Mathematical Methods in Engineering and Science
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Module – IV
An Introductory Outline of Optimization Techniques
Lecture – 01
Solution of Equations

Good morning from this lecture, we start the third module of our course which is on numerical analysis, numerical matrix.

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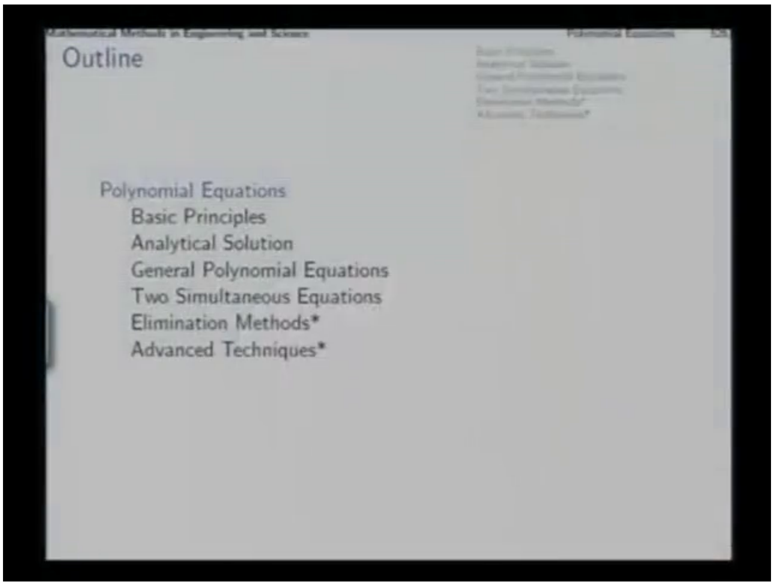

So, in this lecture we will concentrate on a solution of equations and their systems and next lecture onwards we will start our study of numerical optimization techniques. So, first we start with polynomial equations.

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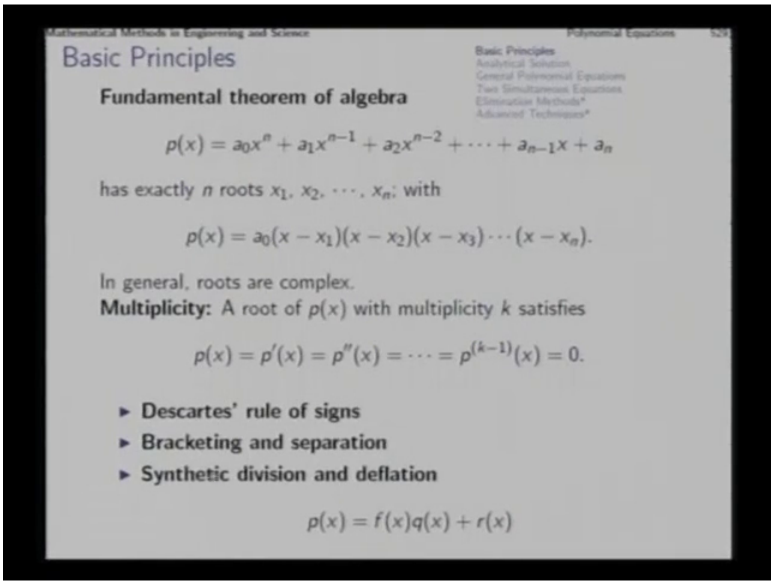

The fundamental theorem of algebra tells us that the a polynomial of n th degree has exactly n roots and with those roots x 1, x 2 up to x n you can factorize this polynomial in this matrix.

Now, in general the roots are going to be complex, now as a special case it may happen that you get real roots; however, with real coefficients if complex roots appear then they always appear in conjugate pairs. So, this is one important piece of information which helps us in many of the methods to solve such polynomial equations. So, this is a polynomial this polynomial equated to 0 gives us the corresponding polynomial equation. So, we can talk of the roots of the polynomial or the solutions of the corresponding equation.

Now, in the equation solving process for polynomial equations there is a lot of theory which helps us or which guides us in several different manners, one of them is the Descartes rule of signs and then the another is bracketing and separation of different roots. And then there is a process of synthetic division, in which if we have a polynomial P x and we have another polynomial q x of lower degree then with q x we can divide the polynomial p x and the quotient is f x and the remainder is r x. Now, f and q they are interchangeable if you take f x as the division then q x will be the quotient and r x is the reminder. So, the reminder is a polynomial of a degree lower than the divisor by which you would try to divided the original polynomial, it is exactly like ordinary arithmetic division.

Now, one more important point is the multiplicity of a root, all the roots need not be different there can be some roots which are repeated still they are counted in this n roots. So, for a cubic polynomial the roots could be 2 3 and 3. So, they are counted as 3 roots, if 3 appears twice in this factorization, now the simplest polynomial equation that you can think of which of course, linear equation which is extremely simple. So, from the next step onwards the thing becomes slowly more and more difficult, above linear equation you have got quadratic equation which all of us studied in school.

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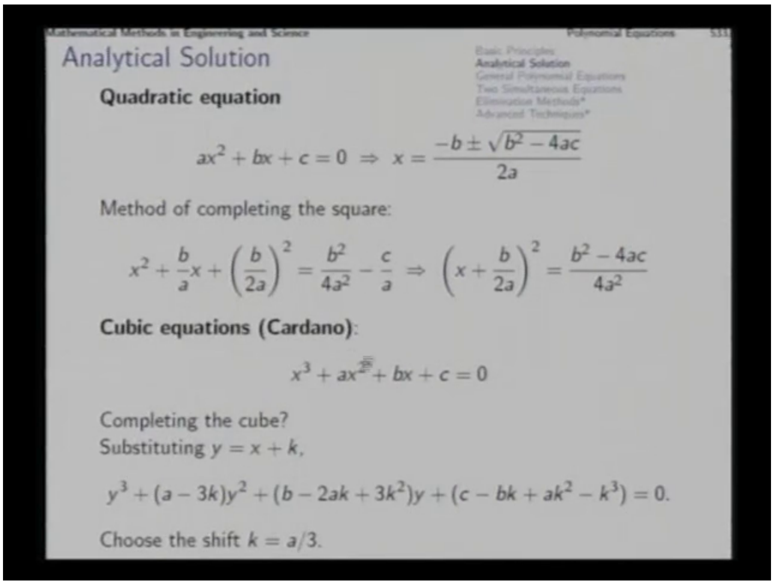

So, this quadratic equation has this solution that is there are 2 solutions with plus minus signs, now depending upon what is b square minus 4 ac the solutions could be complex or could be real. Now, the way this formula is I have write is of course, to completing the squares and from which taking the square root from here we get this expression now the same thing if we try to do in the case of cubic equation then we would be thinking of completing the cube, right. So, that you can take a cube root and get the values of x.

Now, this is comparatively much more difficult compared to quadratic equation and that is why it took a long time for mathematicians to arrive at a good way to solve this problem analytically, Cardano's solution proceeds like this, that is for completing the cube in some sense the process succeeds if this x square term is missing from the equation, now how to make it missing, how to get rid of this x square term. So, the way we handles this problem is by substituting y equal to x plus k here which will transform this equation to y in this manner.

Then we say that let us choose that k which makes this y square term vanish and that k is a by 3. So, by choosing a by 3 as k we will get this term as 0 and then y cube plus something into y plus something equal to 0 that becomes the transformed equation in y. So, in this manner, now solution of this is possible in a process which is something like completing the cube.

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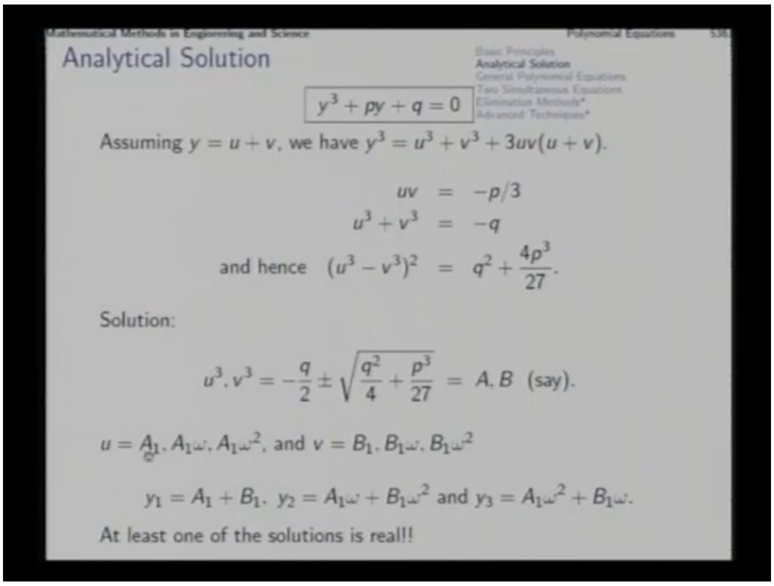

Now, for that y is assumed in this manner, u plus v and if you take the cube of this y cube then you get this expression in which there is another u plus v appearing here. So, if we add this u plus v as y then you note that this equation and this equation can be compared y cube y cube p y and as you bring this on this side you get minus 3 u v y. So, p becomes equal to minus p u v; that means, u v is this.

And then q here and negative of this term for this equation. So, u q plus v q become minus q. So, in u and v you have 2 equations, it is not difficult to solve this because here you have got u cube plus v q and with the help of these 2 equations you can develop another equation in for u cube minus v cube. So, that is a plus b whole square minus 4 a b is a minus b whole square, when you use that formula you get this, the square root of this second one gives you u q minus v cube is equal to plus minus the square root of this.

So, there are 2 equations the first one gives you the sum of u cube and v cube and the second equation will give you the difference of the same 2 things. So, half the sum gives you u cube and half the difference gives you v cube.

So, you solve this u cube and v cube from these 2 and get this, let us call it call these 2 as a and b plus minus you get you get 2 values a and b. So, one of them you can call u cube and the other you can as v cube, after that you take the cube roots of a and b and while taking q routes keep in mind that if this thing becomes complex then you have to appropriately take its angle and in the cube root the angle you will get 1 by 3, 1 by 1 by 3 2 by 3 and full so like that. So; that means, in the cube root.

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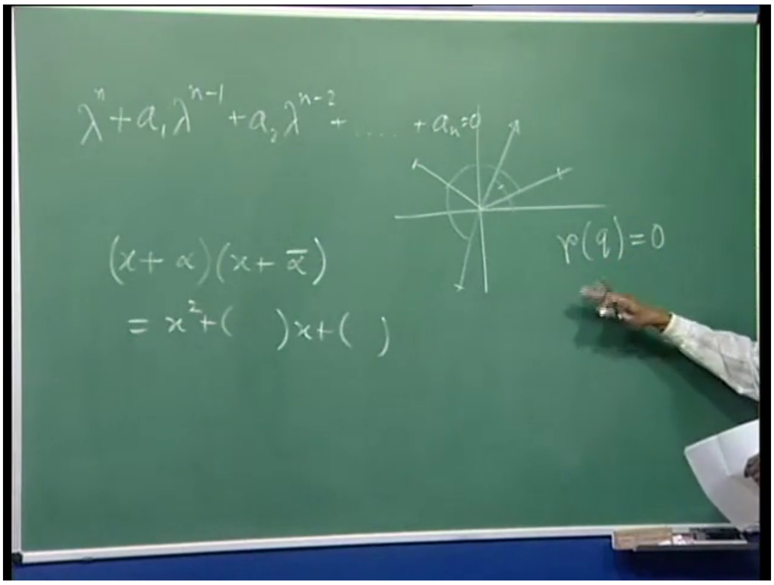

So, if the description of that number in the complex planes turns out be like this then its cube root will have 1 third of this angle plus 120 degree and then plus 120 degree for a vector. So, these will be the arguments of the 3 cube roots. So, even if this is real the 3 cube roots will be 1 which will be real the other will be at 120 degree and the third will be along 240 degree. So, that we take the 3 cube roots of a and b.

So, you can show it as if one of the cube roots is A 1 the other one will be A 1 omega the third one will be A 1 omega square similarly on this side, now with this with u chosen from here and v chosen from here you can combine u plus v in 9 different ways, but all of them will not satisfy this equation only 3 of them will satisfy, you can verify this that.

So, if it happens that A 1 and b 1 taken from these 2 and added together becomes 1 root of this equation satisfy this equation then the other 2 will be A 1 omega plus B 1 omega square and y 3 will be A 1 omega square plus B 1 omega this is the way it is and you will also note that out of these 3 roots one of them must be real because this is a cubic equation. So, the solution of this the roots of a cubic polynomial if they are complex must appear in complex conjugate pairs. So, that will be able to account for atmost an even number of roots.

So, only 2 of them can be complex, the third one the odd member out will be essentially real. So, this is one situation that you will always find when the degree of the polynomial is odd. Now, after the second, after this cubic equation was solved by Cardano then mathematician tried for quite some time to solve the next higher degree equation that is the quadratic equation, 4 degree equation and Cardinol's all students priory found a method again based completing the squares to solve for the 4 degree equation.

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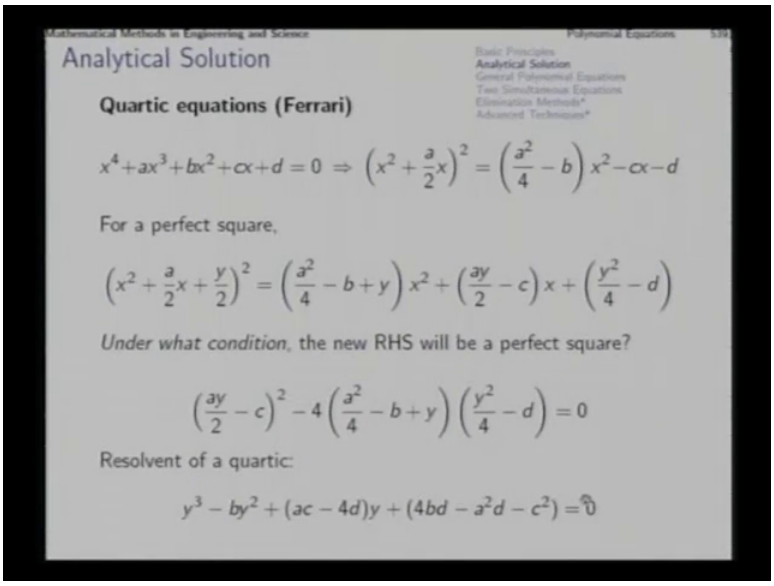

So, suppose this is a 4 degree equation given of course, the leading coefficient can be taken as 1 because whatever else appears there you can divide the entire equation by that number and put it in this manner. Now, if you take if you keep on this side on the left side only the 4th degree and 3rd degree terms and complete that as a complete square compensating by the term a s square s square by 4 on this side and take all these other terms there then you get this equation. On this side you get a complete square it should be nice if at the same time you got a complete square here also, then both sides you could take square roots and reduce the degree of the equation, but then that cannot be always possible because this expression is dependent on the abcd coefficients given in the original problems that is not in our hand.

So, in general this will not be a perfect square of a linear expression. So, we need to make it is square by doing what? By adding certain things here and adding compensatory terms on this side also. So, inside this complete square we add a term some outside we add a term y by 2 and the corresponding compensations, there will be 3 compensations y into x square which is here and then y into a by 2 x which is here and y square by 4 which is here. So, by adding these compensatory terms we extend this perfect square on this side and then ask the question.

What is the value of y for which this side also will be a complete square of a linear expression of x? Here we could not say that because we had no freedom, now we have introduced y so we can ask for y to satisfy some requirement that we supply. So, we ask under what condition this will be a perfect square and that condition we know, that condition is that square of this coefficient is equal to 4 times the product of these 2 terms, parenthetic terms. When you impose that condition like this b square minus 4 ac equals to 0 then and simplify this then we get a cubic equation in y; that means, any solution of this cubic equation taken as y if we insert in this equation then the right side will be a complete square.

So, now, the question is how to solve this cubic equation, we have already seen how to solve the cubic equation. So, by that method Cardanos method or by any other method if we can solve this cubic equation and take any one of the 3 roots, any one of the 3 solutions for y and typically we will take the real solution because cubic will always have a real solution. So, typically we will pick up a real solution from this cubic equation dump it here and then you will find that this right side becomes a perfect square.

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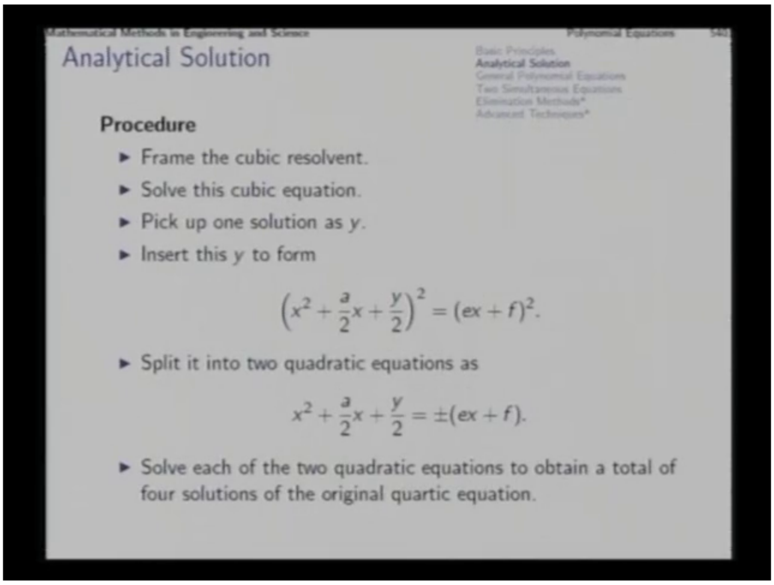

So, these are the steps, first frame the cubic resolvent this is called the cubic resolvent resolvent of a quartic. So, in that a, b, c, d we know from the original problem. So, we insert those things and frame this equation there is the cubic resolvent of a quartic equation, solve this cubic equation pick up one solution as y insert this y into that equation to get the right side in the form of a perfect square, the moment we have come to this point we immediately take square root on both sides. So, this stuff turns out to be plus minus e x plus f right and if we take the plus sign we get one quadratic equation, if we take the minus sign we get another quadratic equations and these 2 quadratic equations we can separately solve we get 2 roots from here and 2 roots from there and that completes the solution of the 4 degree equation. N ow the after the 4 degree equation general quartic is also solved in this analytical manner then people started thinking.

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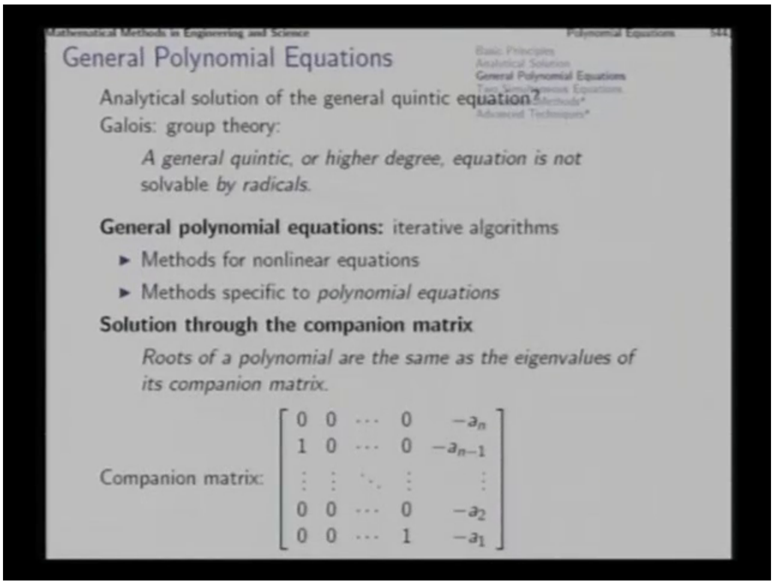

In the next stage that how about higher degree equations till it was proved by the Galois group theory that a general quintic or higher degree equation is not solvable in this analytical time. Only iterative numerical methods one can use for solving general equations of degree higher than 4; that means, we have to rely 1 iterative processes, but then iterative processes are of 2 kind, one kind of iterative processes for equation solving are general they are for general non-linear equations which will apply to polynomial equations also, but then there are some methods which are specific to polynomial equations and they operate with much higher efficiency on polynomial equations. So, when you need to solve a polynomial equation then try to use one of those methods which are specific for polynomial equations, on the other hand for the general non-linear equations you have no option you have to apply the general iterative numerical process. So, the specific methods for polynomial equations are somewhat analytical, they have an analytical base and partly they are numerical partly iterative.

So, one method is through the companion matrix, in one of the exercises in one linear algebra chapter in the text book you might have found the a problem in which it was asked to find the characteristic polynomial of this matrix. If you try to find the characteristic polynomial of this matrix by the standard process by lambda I minus this matrix determinant u equal to 0 and then expand that determinant then you get the characteristic polynomial exactly like this, the typical form in which you would supply a polynomial equation, polynomial equation. So, then you find that if the characteristic polynomial of this matrix turns out to be the polynomial that we have in hand then to find the roots of the polynomial we can try to frame this matrix and call it the companion matrix and then solve an eigenvalue problem for this matrix. So, you find that if the matrix is very small say up to size 4 then the correct way or simplest way to solve the eigenvalue problem is to frame the characteristic polynomial and find its root.

On the other hand if the eigenvalue problem is of much larger size the matrix is very large then that is found not to be a good method for that purpose we studied a lot of other eigenvalue problem methods. Now if we find a polynomial of large degree and we want its roots then one way is to frame the companion matrix and then find its eigenvalue ok.

So, this is one of the best methods for finding the it the solutions of large degree, high degree polynomial equations there is another method which is Bairstow's method.

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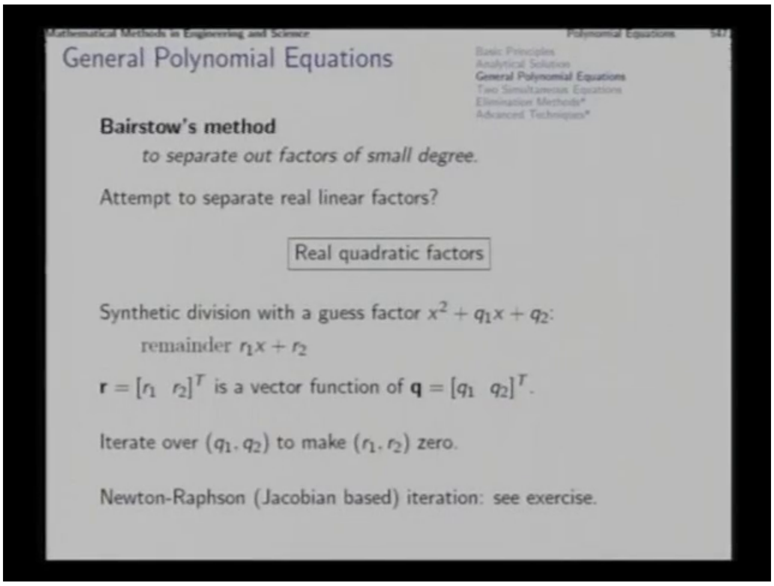

Which also works very nicely on polynomial equations, what it does is that from a given polynomial it tries to separate out factors of small degree. Now, if you try to attempt separating real linear factors like a x plus b then that may not succeed always because a priory you do not know whether the equation has real solutions whether the polynomial has real root, but then you know that if it has complex roots then complex roots always will appear in conjugate pairs. And when you multiply 2 such factors in which alpha 1 and alpha bar are conjugate pairs then the product gives you a polynomial which will be in the form x square plus something x which is real, plus something this is also real.

So, some of the conjugate pairs is real product is also real. So, like that you will find. So, this quadratic factor will certainly have real coefficients if the original polynomial has real solutions. So, we do not attempt to separate out real factors real linear factors, but real quadratic factors. So, for separating out a real quadratic factor what we do? We propose a factor like this x square plus q one x plus q 2 and conduct a synthetic division of the given polynomial. So, after completing the synthetic division there can be a reminder if this is actually not a factor then there will be a reminder, if this is a factor if we, if our guess is perfect then this is a factor and in that case reminder will be 0 and it will confirm that this is indeed effecter on the other hand at the starting point in general we would not be able to make a correct guess.

So, there will be a remainder, reminder will be a linear reminder of degree less than this right. So, the linear reminder will be like this, now let us examine this 2 numbers r 1 and r 2 that will appear in this linear reminder r as the result of these 2 choices q 1 and q 2. So, if we choose different q 1, q 2 values then we will get different r 1, r 2 values. So, that way we can say that r this vector r 1, r 2 is actually a function of this vector q, q 1, q 2; that means, this r is a vector function of this vector variable q this also has 2 members this also has 2 members has 2 members.

So, then we say that now we try to write that as if this is a function. So, r of q equal to 0 this system of 2 equations in 2 unknowns the unknowns are q 1, q 2 and equations are these r 1 equal to 0, r 2 equal to 0. So, these 2 equations in these 2 unknowns we try to solve; that means, iteratively we change q 1, q 2 values in our guess which will make the reminder vanish. Now, this is actually analogous to the typical process of Newton Raphson method, which is Jacobean based and in the case of polynomial equations there is a specific way to develop the Jacobean and apply iterations which is much more efficient compared to a general case. So, in exercise of this chapter there is 1 problem, 1 exercise in which the step by step process has been given to establish this particular method and then use it, I strongly advise you to carry out the operation once to be at home with the method how it operates.

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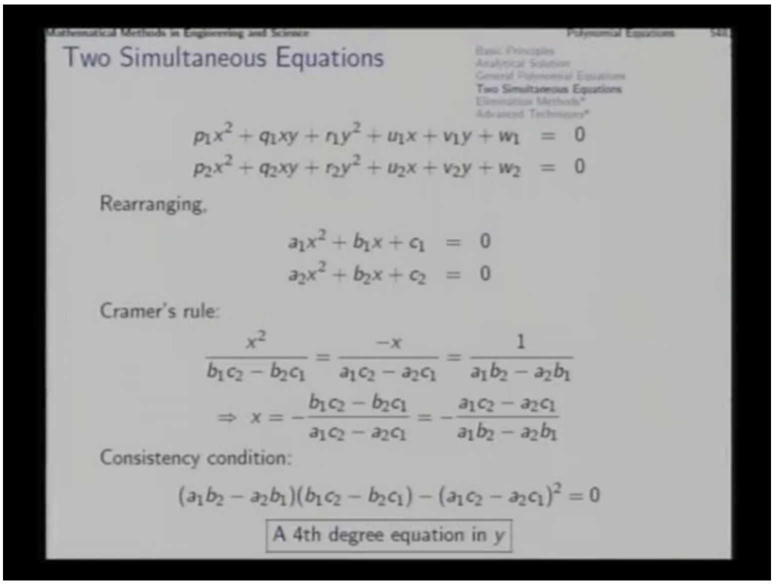

Next we continue into the case of 2 simultaneous equations for that matter one purpose of including the equation solving process in this course is basically to input methods for systems of equations. So, if we have 2 simultaneous equations any 2 unknowns now polynomial based methods operate in a particular way in which we try to eliminate 1 of the unknowns and get a single equation in the other unknown. So, if there are 2 unknown polynomial equations in x and y like this then rearranging we can express these 2 equations as if their equations in x only.

In the coefficients you will get terms containing y A 1 and A 2 are simply P 1 and P 2 as appearing from here and B 1 and B 2 are linear expressions of y including this and this term and C 1 and C 2 are quadratic expressions of y including this term, this term and this term right. Whatever there for the time being we club them together then we approach our school knowledge of common root for these two polynomials, common solution for these two quadratic equations and the result is given by Cramer’s rule x square by b 1, c 2 minus b 2, c 1 is equal to minus x by a 1, c 2 minus a 2, c 1 is equal to 1 by a 1, b 2 minus a 2, b 1. So, that we write here, now if we consider these 2 the equality of these 2 then from there we get one expression of x, that is x square by x that gives us one expression for x x minus this minus this thing divided by this thing that is this.

On the other hand if we use the equality of these 2 terms then we get x by 1 which is minus this by this. So, that is this now the condition for common solution of these 2 quadratic equations for x is that these 2 must indeed be equal and; that means, that product of these 2 cross product cross multiply these 2. So, product of these 2 is the same as product of these 2; that means, this into this is equal to this whole square.

So, this is the condition for common solution of these 2 quadratic equations, now for x that is the condition, but for y what is it as we equate this to 0 and insert b 1, b 2, c 1. c 2 as expressions of y from which they were earlier found then this will turn out to be a 4 degree equation in y. So, if you use that 4 degree equation to if you solve that 4 degree equation for y then in general you get 4 different solutions, each of those solutions as you insert here you get one solution for x. So, that way for x and y combined you get 4 solutions which will be the case because these are both quadratic equations and the maximum number of solutions that they can have together is 4 only.

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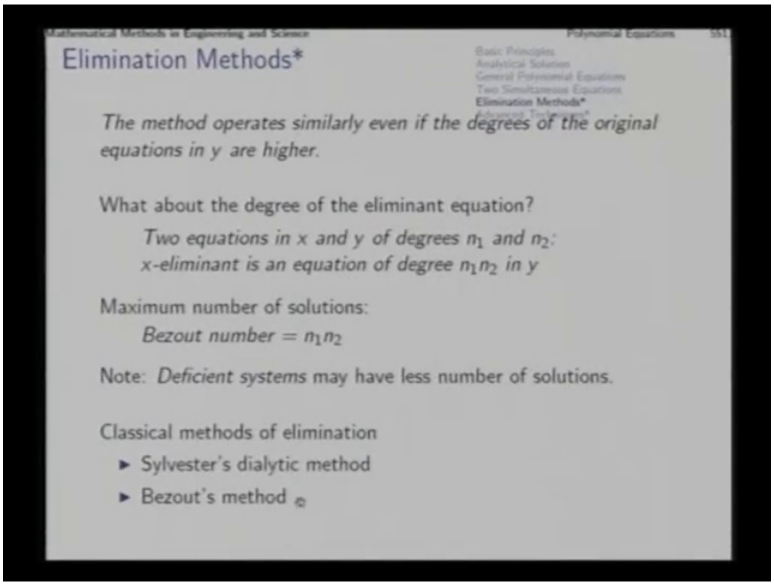

In general you can say that 2 equations in x and y of degree n 1 and n 2 out of that if you can eliminate x, the x eliminent turns out to be an equation of degree n 1 into n 2 in y this number is called the Bezout number. There are several methods for elimination of unknowns from simultaneous polynomial equations two of those methods are quite effective, one is called Sylvester's dialectic method and the other is Bezout's method. We do not go in to that in detail because those who are interested in this polynomial system of equations they can look up particular text and courses for theory of equations itself. Now after this we will continue into the discussion of those methods which are general which will solve polynomial equations as well as transcendental equations they are purely numerical methods.

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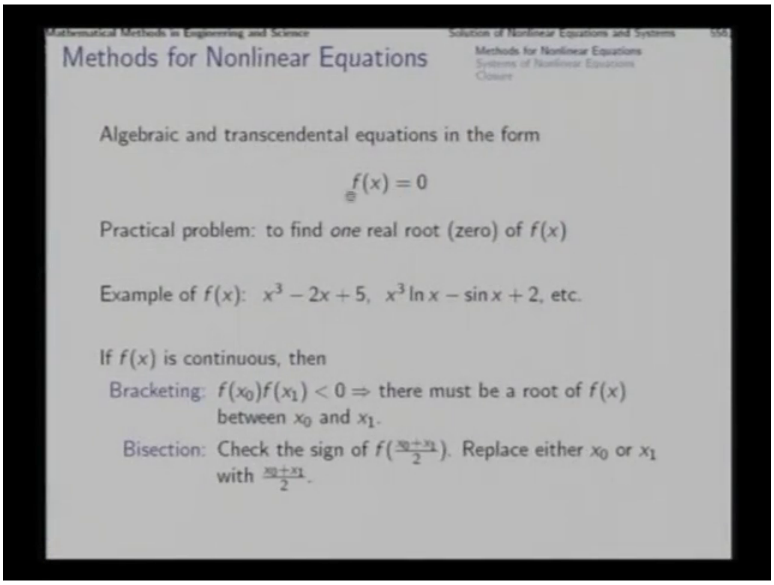

A typical general non-linear equation can be expressed in this manner f of x equal to 0 and the practical problem, now in the case of general non-linear equations transcendental equations we do not ask how many solutions this will have, typically we do not ask because quite often we numerating the solutions that is working out the number of solutions that has that equation has is quite cumbersome.

So, the practical problem is to find one real root of f x or one real solution of this problem. So, this entire study is typically conducted for real solutions, unlike the case of polynomial equations in which case we are quite often interested in the complex solutions as well in the case of general non-linear equations we typically look for real solutions only. So, examples of f x it can be a polynomial or it can be something like this and so on. So, for this kind of problems for solving general non-linear equations if you background processes will help us in tackling the correct interval in which we would expect a solution. So, for that we use the method of bracketing if we can identify 2 points say x 0 and x 1 in which the function value f of x is of different signs then if the function is continuous then we can use its continuity to claim that between x 0 and x 1 there must be a point where the function process 0 and that point gives us a root. So, now, after bracketing the root in that manner all that remains is to hunt out the root in that interval, one straight forward method would be bisection that is given the interval x 0 x 1 which gives a bracket we look for the sign in x 0 plus x 1 by 2. Now, depending upon what is what is its sign this midpoint we replace either x 0 or x 1 and then continue.

So, bisection itself is one viable iterative method for solving an equation like this, but it is costly there are much more effective methods for solving the problem, but you can always consider bisection method as a call back if in some iteration the other (Refer Time: 29:19) methods somehow get into some trouble. So, bisection method remains one method on which you can always relay as long as you have a bracket to start with.

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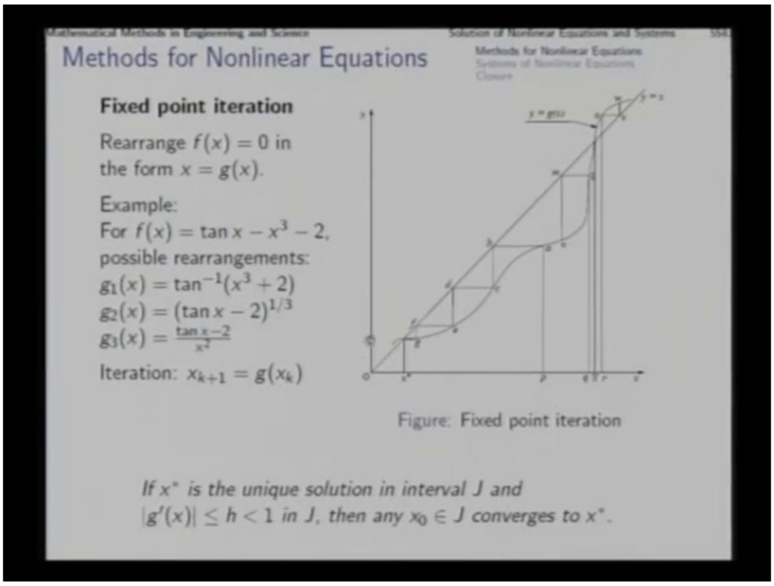

So, what are the other sophisticated methods, one is the fixed point iteration method. So, what we can do is that we can rearrange the given equation in a manner such that x is expressed as an expression of x itself, now for example, take this example of f x. So, the solutions of the equation tan x minus x cube minus 2 equal to 0 is required.

So, f x is this now if this equal to 0 then we can arrange that equation in the form of an expression for x in terms of x itself call that expression as d l, there are several ways of doing it one possibility is to take these 2 on the other side of the equation x cube plus 2 and then say then x is tan inverse that. The other possibility is take x cube only on the other side and then on this side you have tan x minus 2 and then cube root of that will be x or dividing tan x minus 2 by x square you get another expression for x. So, all 3 of these are candidates for suitable g x that you can frame out of these equation and try to iterate now what is the iteration, iteration is very simple the iteration has 2 steps starting from a guess value all these iterative processes we will start from one guess value you see.

So, starting from a guess value x 0 evaluate this function and since we are trying to solve x equal to g x then whatever is the value of this function that becomes the next value of x, x 1. So, then that x one goes here whatever comes out that becomes the next iterate for x that is x 2 and so on. So, this continues hoping for convergence, now as I say hoping for convergence, for an arbitrary arrangement like this there is no guarantee that the process will converge. If the solution is say at 5.2 then if you make a wrong kind of re organization then it may happen that even starting from 6 it might go to 10 and then 16 and then 79 and it may diverge. So, convergence as it is its not guarantee, then we ask what is the cost what is the condition under which the convergence is guaranteed there is a condition.

So, through discover the condition see the way this process will work, suppose after organizing x equal to g x we have got this curve as the graph of g x and so y equal to g x is this graph this curve and y equal to x is this 45 degree straight line. Now, y equal to g x and y equal to x intersect at a point where x equal to g x is satisfied, those are the points which we were looking for as shown in the figure there are 2 such points 1 is x star and the other is x bar at which this curve and this straight line intersect. Now, suppose we start from this value x equal to p, as we were just discussing the iteration is just 2 steps at the current point evaluate the function g of x and then whatever is the value assign it to x and this process is continued.

So, at this point x equal to p evaluate the function that way we get this value which is a. So, then from there this value now has to be assigned to x; that means, parallel to the x axis that is the horizontally if we move and then meet the y equal to x line; that means, this y value will be here same as the corresponding x value.

So, at this x value then we evaluate the function g x along at the point on the curve and then again horizontal, vertical, horizontal, vertical we do converge on the other hand if you start at q extremely closed to the other solution x bar you would expect that quickly it would converged to x bar, but that does not happen. See from q we evaluate the function reach this point l that value of g x that is y we assign to x; that means, we come here and that value we evaluate g x again we come here.

Look we are not going from q towards x bar, but we are going away if we start on the other side of x bar say at r again evaluate it here come here go there u v w we are again diverging on the other side. So, what is wrong here? So, closed to x bar we are not converging to x bar whether you start from this side or that side on the other hand so far away from x star we could happily converge. So, what is the difference? So, if we analyze the situation you will find that if there is an interval bracketed situation in which there is a unique solution x star, then and apart from that if the reorganization has been done in such a manner that g prime has an absolute value which is less than one in that interval then.

Any point started with in that interval we will converge to x start; that means, the slope of this curve in the interval if the slope of that is not greater than this slope of this line then the process will converge with guarantee otherwise not.

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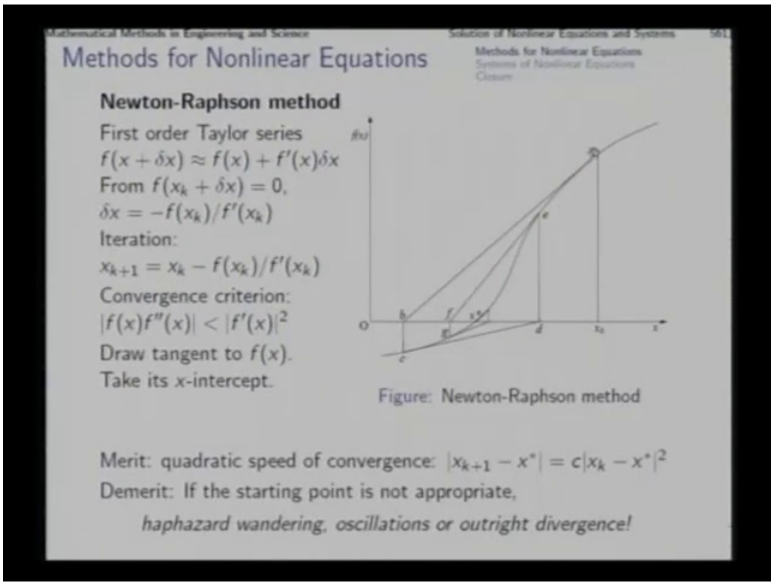

The next method for solving a non-linear equation is the Newton Raphson method this has a connection with the Newton's method of optimization theory as well. So, this method relies on the first order Taylor series of the function f x which is like this, this is the first order truncated Taylor series. Now, the understanding is that we have got a value of x where we have evaluated the function and its derivative and we want to move to a new point x plus delta x where the function value is 0.

So, if we want f of x plus delta x to be 0 then further purpose we equate this right hand side to 0 and from there we get a value for delta x which is minus f x by f prime x. Now, when we add this delta x value to the original current x x k then we get this expression for x k plus 1, this is the typical update formula. If you examine this formula carefully you will find that this is one way of writing x equal to g x, on the left side you have the next value of f x on the right side you have an expression of the current value of x. Now, for that g x if you work out the same convergence requirement as the (Refer Time: 36:51) iteration formula then you get the convergence criterion to be this, if this is satisfied in an interval then the starting point from within that interval will converge to the root in that interval.

The geometric interpretation of the working of Newton Raphson method based on this formula is the following you start from a guess point and evaluate the function and reach this point, at this point make a first order approximation of the curve that is a tangent approximation, tangent based approximation. So, that means, draw a tangent to the curve at this point and wherever that tangent crosses the x axis take that as the next point, again from there evaluate the function reach the point on the curve draw the next tangent and so on. So, started from x 0 you will travel as a, b, c, d, e, f and like this you can converge to the value x star which is the root of this particular function fx.

Now, the merit of this method is that it has a quadratic speed of convergence that is at every iteration the accuracy gets improved by somewhat 2 orders, this the accuracy gets improved by 2 orders at every step. So, this is called the quadratic speed of convergence the demerits that if the starting point is not appropriate then there is no guarantee that the process will converge. So, sometimes Newton's method is found to wonder haphazardly and go away from the solution, it might divergent diverge or sometimes it might oscillate from here to on this side then on that side again on this side and sometimes the Newton's method Newton Raphson method is found to oscillate around the solution also rather than converging.

So, these are some of the demerits of Newton Raphson method has, but when it works if started close enough and if the function derivative is good the rate is good, the slope is good then when it convergence it converges extremely fast. Now, in this particular iteration formula if we replace the derivative by a finite difference kind of first order derivative then what you get is the method of secant method rather than tangent based method you get a secant based method that is through a chord cut through 2 points on the curve.

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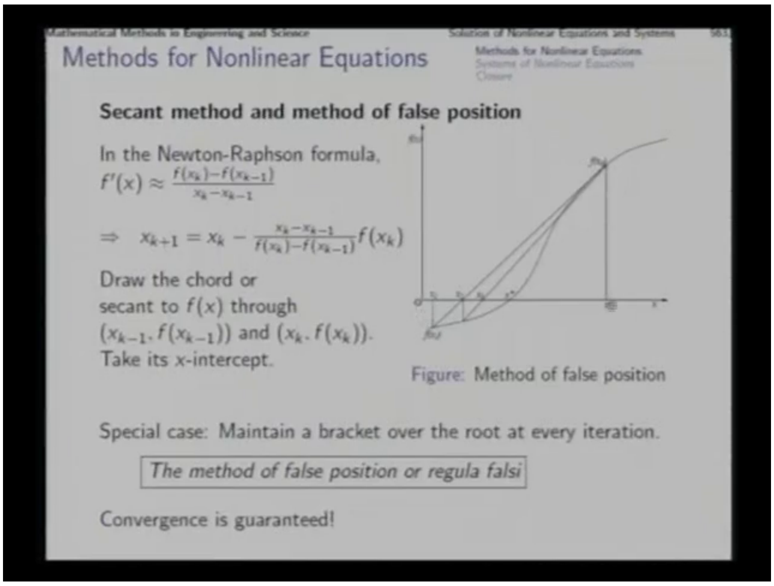

So, replacing f prime by this expression you get this iterative formula that is called the secant method. So, the way it works is that start from 2 points x 0 and x 1 at these 2 points evaluate the derivative, evaluate the function values no need of derivative here because derivative is replaced by this then through these 2 points draw a chord of the curve, wherever this chord cuts the x axis take that as the.

Next point then out of x 0, x 1 and x 2 retain 2 points and continue the process, now this is the secant method and a particular case of secant method in which the initial 2 points are taken in such a manner that the function value is positive on one side negative. On the other hand at every step that bracket is maintained that method is called, that method is a special case of secant method and that is called regula falsi or method of false position, in this particular case since we are maintaining a bracket always.

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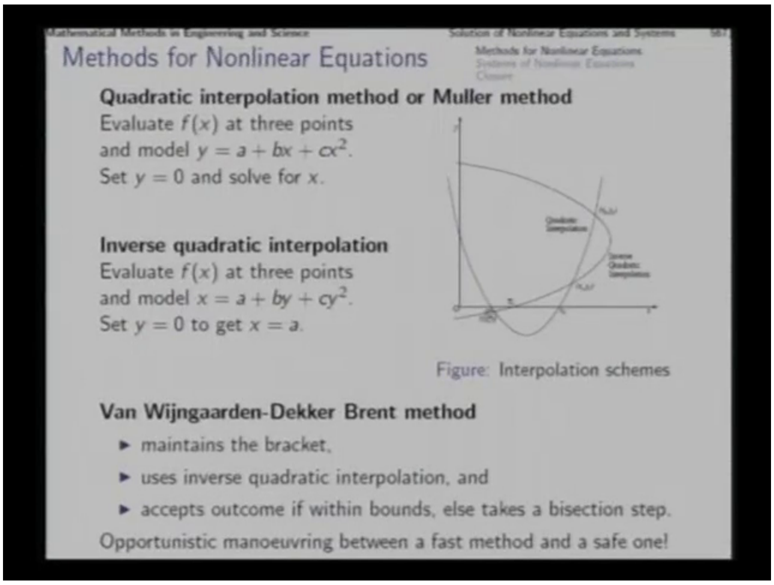

So, convergence is guaranteed this is a very reliable method now there are other methods also to solve non-linear equations one of them is quadratic interpolation or mullers method in that what you do is that you do not talk of derivate at all for evaluation you take 3 points at which the function is evaluated and through 3 points you can develop a quadratic model for the function.

A local quadratic model and then say we want to find out at which value of x this is 0 and that point is the next point that we have so; that means, started from 3 points this model gives a quadratic approximation for the function in the local neighbourhood. So, we equate that to 0 and solve for x and then with the old 3 points we have got a 4th point out of these 4 points depending upon the function values we retain 3 and throw away the worst point and continue this process. So, this is quadratic interpolation of course, in this we have solve a quadratic, solution of a quadratic is not difficult, but then questions that arise is that what if that quadratic has 2 solutions which one to take then if both the solutions are complex then we have nothing in hand.

In comparison to that there is another method which is called inverse quadratic interposition which is found to be much more straightforward, from the same 3 points that you start with do not frame y as a quadratic expression of x, but x as quadratic equation of y and then for y equal to 0 you get x equal to a. Say that value of x as the fourth point and again out of the 4 points available drop one retain the 3 best and continue this is called inverse quadratic interpolation. Combining this there is a combining this there is many other aspects you have got a very good professional method for non-linear equation solving that is called van Wiijngaarden Dekker Brent method or in brief Brent method. So, see here through the same 3 points x 0, x 1, y 1, x 2, y 2 the quadratic interpolation will give you this curve from which you might take this point or this point as the next point the inverse quadratic interpolation on the other hand would give you y quadratic interpolation gives you this curve y as a quadratic expression of x and inverse quadratic interpolation will give you this parabola both are parabolas.

So, the same 3 points same 3 points this is the parabola that you would get in the case of quadratic interpolation this is the parabola that you would get in the case of inverse quadratic interpolation which will cut the x axis only at 1 point. Now, in the Brent's method there are quite a few aspects that are combined, first of all it starts with the bracket out of the 3 points that we have in hand, the 3 points must fall on 2 sides of the solution that is at least in one of the point at 1 of the point the sign of the function must be opposite to what is the sign in the other 2 points. So, and that bracket is always maintained.

So, you never drop the solitary point on the other side retaining all 3 points on one side of the root you never do that in Brent method you always maintain the bracket second you use inverse quadratic interpolation that is this model and whenever you find that the next suggestion of a point that is that you get out of this turns out to be within the bracket interval you accept it. On the other hand if you find that the suggestion x equal to a turns out to be outside the current bracket; that means, rather than squeezing the bracket it would enhance the bracket which is not desirable then you say that for that particular iteration we do not accept the suggestion of inverse quadratic interpolation rather for that particular iteration we replace that step by a straight forward bisection step. So, whenever you find that the sophisticated method suggests something good we accept it and if it suggest something undesirable then we do not accept it and for that particular iteration we use our time tested reliable and guaranteed method of bisection for only that step.

So, when the bad time comes we look for a slow, but assured step and when good time is proceed we approach the solution faster. So, this is a kind of opportunistic measured in between a fast measured and a safe method this kind of situation we will come across quite often when we go into the optimization methods. In this context it can be noted that many of the equation solving processes are deeply related to many of the corresponding optimization processes they have the same background of the iterative schemes.

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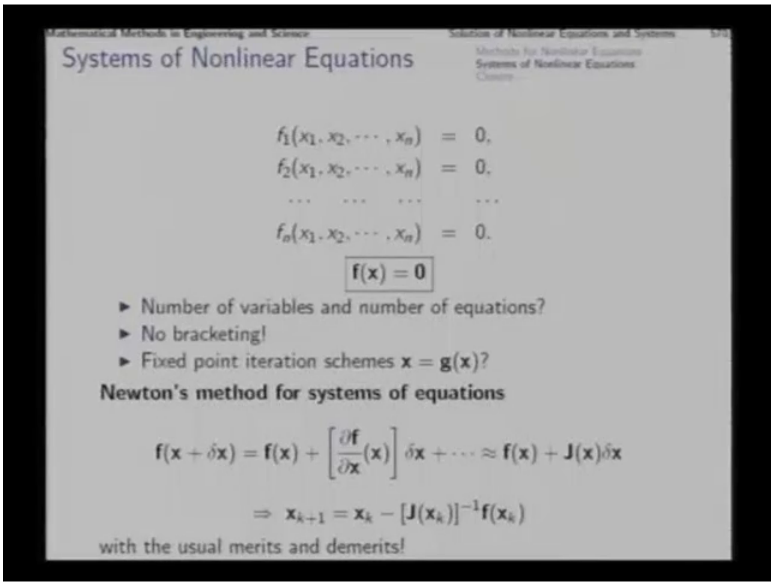

Now, we proceed to the system of non-linear equations if you have got a number of equations in the number of unknowns then concisely this system can be written like this, where f is a vector function of a vector variable x. Now, in general it could be that the dimension of x is greater than dimension of f or the other way around if the dimension of x is grater; that means, you are trying to solve less number of equations in more number of unknowns that is an undetermined problem and you would expect infinite solutions if there are solutions at all. If the system is consistent then you would expect infinite solution on the other hand if the number of x s is less then in general you would expect conflict and then you talk of least square problems that is through the extent possible you try to satisfy all way all the equations if 3 number of equations is larger.

The most often encountered situation is when you have n equations in n unknowns so; that means, in which case you will expect a finite number of solutions, not necessarily unique, but a finite number of solutions that is what you would expect, but you might find one of the other cases also in that case, but typical expectation is a finite number of solutions in that situation. So, now, one comport of a single equation is lost, you cannot talk of bracketing at all because it is not a one dimensional line anymore it is a multi dimensional space in which you have to make the curve and if you try to work out fixed point iterations schemes like this then there will be so much of possible reorganizations of these equations that you cannot enumerate them out.

So, you try to establish those kinds of re organization schemes which come from some straight forward theory, rather than haphazardly trying the recombination's the most obvious candidate for that purpose is again Newton's method or Newton Raphson method which we get from the first order truncated Taylor series of this vector function of a vector variable and that is f of x plus delta x is equal to f of x plus the Jacobean multiplied with delta x. The higher second order term for the time being we neglect, now if f of x plus delta x is roughly equal to this and if we have a starting value starting vector x, x 0 and we want at the next point this function value to be at all 0 then we equate this to 0 0 vector and then take f x on the other side pre multiply with the inverse of j x and that gives us delta x that delta x.

We add to the current x and get the next x right. So, the next value of the vector x is current one minus j inverse f. So, this is the typical iteration of Newton Raphson method for solving a system of non-linear equations, this matches exactly with what we had earlier.

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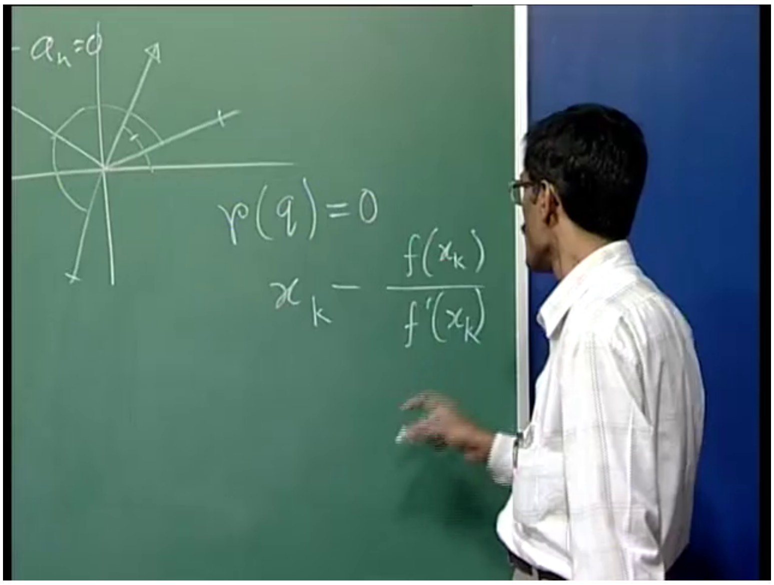

For single equation for a single equation we had this, now again f of x k is here, f of x k and division by the first derivative is now replaced with pre multiplication with the

Jacobean inverse division by a matrix you cannot conduct. So, the corresponding operation here is pre multiplication with inverse. So, further you could write it as minus 1 by f prime into this that is f prime to the power minus one into this. So, that j inverse is here Jacobean so that is the first derivative.

Now, this method of solving a system of linear equation non-linear equations has it is usual merits and demerits, merit of fast convergence and demerit of the lack of guarantee of convergence, it might wonder away it might oscillate it might diverge such things may happen and you can figure out that such things are most likely to happen when this matrix is closed to singularity because in that case the corrections will turn out to be. So, large up to which you cannot rely on a first order truncated Taylor series.

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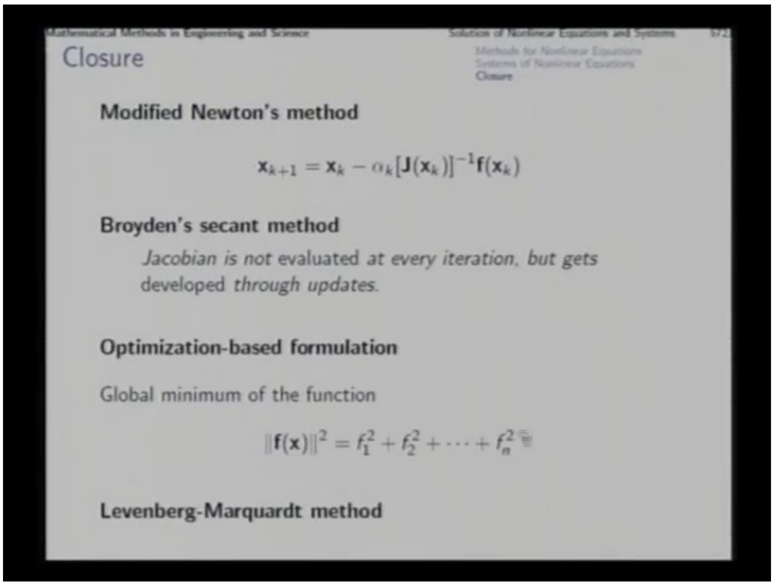

So, what are the other methods, there is a modified Newton's method and then there is a Broyden's secant method. So, just like the secant method of a single equation for multiple equations also for a system of equations also you have a secant method in which this same Jacobean theme is used, but Jacobean is not evaluated at every iteration because you see that it is costly because for n functions you have to determine n derivative. So, n squared derivative you need to evaluate. So, that is costly. So, in broydens secant method the Jacobean is not evaluated at every iteration, but it is developed through some suitable updates at every iteration through the steps that we conduct we get some idea about the derivative and based on that idea we update we keep on collecting derivatives. We keep on updating the Jacobean matrix and that way towards the later iterations we expect to find the Jacobean without actually evaluating derivatives we try to develop a Jacobean which is closed to accurate.

The best formulation for non linear equation solving is through the method of optimization, you see that rather than asking for f x is equal to 0, f 1 0, f 2 equal to 0 f 3 equal to 0 and so on. If you pose the question as an optimization problem and you say we want to minimize this function, we know that a global minimum of this function is at that point where each of the squares is individually 0 right, because it is a sum of square. So, this kind of an optimization based formulation find on near system of equations is typically the best among the non-linear equation solving processes and for this there kind of after we formulated like this, in principle we can use any optimization method to minimize this function.

But in practice we find that some methods perform better in this kind of problem some other methods do not do. So, well Levenberg Marquardt method is one optimization method which is particularly suitable for this kind of equation solving processes and least square minimization problem which we will discuss later, because the proper appreciation of this method will be possible only after some discussion on the optimization method, methods of non-linear optimization a topic which we start from the next lecture.

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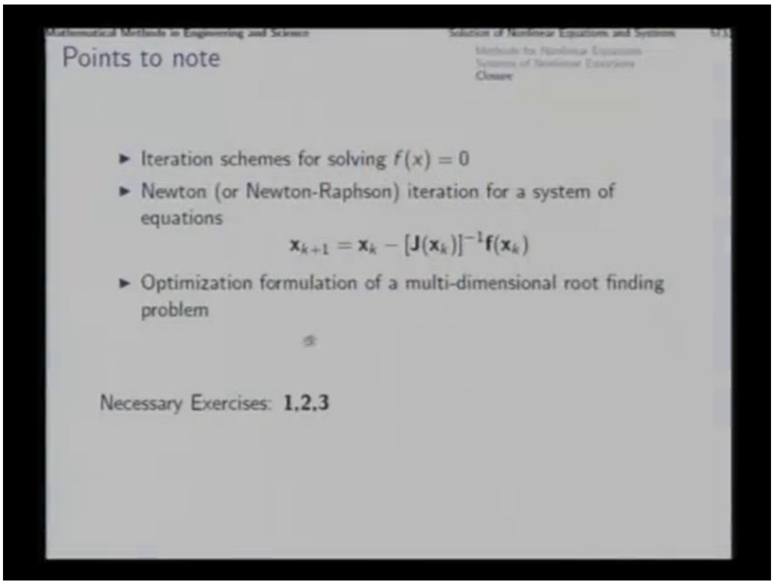

In the current lesson these are the important points that we should keep in mind the iteration schemes for solving single equations and Newton Raphson method which in most situations turns out to be adequate. If it is not adequate then you look for certain other optimization based approaches and other methods also have a connection with non-linear optimization method. So, next lecture we start our study of optimization methods.

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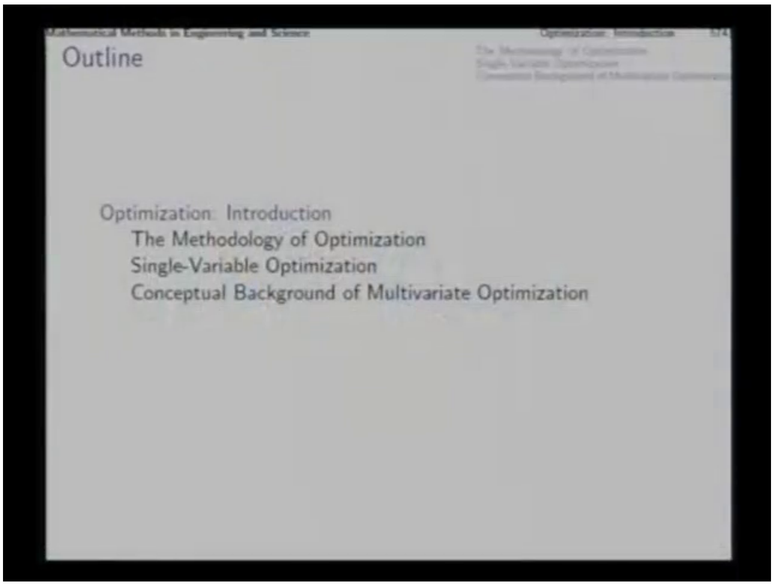

And along that we will see at the appropriate time what is the optimization based formulation for solving systems of non-linear equations.

Thank you.