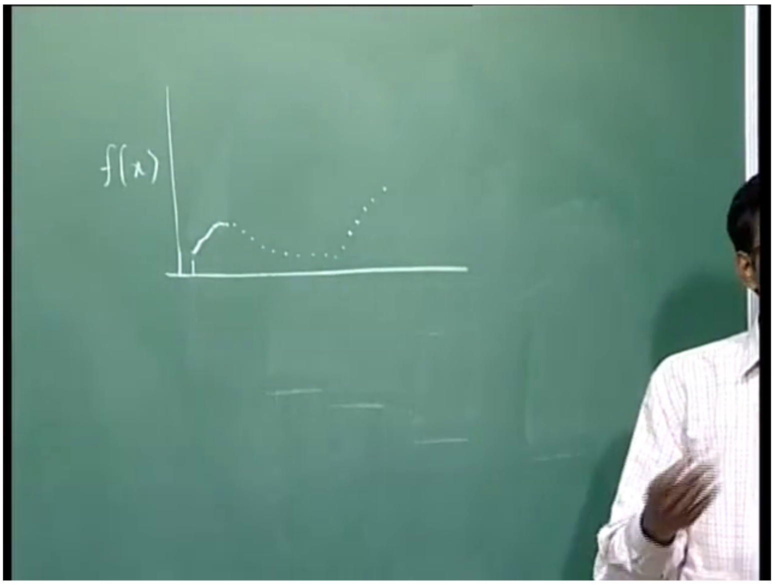

Simplest pieces wise polynomial interpolation is piece wise linear interpolation so; that means, that if we have a large number of data points like this then one state forward sensible way to find a pieces wise interpolation of this could be to join these with straight lines and say that is a function representation which is good enough for many purposes. So, for dels data where function values are available at a large number of close by points there quite often we conduct pieces wise linear interpolation, but we have to keep in mind that this representation of the function will not be differentiable because at the corners from this side there will be some slop and from the side there will be some other slope.

So, this is not be differentiable. So, piece wise linear interpolation is not differentiable you can see that this formula gives you the piece wise linear interpolation between x i minus 1 and x i, you will get this formula which matches x i and x i minus 1 exactly and it is a linear expression next higher interpolation typically used and very popular representation is pieces wise cubic interpolation now this is typically Hermite segments what you do in this pieces wise cubic interpolation is that at the data points you have got the function values and the derivative values and then for every pair of point you try to frame a cubic with the help of two values here function and derivative and 2 values there you see function and derivative value at the 2 end points of every segments give you 2 plus 2 4 conditions and with those 4 conditions you can determine the 4 coefficient of a cubic exactly and that you go on determining for every segment, so far that what you will get.

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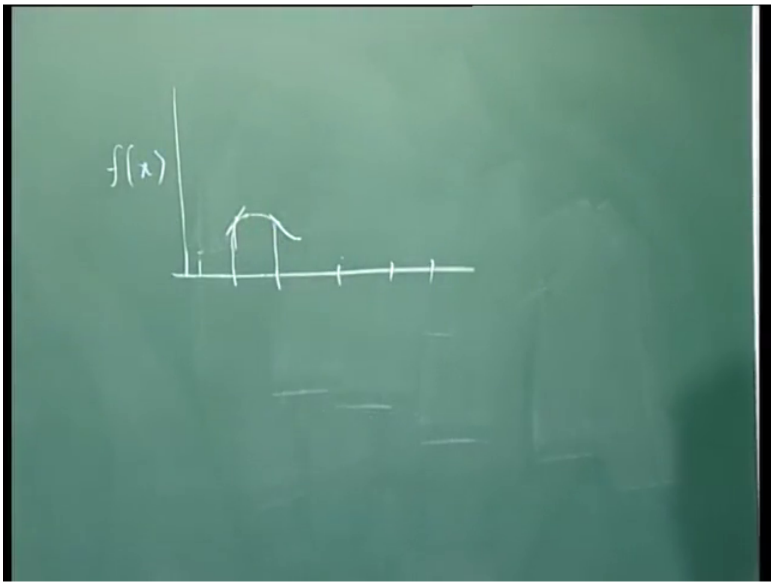

So, for every segment you will get here you have got the function value and the derivative here you have got the function and the derivative.

So, the cubic that you will be framing will perhaps be this cubic, but then the part of it which is outside this sub interval that will not be used and you will have only this much right and note that is since the adjoining segment between these 2 points will also use this function value and this derivative value and match that data exactly the trend of the function value here will match this segment up to the first order derivative and this way you will get a curve a function which is differentiable up to first order. So, you will have an exact interpolation with first order derivative. So, the data for the j-th segment will be functions value at point j minus 1 at point j and derivative values at the same 2 points with the help of these 4 pieces of data you will get the 4 coefficient from here because as you force this data values on this interpolation you will get 4 linear equations in a 0, a 1, a 2, a 3 and then as you solve them you will find that the coefficient will turn out to be linear combinations of these 4 items of data this and if the coefficient are linear interpolation of these then you will get a composite function which is c one continuous at note point; that means, at the junction of the intervals.

They will have first order continuity that is the derivative is also first order derivatives also continuous in between the naught points that is during the segment of course, it will have infinite order differentiability because they are single polynomials it is the differentiability at the junctions which we are trying to establish the differentiability in the interior of a segment is of course, up to whatever order you want. Now this same formulation can be made in a general setup if we reparametrize in a normalized interval say 0 to 1.

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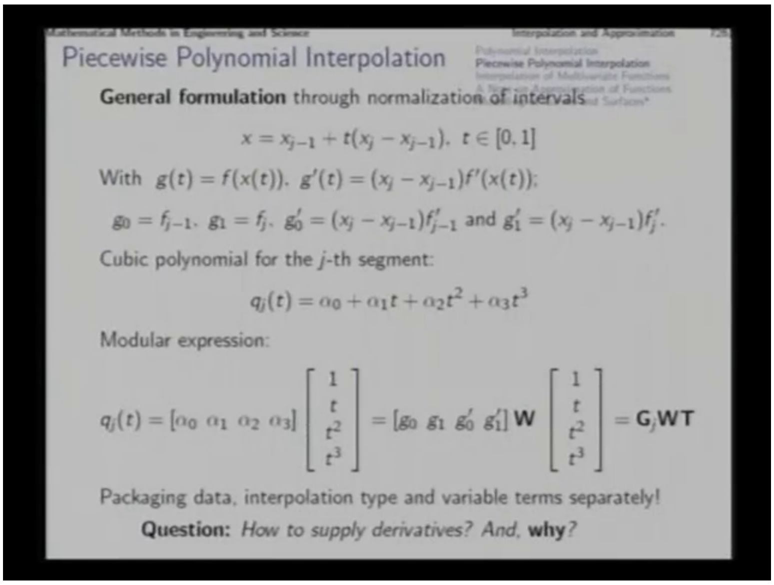

So, that is quite often done and the result is something quite useful say the variable x for that you have the interval say x 3 to x 4 j equal to 4. So, x 3 to x 4. So, now, between x 3 and x 4 if you put another variable t in terms of which you will reparametrize the scale, then you are at t equal to 0 you will have x 3 x j minus 1 at t equal to 1 you will have this and this will get canceled you will have x j.

So, the interval 0 to one will scale up to the x interval that is x j minus 1 to x j and then the function f of x can be talked of highest f of x of p which you can call as g of p and then g prime from this expression will be derivative of f with respect to x and then into derivative of x with respect to t which you get from here x j minus x j minus 1. So, whatever is the derivative given in terms of f prime that can be mapped in terms of g prime? So, with g mapped like this and g prime mapped like this both functions of t you will have this as the representative data. So, this with this 4 pieces of data you determine the cubic polynomial for the j-th segment right.

Once you do that you can note something interesting this q j t you can write as the rho vector alpha 0, alpha 1, alpha 2, alpha 3 into 1 p t square t cube like this and we have already seen that alpha 0, alpha 1, alpha 2, alpha 3 will be linear combinations these 4 pieces of data; right and with this fixed interval 0 to 1; these coefficients of these linear interpolations this is linear combinations are going to be constant and therefore, you can have a 0, alpha 0, alpha 1, alpha 2, alpha 3 as constant in terms of numbers non-linear interpolations of these 4 pieces of data and that those linear interpolations those coefficients will turn out to be a matrix we will constitute a 4 by 4 matrix W.

In the text book there is a small exercise in which the steps to evaluate this matrix W is given and I suggest that you attempt at problems. Now with this representation we can write for the j-th segment g j into W into T. Now this way we have packaged the data interpolation type and variable terms separately; that means, g j this vector as nothing other than the items of data this W has just constant numbers which are which reflect the type of interpolation that we have decided upon that is in this case pieces wise cubic. So, that Hermite taking function values and derivative values at both the ends of the sub intervals based on that decision of the interpolation type we get this matrix W.

If we had taken some other interpolation type you would get some other matrix W there and in the third segment t we have the variable terms which will get their values at the time of a valuation of the function at a particular point. So, this modularity of representation we get if we convert if we rescale every sub interval to the normalized interval 0 to 1 and the same expression the same formula we can use for every segment of one composite curve and for every segment of every other composite curve for that matter everywhere we use the same matrix W of fixed numbers and the data changes and at the time of evaluation we have to give this values of T.

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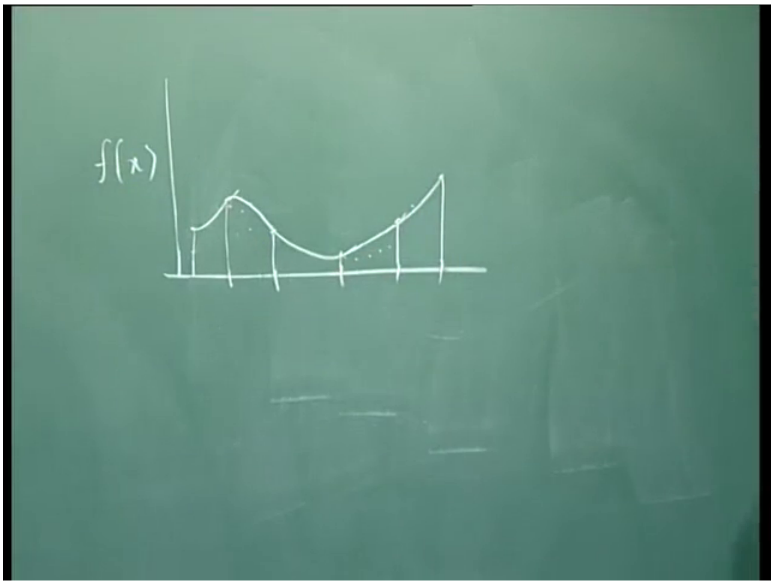

Now, the question is arises that is there are many situations where we have got the function values at several points through which we want a smooth curve smooth or say continuous curve differentiable; differentiable function passing through all this points now at these points if we do not know the derivative values then how do we supple it to a Hermite interpolation program Hermite interpolation algorithm the requirement of the first derivative becomes a problem that is if the problem as it is does not come with the derivative data, then how do we supple them and also the question arises why should we supple can we not say that we do not care what derivative is here as the curve goes from this segment to this segment we do not care in which direction it goes whether it goes like this or this all that we care is that whichever way it terminates at the end of this segment exactly in that direction is should comments at the beginning of the next segment and so on only this much we want to say.

So, in that case we do not need to give the derivative values, but we can demand first order continuity and if we do not want to specify the derivative values there then we can demand not only first order continuity, but we can demand second order continuity as well note that when we were earlier giving the value of the derivative here at this points then through that value actually we were supplying 2 conditions one is that as part of this segment the function should have this slope here and as part of this segment the function should have this slope here. So, at this point through the derivative we were actually giving 2 conditions now that we want to Hermite that piece of data all that we say is that we are going to demand first order continuity of derivatives at this point that is whatever is the derivative here.

We do not care similarly whatever is the derivative on this side we do not care all that we need is that the derivative at this point from this segment and the derivative at this point from this segment should be equal; that means, that the derivative value we are not concerned about, but derivative continuity we are concerned about. So, when we ask for the derivative continuity at this point then we are actually specifying only one condition now we have a slot for another condition, so from where to get that condition. So, we can say that at this point we want the continuity of the second derivative also. So, that way with the help of less data that we supple from our hand that is derivative value we are not supplying, but we are demanding 2 conditions first derivative continuity and second derivative continuity, if we do that that is we can do that when the requirement does not specify the derivative value.

So, when we do that we can develop a function representation which is not only first order continuous, but second order continuous across the junctions and that kind of an approximation is called spline interpolation.

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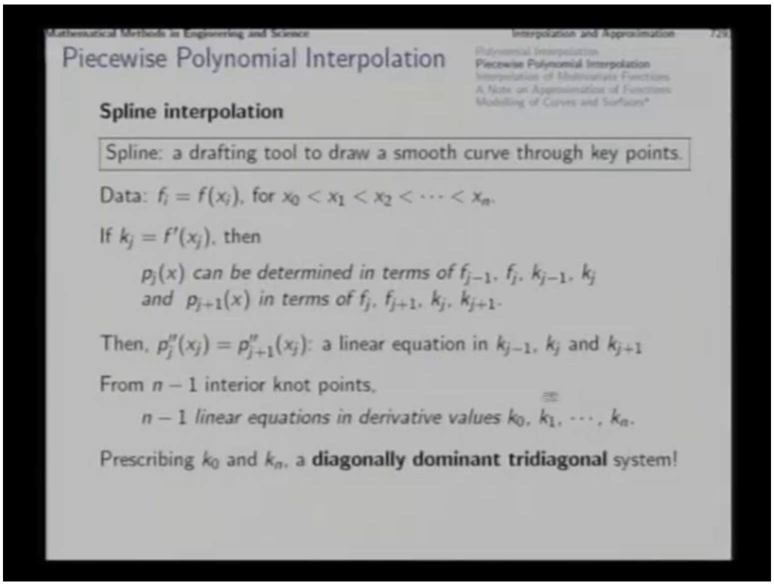

Spline basically it is a drafting tool to draw a smooth curve through key points which are already marked on the drawing board now from that particular drawing drafting tool the spline interpolation as got its name how do we do this. So, there are n plus 1 points at which the data values are given.

Now, if the derivative value at the junction points are kj unknown to us currently then p j the j-th segment say; let us take j equal to 0; that means, the first segment. So, this segment this segment can be planned or determined in terms of f j minus 1 f j k j minus 1 k j; that means, function values at this point at this point and the rate here the rate here right. So, in terms of these 4 quantities we can we can determine this segments right similarly for this segment we will required the function values here here k j and k j plus 1 right. So, this segment is 0 minus based on function values at these 2 points which are known and k j minus 1 and kj. Similarly the segment here this one can be determined based on the function values at this point this point kj and kj plus 1, right if. So, then the derivative at this point can be determined based on the same 4 pieces of data and similarly on this side.

So, apart from the known quantities the slope here can be determine in terms of k j minus 1 and k j from this direction and k j and k j plus 1 from this direction all that we want to do is to equate these 2. So, the second first derivative is is going to be same because k j is used on this side and as well as on this side now the second derivative also you can evaluate and equate on both side that is wherever is the second derivative here that is in terms of k j minus 1 and k j and the second derivative from this side will be in terms of k j and kj plus one. So, these 2 quantities we determine and equate.

So, p j is determined based on these 4 quantities p j plus 1 is determined in terms of these 4 quantities. So, the derivatives second first derivative continuity is ensured because k j is used here and the same k j is used here. Now if we develop the second derivative from this expression and the second derivative from this expression and evaluate both of them at the same point at the junction point that is at the end point of this interval and at the beginning point of this interval and equate these 2 that will give us one equation in k j minus 1 k j and k j plus 1 in these 3 unknowns.

So; that means, that at the junction 1, 2, 3, 4 will get equations in k 0, k 1, k 2, k 1, k 2, k 3, k 2, k 3, k 4 and so on. So, we will get n minus 1 such junction points and at that will get n minus 1 such equations in k 0 to k n plus 1 and this equation system will be a tri diagonal system and as it happens it gives us tridiagonal diagonally dominate system of equations. So, from the n minus 1 interior junctions we will get n minus 1 linear equations in the derivative value these are n plus 1 values.

So, to determine them completely, we will need 2 values 2 of these n plus 1 unknowns we will need and then the rest of n minus 1 can be determined typically the derivative value at the beginning and at the end are supplied and then the rest of them can be determined from this n minus 1 linear equations together framing a diagonally dominant tridiagonal system. So, this way we get a smooth curve like this which matches the second derivatives at the junction points and in between the junctions in a sub interval of course, it is differentiable up to infinite order that is of course, known. So, this is called spline interpolation with c 2 continuity.

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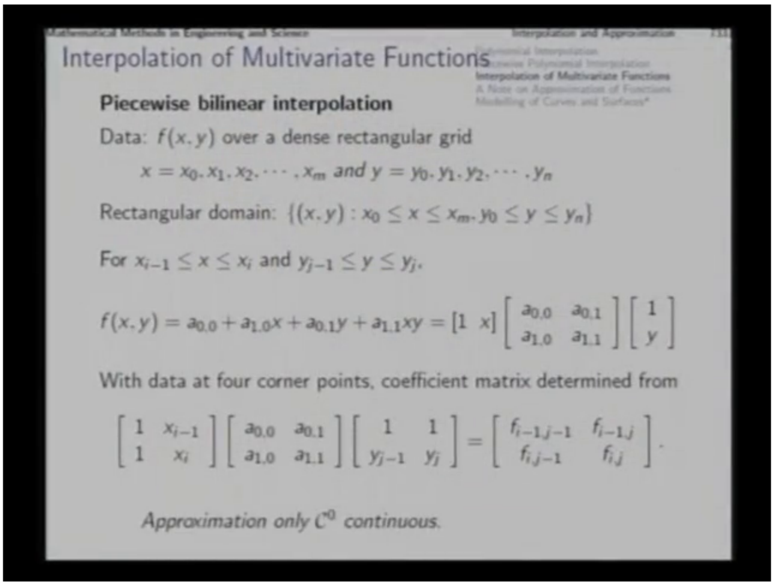

Now, in this same manner we can talk of multi varied functions also and interpolate them that is if we have a function of 2 variables. So, for that also, we can have by linear interpolations by cubic interpolations by quadratic interpolations equation and pieces wise by linear interpolation whatever we did for one variable, we can have for multiple variable the spline also can be set in that same manner now we will take one example that is pieces wise by linear interpolation for piece wise by linear interpolation we have the domain structured in the form of a rectangular grid, right.

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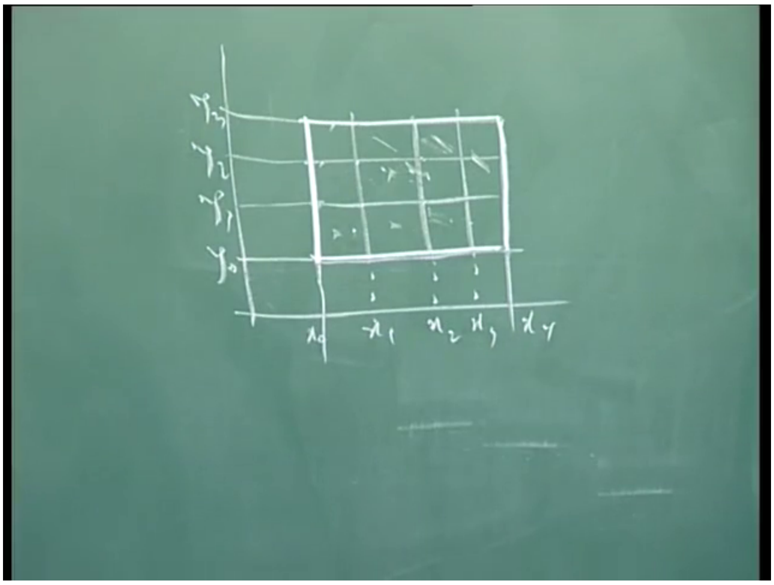

So, we typically have x 0, x 1, x 2, x 3, x 4 and so on; similarly on this side y 0, y 1, y 2, y 3 and so on. So, suppose our domain is this over which we want the function representation for this region this rectangular region. So, for that we can have a grid like this and typically for by linear interpolations we typically want the grid to be quite dense and then the function expression will look like this constant term plus something into x plus something into y plus something into x y which can be represented in this manner.

1 x here 1 y here and the coefficients here now the coefficients form a matrix. So, with the data at 4 corner points function values only with 4 pieces of data we can determine these 4 coefficients. So, as we put these 4 pieces of data with the corresponding values x values y values the we will get this will be a known matrix this will be a known matrix this will be a known matrix and we need to determine this matrix. So, pre multiplication of universe of this and post multiplication of inverse of this we will determine this matrix for us which means the coefficients here.

So, like this for every small rectangular region we will get the function representation and this will be continuous representation only no differentiability. So, differentiability can be ensured if we have a pieces wise by cubic interpolation and so on.

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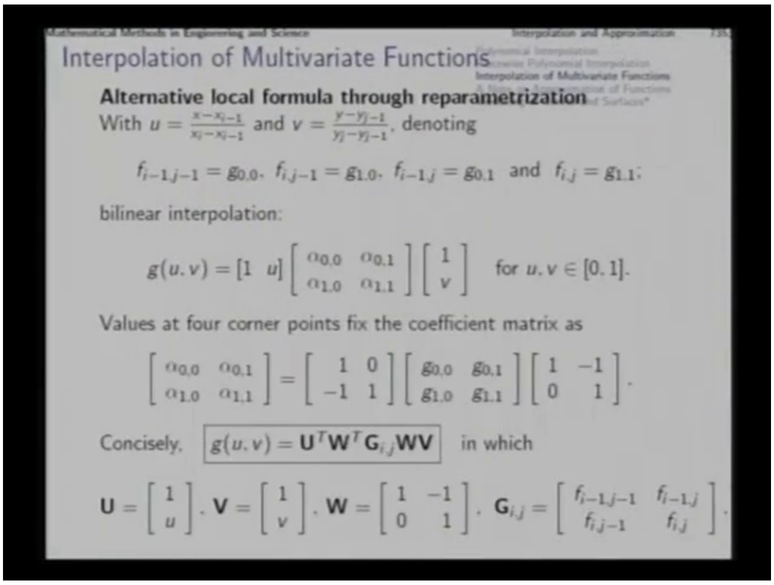

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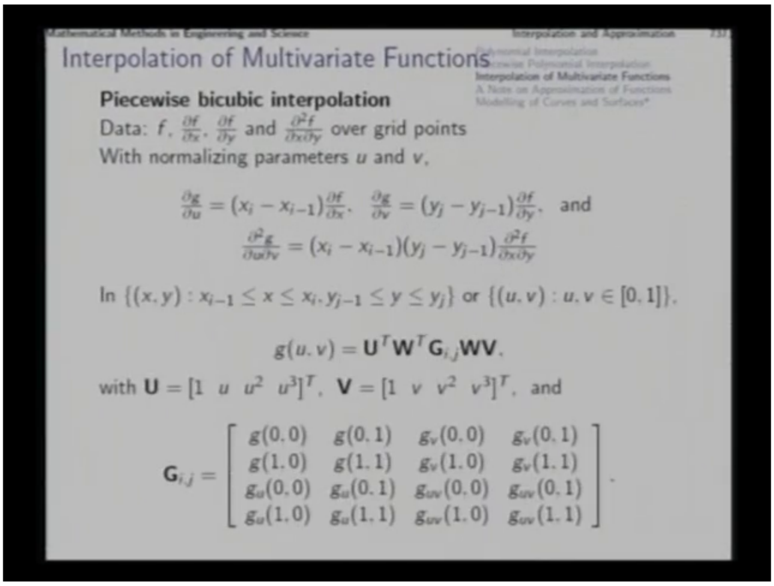

So, we will omit all these details because in the same manner you has representation of single variable function you can develop this and everywhere will have coefficients in the form of matrices data also in the form of matrices this is pieces wise by cubic interpolation, here you need the derivatives; these derivatives which again if you normalize, then you will scale in this manner as we did in the case of single variable function.

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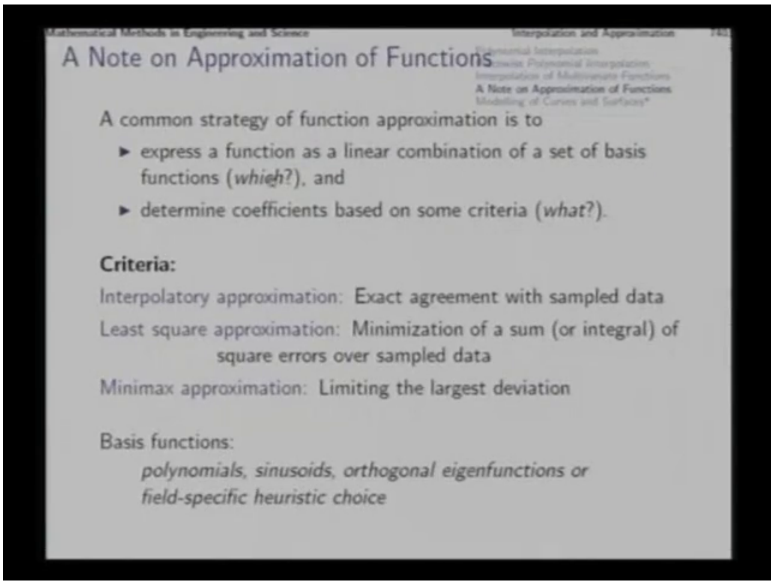

Now, what is important for us at this stage is to have a quick look of what we have been doing all this while now typically a common strategy of function approximation is to express a function as a linear combination of several known function that is a set of known functions which are called the basis functions.

Now, the question arise is which basic functions to use, similarly we determine the coefficients based on some criteria what is the criteria that is a second question. Now if we decide that our criterion is exact agreements which sample data then we get one kind of approximation which is interpolatory approximation, right which we have been doing in this lecture, there is another kind of criterion that is least square approximation that is we do not mind the some errors in the at the data points, but then what we ask for is that the errors all over should be limited that typically; we do when we have supper abundance of data that is more data than necessary to fix the coefficients in that case we expect error and then we say that error square sum should be minimized that is the least square approximation earlier in the linear problems; in the linear algebra segment and then in the optimization; optimization segment we have seen how we conduct least square approximation in the modeling of functions least square approximation is based on minimizing the sum of squares of errors. Now when these sums are sums of infinite terms organize arranged very close to each other than that sum is replaced by integrals.

So, even that is a kind of least square optimization which we will study after our study of differential equations and I have the same time we will study the third criterion of function representation also that is mini square optimization which also as a certain advantage which also as a certain application, there we do not try to make exact agreements with certain data points we do not want the overall least square of approximation what we want is to keep the largest error within a limit; that means, we try to minimized the maximum error and that is why that kind of an approximation is called minimum max approximation. So, after the study of ordinary differential equations when we take up our topics on approximation theory there we will consider this kind of an approximation also now this is the kind of approximations that we can talk by using different criteria.

Now, which set of basis functions we use one possible choice which we have been doing in this particular lecture in this particular lesson are polynomials. So, in this lecture we have done single polynomial with which with basis functions as 1 x x square, etcetera then Lagrangian polynomials then newton approximation all pieces wise polynomial. So, polynomial is one large class of basis functions that can be used sinusoids sines and cos coses sins and cosines are another set of popular basis functions which give us Fourier series orthogonal Eigen functions which will be encountered here are another vertical area choices of basis functions and then sometimes field specific heuristic choices are also possible.

The kind of choices we have been talking about when we are talking about least square approximation in the context of optimization now all these are sensible all these are proper choices of basis functions and we choices the set of basis function and we choice the criterion depending upon what is our field of application now one important issue here is that we have been talking about the approximation of functions of this kind f of x right and then f of x y, right.

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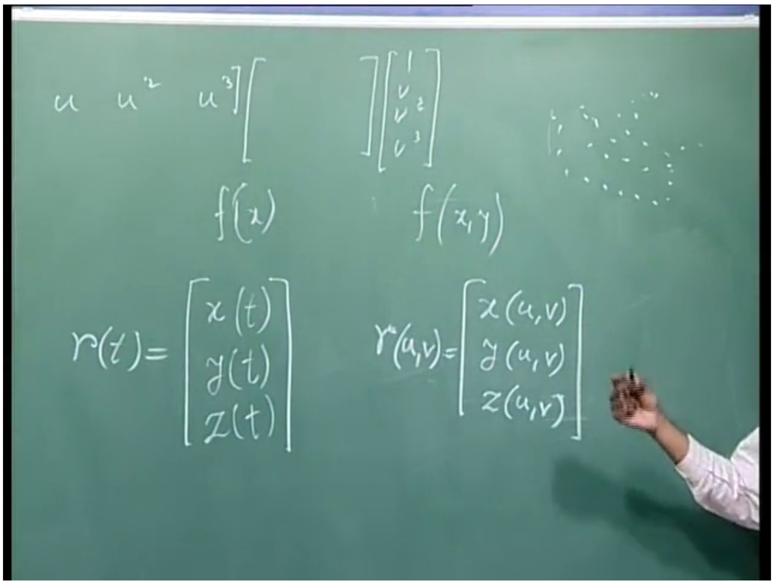

Now, consider this independent variable we will just change the names and then see what kind of interpretations we can make in place of this independent variability p a x suppose we put t and for this function in place of f, we put x x of t will be a function of parameter t. Now, similarly y of t can be another function of parameter t and z of t can be another function of parameter t now all these we can do exactly the way we have been doing for function f x and now these 3 functions can be taken as x coordinate y coordinate and z coordinate of a point or a curve as t varies.

So, enclosing it like this we get this right. So, this is a vector function of a scalar variable t now with this kind of a set up with the same interpolation tools we can actually model curves in 3 d space similarly the bi variate functions that we have been talking about in terms of such bi variate functions we can model triplets of such functions that is we can model vector functions of 2 parameter 2 parameters u and v and that way with a lot of data over the surface we can make a representation of the surface like this surface as functions of 2 parameters u and v and with the help of that we can model surfaces.

So, typically curve and surface modeling is one large area which stems from this kind of interpolatary or other function approximations for example, suppose we have got a large number of data points in 3 d space and you want to find out a the analytical representation of a surface on which all these data points lie then what you can do you can say that suppose i have go a by cubic expression of u and v and a by cubic expression not just one expression by 3 expressions; 1 for x coordinate, 1 for y coordinate; 1 for z coordinates. So, you can say that we have got a by cubic vector function r x, y, z; that means, 3 scalar functions and each of them by cubic with 2 variables u and v in hand; that means, you have got 4 by 4 matrix like this.

So, this will give you a by cubic expression now you have got 3 such expressions for x of u v y of u v and z of u v and then you want to determine the 3; 4 by 4 matrices sitting here for the x coordinate y coordinate and z coordinate and with all the points that are given to you as points on the surface you can use that those points as the set of data and then you can frame equations for x coordinate y coordinate and z coordinate and from that you can determine these 3 4 by 4 matrices once you determine these 4 by 4 matrices you have got x coordinate y coordinate and z coordinate as functions of u v with known coefficients here and; that means, that you have got a complete analytical representation for the surface for which the data was just a lot of tray points. So, this way you can do a lot of curve and surface modelings with the help of these kinds of interpolatary formulas.

So, this particular problem you can also do a by cubic approximation based on the function values and that derivatives at the end points. So, in that case you will have the data points the values of the function and then their u derivatives their v derivates and the cross derivates that is del 2 f by del x del y this kind of data you can use to model bi variate functions and when you have vectors functions like that with 3 scalar bi variate component you got a model of a surface. So, such curve and surface modeling are often done in the case of compute aided engineering design or computer aided geometric designs.

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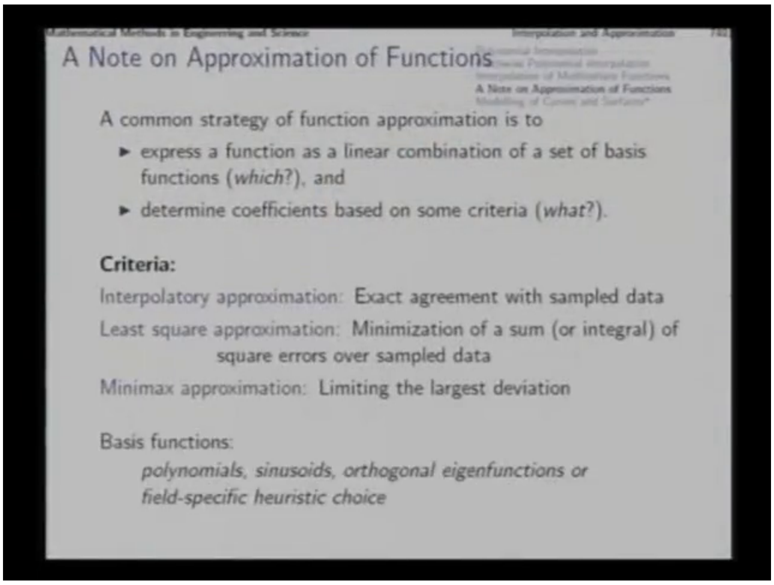

So, some of these examples such some of such examples are there in the exercises.

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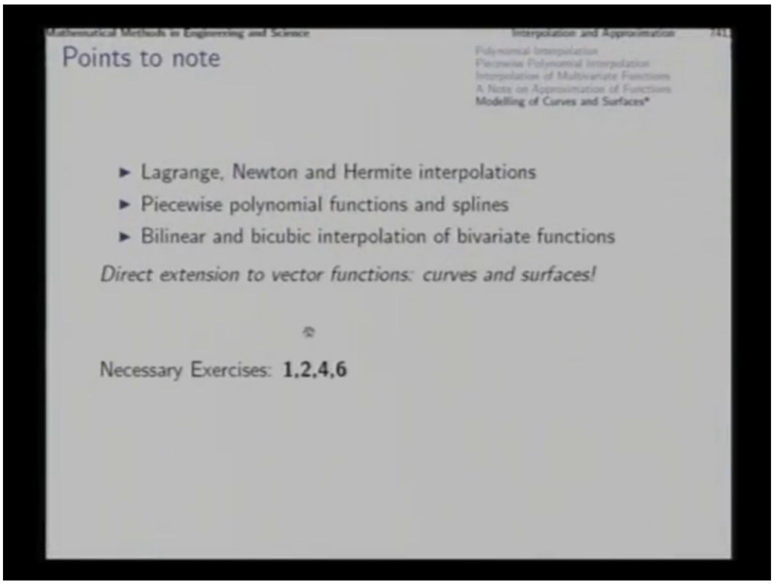

In the chapter of the text books and I will suggest you strongly to attempt some of these exercises to develop an ability to work with such approximations.

Thank you.