Mathematical Methods in Engineering and Science
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Module - III
Selected Topics in Linear Algebra and Calculus
Lecture - 05
Vector Calculus in Physics

Good morning. Towards the end of the previous lecture, we saw the del or nabla operator operate on ordinary scalar and vector functions. Now, we will consider composite operations involving the del operator.

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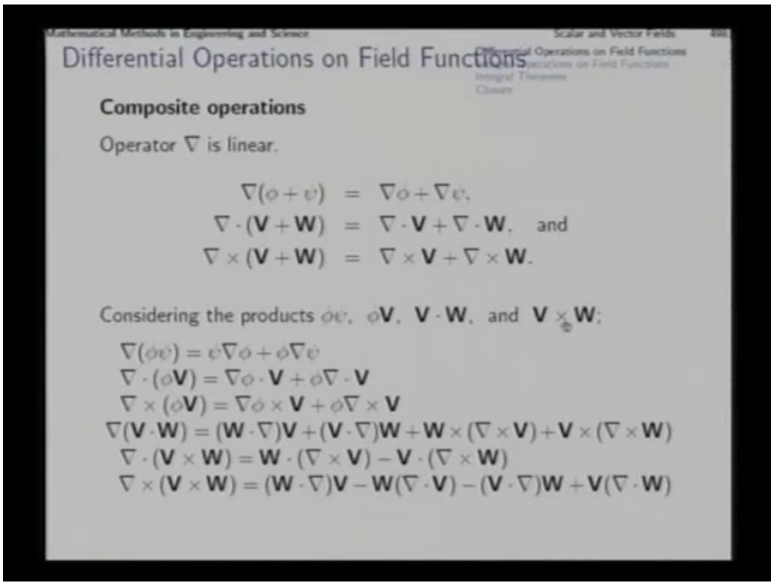

Always keep in mind that the operator del is a linear operator and therefore, whether applied as gradient divergence or as curl, they distribute over a sum of functions.

Next we consider the del operator applied over products of field functions. Four cases may arise. The product of 2 scalar fields is again a scalar field, the product of a scalar field and a vector field is a vector field and the product of 2 vector fields can be in 2 ways; one is the dot product which is a scalar and the other is a cross product which is a vector. When we apply the del operator on these four composite functions, these are the way to apply the operator. So, the gradient of phi into psi turns out to be this. The divergence of phi into V turns out to be gradient of phi dot V plus phi into divergence of V.

The cross product, a curl of phi V turns out to be gradient of phi cross V plus phi into curl of V and so on. So, these representations, these expressions you can work out. If you open, if you expand these expressions term by term and then simplify, when we go further and operate the gradient divergence and curl of a scalar or a vector function by the del operator once more, then we get what we call as the second order differential operators.

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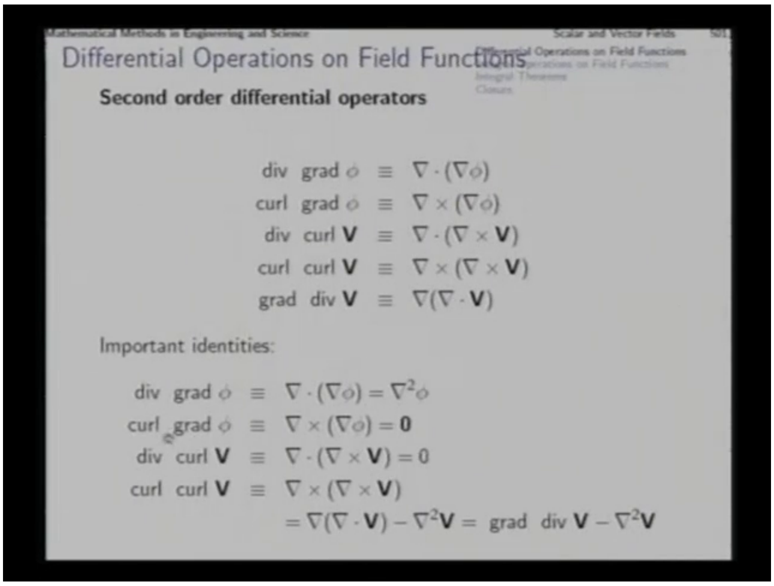

Now, grad phi is a vector quantity. It is a vector function. So, you can apply del in 2 different ways through the dot product or the cross product and accordingly, you get divergence of grad phi and curl of grad phi. When you consider the curl of V and consider applying the del operator over that curl of V, then again you can apply it into 2 ways; one is to dot product and the cross product. Therefore, you get 2 further second order operators; one is div curl and the other is curl curl.

On the other hand, the divergence of V happens to be a scalar function and the only way you can apply del over that is through gradient. So, that gives you these 5 second order differential operators and 2 of them give us very important information that is curl grad phi is identically 0 whatever may be phi. Similarly, div curl v turn out to be identically 0 whatever is the vector function V. So, what these 2 mean is that curl of a gradient is always 0 and divergence of a curl is always 0 and then, divergence of the gradient function grad phi turns out to be the Laplacian del 2 phi. The curl curl and grad div, these

2 have this relationship between them. So, this give you some of the relationships that the second order differential operators always the first is the line integral along a curl.

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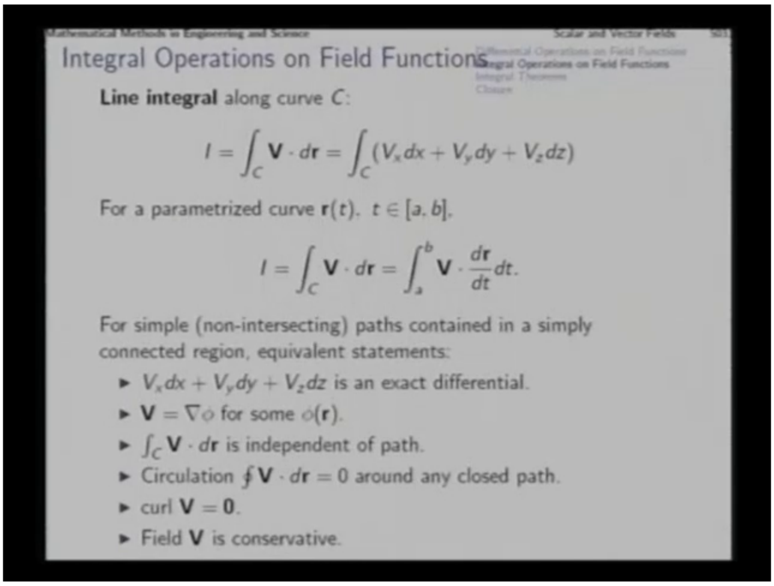

So, if you have a vector function V, then along a curve if you take its integral, that means you have this curve and from this point to this point.

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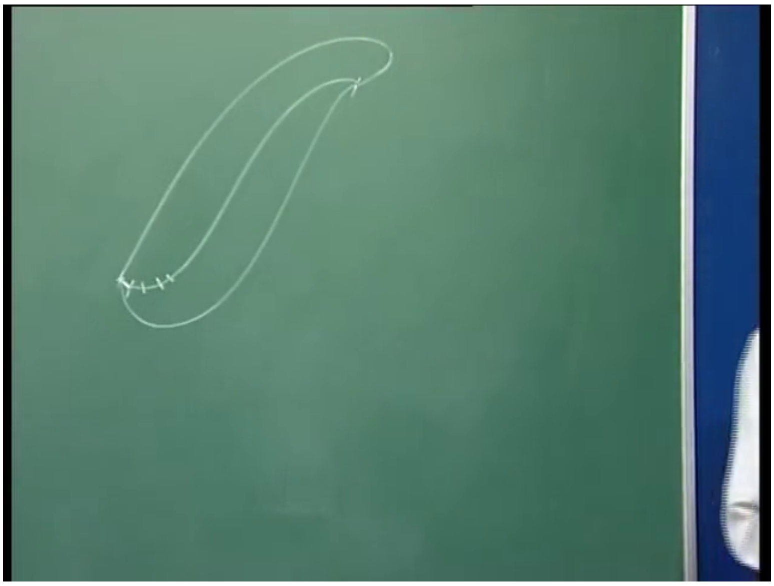

If you want to take the line integral of a vector function, that means you take V dot a small length element, a small vector element replacement element along this curve. So, V dot dr, so such V dot dr components if you keep on adding from the starting point to the end point, then you get the line integral along a curve of this particular vector function along curve p. So, that is defined like this V dot dr which will be V x dx V y dy plus V z dz and this will be integrated all over the curve continuously. So, if the curve is parameterized in this manner over an interval a to b for t, then this line integral reduces when ordinary definite integral from t equal to a to b like this. So, V dot d ir becomes V dot dr by dt into dt. So, V dot dr by dt comes out to be a function of t which you can integrate from t equal to a to b. These are some important statements which mean the same situation.

So, all of these are equivalent statements for simple non-intersecting curves or path contained in a simply connected region. All of these are equivalent statements V x dx plus V y dy plus V z dz. This quantity here, this differential quantity here is an exact differential that will mean the same thing as that the vector function V is the gradient of some scalar function, some scalar field phi and that also means that if this is a perfect differential, that means it can be integrated and the integral of this will be something which does not depend on the path along which the integral has been performed. So, for that kind of a function V which is the gradient of some scalar function whether you integrate from this point to this point through this curve or along this curve or along this curve, the result will be same and this also means that if the curve field is a cross curve, then the point from where you start, then it is the same point where you end.

That means, the integral line along a closed curve circulation represented with this circle on the integral sign that turns out to be 0 around any cross path. It also means that curve V is equal to 0 that is clear because V itself is the gradient of some scalar field. So, it is called must be 0. So, this in terms of relevant physics means that the corresponding field V is a conservative field.

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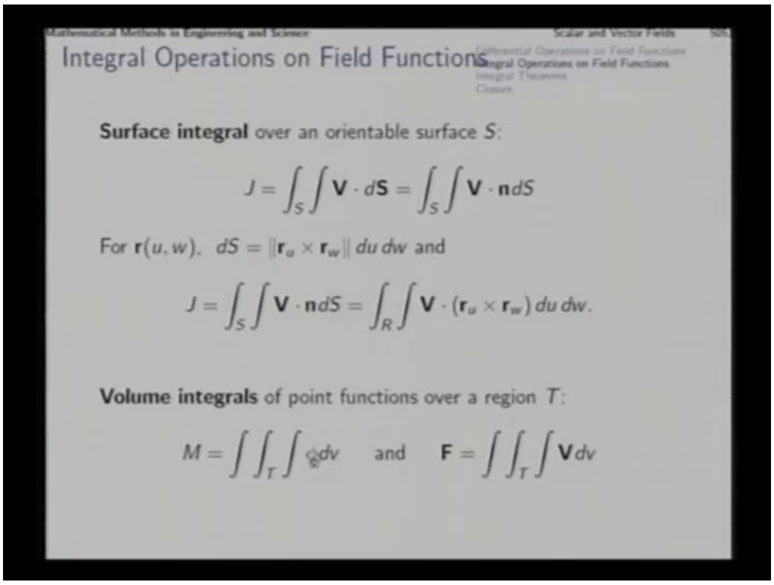

The second integral that we define is a surface integral. This is defined over surface element. Now, dS is a differential surface element and its magnitude is the area of that small surface element and the direction is along the normal to that surface element. So, if you affect this dot product and an integer that over a surface patch, then you get the surface integral of the vector field V over that surface patch S and for this the surface patch S must be orientable, that is which should be clear which side of the surface we are talking about. That is only those surfaces for which one side of the surface and the other side of the surface are clearly identifiable.

Now, there is a parametrization. All the surface in terms of 2 parameters u and w, then this small surface area element dS can be found from this and we can work out the normal. Also, unit normal which is r u cross r w divided by this magnitude and therefore, when we insert the relationship between this expression, then V dot n dS in place of n, we get this cross product divided by its magnitude and this dS the magnitude of that small differential area element, you get this magnitude into d u d w. So, that magnitude gets cancelled and finally, we get this as the indigent and we can integrate over the region are in the u w plane, the parametric plane.

Finally, we have got the volume integral which operates over a differential volume. So, volume integral you can evaluate for a scalar field functions as well as vector field functions.

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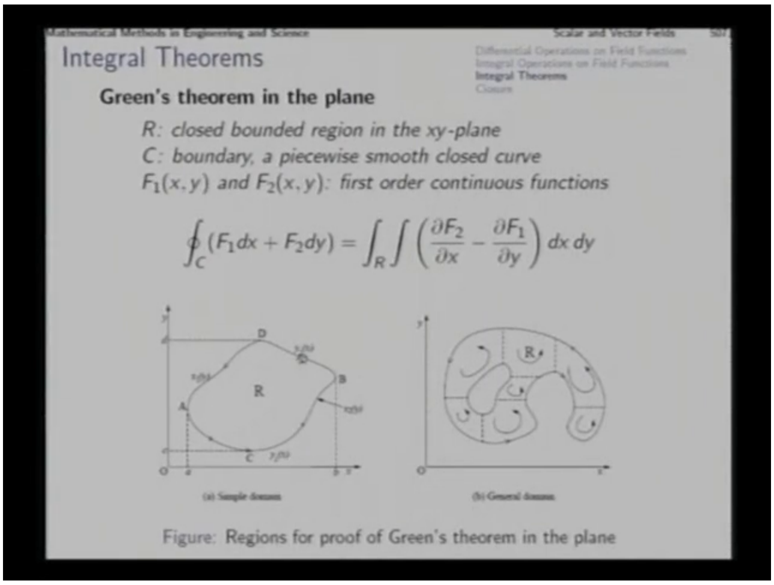

So, in terms of these integrals, we have quite a few important theorem which are called the integral theorem. The first among them is Greens theorem. In the plane consider R as a closed boundary region in the F y plane like this. This region is R bounded by a closed curve C. C is the boundary of R. This curve a, c, b, d is the curve V. Now, F1 and F2 are 2 first order continuous function. First order continues means that F1 and F2 are continuous functions of x and y and they are first order derivative also continuous. Then, Greens theorem in the plane states that the line integral of the vector function F having F1 and F2 as components turns out to be same as the double integral of this quantity on the entire region R. Now, this gives you a relationship between the double integral over a region and a line integral over along its boundary. Now, among the three theorems that we are going to discuss the line of proof for the first of these the green theorem in the plane.

So, we will discuss in detail to give you the proof of these theorems and for the rest of them, we will summarize the theme of the proof only. So, the way you try to prove, the way you tend to prove Greens theorem in the plane is first by considering a simple domain in which any line parallel to the coordinate axis parallel to y axis or parallel to x axis cut the boundary of the region or cuts the curve c only at most 2 points, say if you draw a vertical line like this, it will cut here and here, 2 points only at most 2 point because if you draw this line, then you get a single point and if you draw like this, you get no point because we consider such a simple region first. So, that will mean that the entire curve C can be split into 2 parts; one is acb and the other adb. First one can be called the lower half and the second one can be called upper half because any line parallel to y axis cuts it only at 2 points; one is a lower point and the other is the upper point.

Similarly, since a line parallel to x axis like this also cuts it at most 2 points. So, in another way we can subdivide the boundary into 2 parts; one is cbd, the right half and the other is cad, left half.

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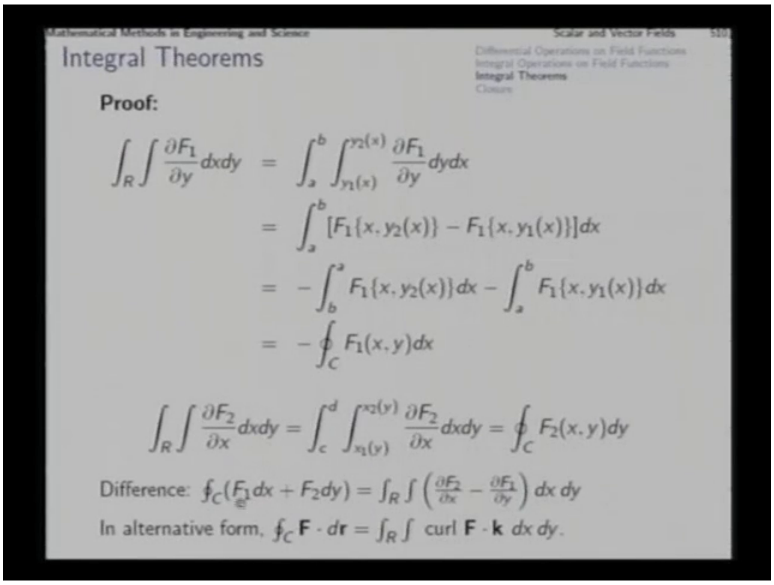

Now, if we can do that, then consider one of the double integrals from here say del F1 by del yd F dy. So, first we consider that and del F1 by del y dx dy, this double integral over the region. So, what we can say is that first we will integrate it respect to y and then, x. So, we interchange the order of these differential dy dx and as we do that first integral is with respect to y and y varies from the lower part to the upper part. So, let us represent this lower part as a function y of x, call it y 1, x the lower part of already is a curve that can be represented as y 1 of x, the upper part as y 2 of x. So, that means for the first integral with respect to y, the lower limit is y 1 of x and the upper limit is y 2 of x. So, that is this y 1 x to y 2 x and this integral will be next integrated with respect to x from x equal to a x equal to b. That means, all these verticals will be then added together from this end to that end, right.

If we do that, then del F1 by del y integrated with respect to y. So, that will simply give us F1 at the upper limit and minus F1 at the lower limit. So, F1 at the upper limit is F1 of x and y 2 and F1 at the lower limit is F1 of x and y 1, right. Now, note this that this has to be integrated with respect to x from a to b over along y 2, the first one.

So, first one along y 2 has to be integrated from a to b. We can say that we will integrate its negative from b to a. That means, F is equal to b x equal to a. If we do that, then this gets changed to negative of this with the corresponding swapping of the limits of integral, right. Now, see F1 of x y 2 and here F1 of x y 1, both signs negative. So, you find that b to a is this integral and a to b is this integral. This is along y 2 and this is along y 1. So, what this is going to mean from a to b along y 1 and then, from b to a along y 2. So, you get the first part gives you this line integral and second part gives you this line integral. So, you have got a closed line integral over the entire boundary, right. So, that means these 2 terms together mean minus, common minus sign the line integral of F1 over the entire curve C. That chose that del F1 by del y double integral over R turns out to be minus the cyclic integral or circulation of F1 dx, right.

So, the second term here turns out to be equal to the first term here. Similarly, by dividing this curve into 2 parts, the left half and the right half as x 1 y from here to here and x to y from here to here, you can establish the equality of this part with this part. That is the first double integral from here as the same as the first line integral from this side and second line integral from this side. So, as you do that in this manner, first integrate with respect to x from this limit to that limit and continue together next integral with respect to y and then, we add them together, then you find that there this turns out to be same as this and as you take the difference of the two, you get the final result which is this.

Now, if you carefully evaluate, if you carefully check this, you will also find that this turns out in alternative form. This one is the line integral F dot dr. F is F1 i plus F2 j n the plane. There is no k component. So, then this is F dot dr and on this side you find that this turns out to be the magnitude of the curl F and its direction is k because i and j are both in the x y plain. So, curl will turn out to be in the direction k and if dx dy is an area element the xy plane, then its direction as a as area is vector quantity. The corresponding direction will be again in the k direction. So, the magnitude will remain and k curves k dot k will turn out to be unity. So, curl F dot k you will find turns out to be the same thing.

So, in alternative form this same relationship, the result of Greens theorem in the plane means that the line integral of a vector function turns out to be along a closed curve. C turns out to be same as the surface integral of the curl of f over a surface element bounded by this same closed curve C. Later we will see that this n, a more general form turns out to be the statement of Stokes theorem. So, that way Greens theorem in the plane is actually a special case of Stokes theorem which is more general. Now, recall that we consider this entire proof for a simple region. Simple region in the sense, simple domain in the sense that any line parallel to one of the any of the coordinate axes cuts the curve C in at most 2 points.

Now, if that is not the domain, the domain is like this not only general, but also multiple connected. There is a whole. Also this part is not included in the domain. So, for that also we can have Greens theorem and the proof is not very complicated because for this kind of a domain, we can always decompose this domain into simpler regions, simpler domains in such a manner that each of the component domains satisfies this kind of a requirement and then, over every component domain we can prove this and then, we just some up all these components; 1, 2, 3, 4, 5, 6, 7 components we sum up together. The double integrals are directly additive. They add up totally and we get the complete double integral by the simple sum.

Further, line integrals here what we find is that as we add them up, then the actual boundary of the original domain, both the outer boundary and the inner boundary is covered only once. On the other hand, the spurious boundaries which was due to our subdivision of the domain gets actually circulated twice. As a part of this sub domain, it got circulated ones from lower end to upper end as a part of this region. It got circulated once, it got included once from upper point to lower point. That means, this inner boundary, this spurious boundary between 2 sub domain which was not part of original boundary gets included in the integral twice; once along this way and the next time along this way and in the algebraic sum, they get canceled for each of the inner boundaries.

The spurious boundaries that is the same thing that is going to happen. So, in the final sum, the line integral that remains status to the actual original boundary of the given domain and removes all the contributions from the spurious boundaries because they are traversed twice in opposite senses. So, they cancel each other. So, this way Green theorem in the plane can be applied to these kind of domains also.

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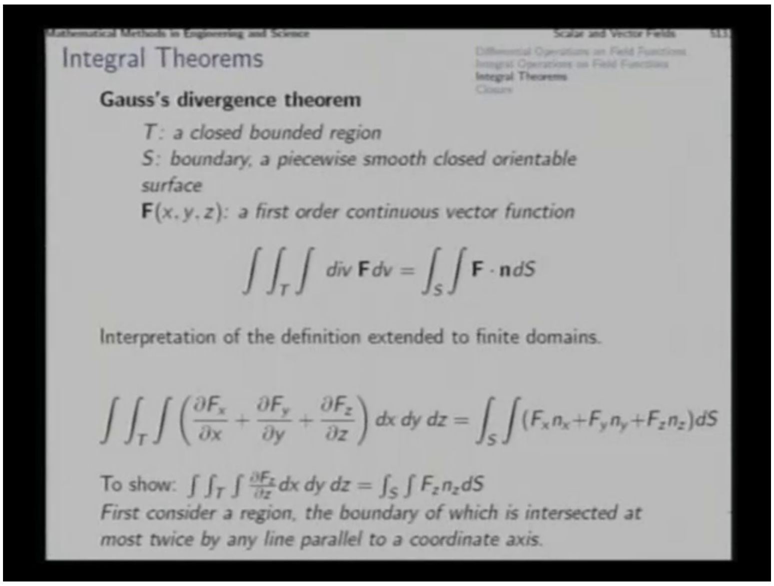

Now, the next important theorem is Gauss theorem which has a lot of application, lot of fundamental interpretation in physics. So, for that we have a closed boundary region represented as T and its boundary S which is a piece twice smooth close orientable surface and over the entire region, we have got defined a vector function F which is first order continuous. That means, it is continuous and its first order derivatives are also continuous. So, for that kind of continuation, the Gauss theorem are the divergence theorem says that the volume integral of the divergence of the vector function F over the entire volume is the same as the surface integral of the function itself of the vector function itself over the boundary of the region that is the surface integral is over the boundary's.

Now, this is actually direct result of the interpretation of the definition of divergence to the finite domain, that is whatever is the meaning or interpretation of the quantity divergence for an infinitesimal domain around a point, the same theme when extended to finite domains, the corresponding extension turns out to be this divergence theorem. So, when you open these expressions, then you get this divergence of F is this. The volume integral of that is this over T and F dot n gives you this scalar quantity and you get the surface integral of that with scalar surface area element dS. So, this equality is the result of Gauss divergence theorem. If you want to establish this result, what you try to do is that you try to establish the equality of this term. By term is a third triple integral from here is going to be equal to the third double integral from here, the z component to z component and so on. The term by term equality you can establish that is del F by del z triple integral will be equal to the surface integral of this part and so on for all three parts.

So, for this also first we consider a region, a volume metric region such that the boundary of which is cut by any line parallel to x axis, y axis, z axis at most 2 points and not more than that. First we consider that kind of a region over that we establish the equality and then, for extension to general regions, we again sub divide the general region into many such simple regions which satisfy this requirement and sum up the contributions and in the case of Greens theorem, in the plane the direct additive sum was on the double integrals here. The same thing happens for the volume integral and there the boundaries where segments of lines or curves which were traversed in inner spurious boundaries where traversed twice in opposite senses here, the boundaries will be surface elements and then, whatever spurious element is going to be used from one side, in one of the double integrals, one of the surface integrals for one region, one sub region, it is going to be considered in the other sub region as from the other side and that is why those spurious surface elements in the final sum get canceled out and you get actual surface integral over that surface which is the part of the original domain.

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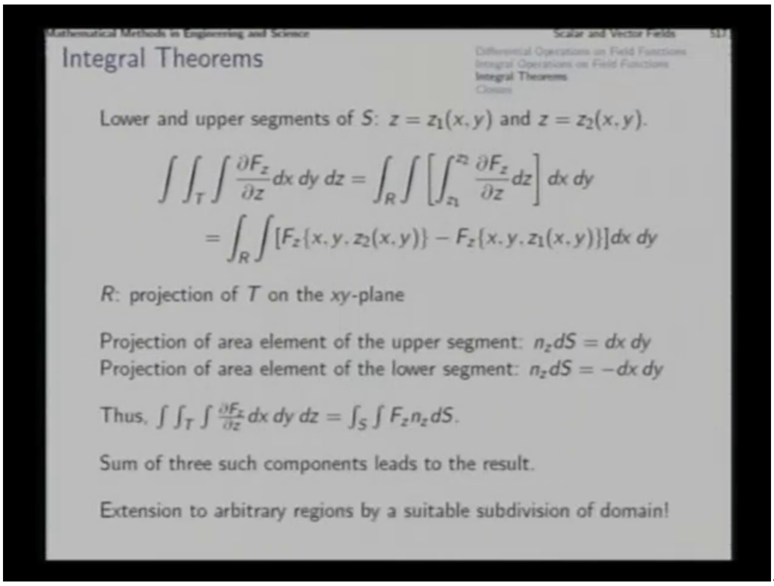

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Now, we will omit the detailed proof of this step by step and you can follow the proof at leisure later and work out the proof in the same lines.

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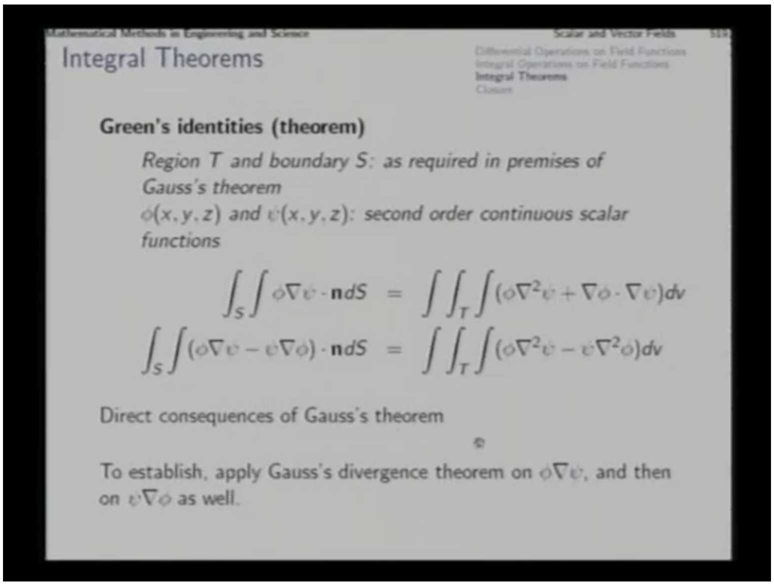

There are 2 further important identities or results called Greens identities which also work with a region, volumetric region T with its closed surface boundary S as required in the premises of Gauss theorem and when we apply Gauss theorem over certain functions in certain manner that is once on phi grad psi and then, on psi grad phi, then as direct consequences of Gauss theorem, we can establish these relations which have a lot of important applications in many field.

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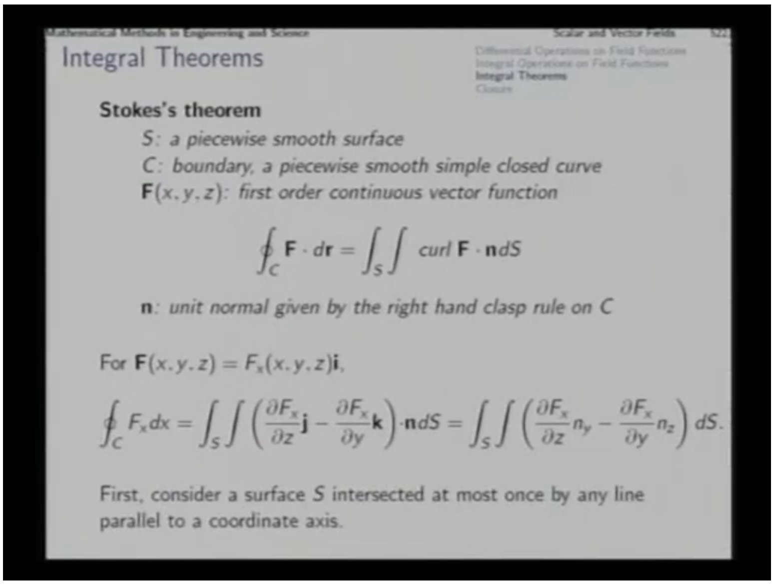

The third important theorem of vector calculus is the Stokes theorem that makes reference to not a closed surface, but an open surface.

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So, suppose this is an open surface S and this is the boundary of the open surface. Now, note that this same boundary of the open surface with the same boundary being a closed curve, you can have many different surfaces. For example, if you have a ring and then, here is a net with which you try to catch something. So, now you can go on changing the shape of this net, but the ring which is the boundary of the net remains the same. So, all these surfaces will have the same boundary and these are all open surfaces. This side is open. So, that is why you get these boundary. Note that a close surface will have no boundary where in open surface, there is a boundary.

So, even the planar region if this curve is a plane curve, then the planar region bounded by this curve is also one such surface and that kind of a surface we encountered in the case of Greens theorem in the plane. So, this is in surface S1 of these is taken as S and the boundary of the open surface S is this curve, this closed curve C. Now, if there is a field function first ordered continues vector function F defined over this entire region, then Stokes theorem tells that the line integral of the function F over this closed curve like this turns out to be the same as the surface integral of the curl of F over this entire surface S. That is a statement of Stokes theorem in which the unit normal that you need to use here is given by the right hand class rule on C that is since it is an open surface. So, whether to take the normal in this way or in this way has to be decided. Both are valid, but it will depend on which way we will be traversing C. So, as we are traversing C, so if we put our right hand along the arrow like this, then whichever way the thumb will point out that works out to be n ,not this, ok.

So, if we point through the arrow by the clasping of the right hand, in whichever way the thumb points through the surface, that is the direction of n to be used here in this context. Now, with again omit the proof of this and as a special case, the Greens theorem in the plane, the proof of that we have already seen.

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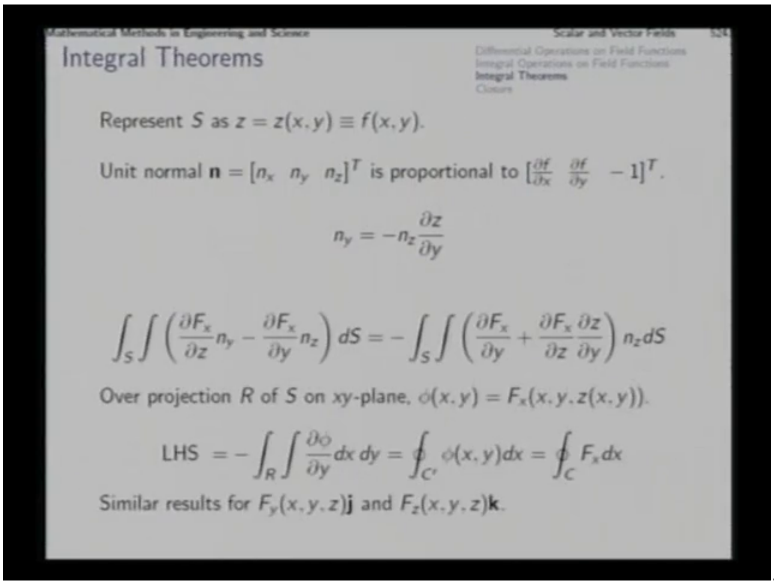

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So, here you will find that if we consider the surface region S to be a region in the x y plane itself, then you will find that we will recover the Greens theorem in the plane as we have studied earlier. So, we will bypass the proof of this.

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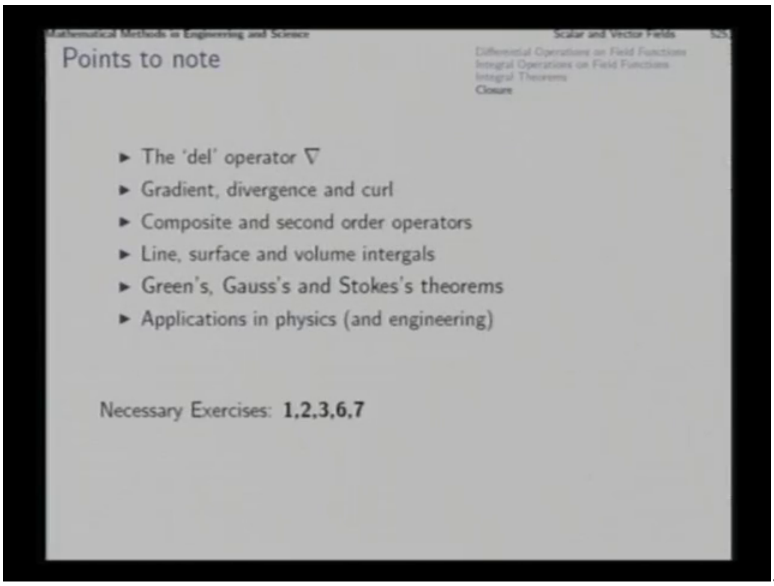

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So, the important points to be noted from this lecture are, this lesson are the del operator. The del operator applied in three different ways on scalar and vector functions. On scalar function you get gradient, on vector functions through dot and cross product you get divergence and curl the way we have the composite and second order operators on field functions.

Then, next the line surface and volume integrals and the three important theorems which are Greens, Gauss and Stoke theorems and these theorems are important in Physics and Engineering of large number of systems through their applications. So, in the exercises of this chapter in the book, you will find several important problems from physics as well as from applied mathematics where some of these theorems are applied directly to break the problem or reduce the problem into much simpler situations. In these 2 lectures on the vector calculus by avoiding some of the long proof, we have saved some time. So, let us take some examples on vector calculus, some examples on physics and engineering problems as well as on applied mathematics problem pertaining to the scalar and vector field are given in an exercise in the text books in chapter 18. I strongly advise you to attempt those exercises, particularly the exercise on Maxwell's equation because that takes you through a complete exposure to almost all the important issues discussed in this lesson.

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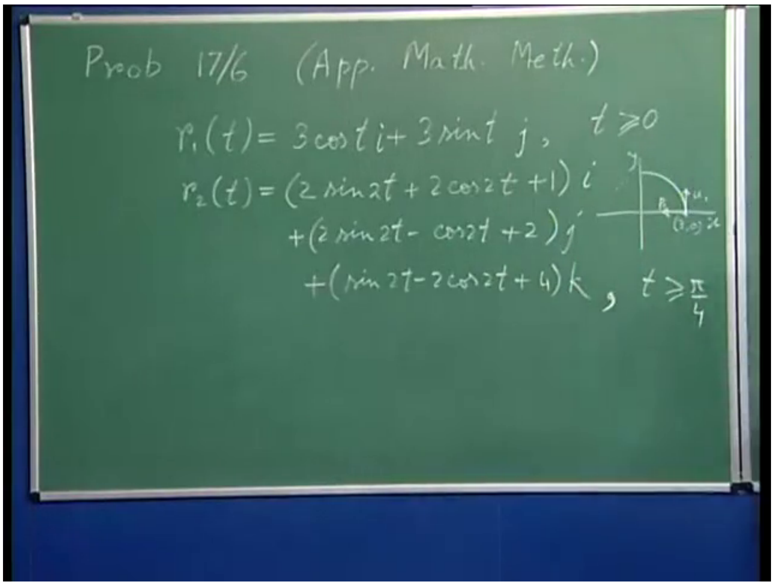

Now, we considered one problem on parametric curve. This is actually the problem 17/6 of the text book that we are following in the appendix of the corresponding to this problem. Three approaches have been outlined here. I will elaborate one of the approaches. The problem is to verify or check whether these 2 parametric equations represent the same curve or not.

This is the first curve and this is the second curve. Now, you will note very easily that the first curve is evidently a circle with origin as the center and radius three lying in the expression y plane. That much is very clear. Now, we need to check whether r 2 of t also represents the same curve. So, for that we try to transform r 2 in such a manner that it gets started from the same point, where the first curve starts. So, first for r 1 for the first curve.

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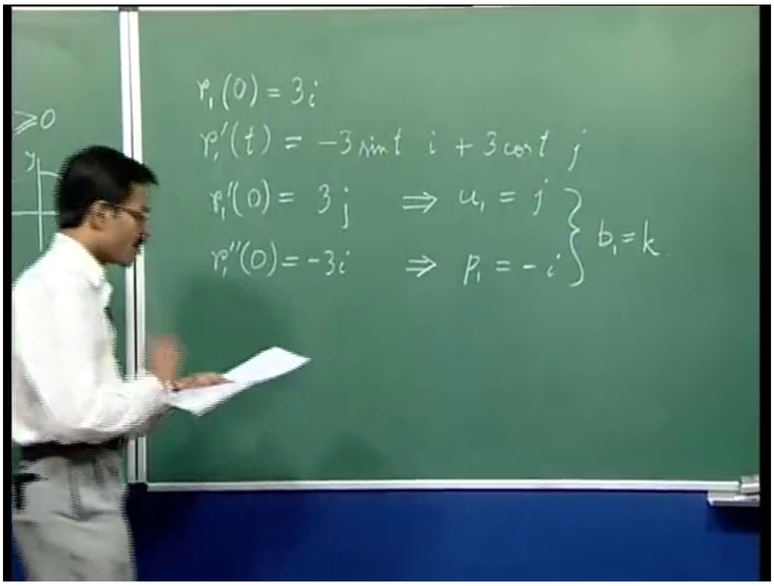

Let us see what r 1 at t equal to 0 at starting point is. So, if you put t equal to 0 here is i, yes if we put t equal to 0 here, then we get r 1 0 as 3 i and its derivative r 1 t. The derivative of the first curve, the tangent vector that you will get as 3 sin t i plus 3 cos t j at t equal to 0. If we evaluate this tangent vector, then we get this as 0 and from here we get c j. That means, that the curve starts here at 3 0 3 0 0 actually in 3 d 3 0 0.

So, in x y plane this curve is in planar curve in x y plane. So, we are making the diagram only in the x y plane. So, it starts from 3 0 and since it is a circle, it must start like this and that is why the tangent vector is found to be in the direction j. So, this s, the vector circles cross ends. So, this is the point of our interest right now. Now, if this is a tangent vector, then what the unit tangent is. So, the unit tangent at the starting point for the first curve is j, right. At the starting point, note that you can write u 1 of 0, but to make the notation simple, currently we are writing here. Understood it is the fact that it is the starting point which is being analyzed right now. Similarly, we can find out the second derivative of r 1 t and then, we put the value t equal to 0 in the second derivative which will give us this as minus 3 i which will mean that the unit principal normal is minus i for the first curve. So, if unit tangent is there, unit principle normal is there, then from these 2 we can work out what bi normal is, right. U 1 cross p 1, so j cross minus i will get that as k. Now, this much we have in hand. Now, we will try to find out the same quantities from r 2 and then, try to work out the transformation which will bring u 2 to u 1, p 2 to p 1 and b 2 to b 1 and r 2 r 1 at pi by 4 to r 1 at 0. That means, we want to match the starting point and we want to match the Serret Frenet frame at the starting point.

So, we want to match the point r 1 and u p b point r u p b at the starting point of both the curve. So, you leave the first curve as it is placed and the second curve we try to bring here. So, for that purpose please make note of this and let me mark here unit tangent is here, unit principle normal is here and of course, b 1 is perpendicular out of the board, right because of the cross product. So, now we get rid of this because all the information from here is actually available here as well this is the starting point. Now, the same thing we try to evaluate from here. So, for the starting point we put t equal to pi by 4 as we put t equal to pi by 4 here.

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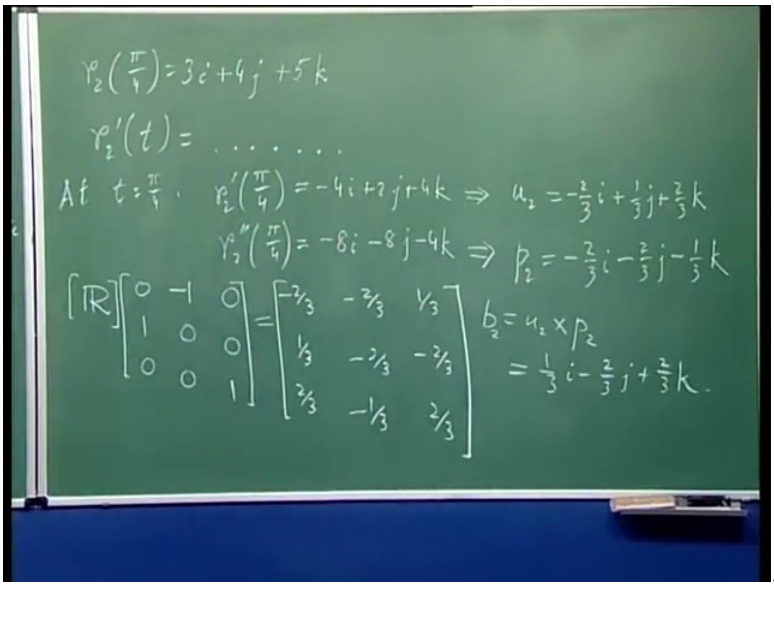

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So, pi by 4 into 2 that s pi by 2. So, sin pi by 2 is 1 cos pi by 2 is 0. So, from here we get 3 i from here, we get 2 plus 2, 4 j and from here we get 1 plus 4, 5 k. So, this is the starting point for the second curve as it is given.

Now, if we develop the second derivative, first derivative and second derivative that also you can similarly do finding derivative is not very complicated. So, I am omitting that and giving you the result. If you evaluate the first derivative of this that is r 2 prime t and then, at t equal to pi by 4, you evaluate that, then you find that you get this as minus 4 i plus 2 j plus 4 j y. You must evaluate this expression for the derivative and then, verify that this indeed is what you get.

Similarly, differentiating that again for the second derivative and inserting the value pi by 4 in case of t you would find minus 8 i minus 8 j minus 4 k. So, from here you can work out the unit tangent and unit principle normal between these two, right. So, from here you will get u 2 at the value pi by 4 at the value t equal to pi by 4, that is at the starting point as now this is minus 4 to 4, right. So, what will be the magnitude? Magnitude will be 6, right. So, you divide by 6. So, you get minus 2 by 3 i plus 1 by 3 j and plus 2 by 3 k. This turns out to be the unit tangent at the starting point for the second curve.

Now, this in the first case as you saw it was very clear that the tangent vector came to be in the direction j and the second derivative appeared in the direction minus i which are orthogonal to each other, perpendicular to each other anyway. So, you did not have to subtract the component of r double prime from the direction of u. If the same thing happens here also, then it is fine otherwise we will have to subtract that. So, whether this r prime and r double prime are perpendicular to each other that you can check, otherwise we would have to subtract.

So, you get 32 minus 16 minus 16, it is 0. So, the dot product between these 2 is 0. That means, this is indeed perpendicular to this. So, that makes our life easy. So, the unit vector as along this direction itself is the unit principle normal. So, you get the unit vector in this direction and divide by 12, right. Yes the magnitude of this is 12. So, you divide by 12 and you get minus 2 by 3 i minus 2 by 3 j and minus 1 by 3 k. So, this you get as the principle unit, principle normal and from these 2 you can work out b 2 as the cross product which turns out to be 1 by 3 i minus 2 by 3 j plus 2 by 3 k.

Now, we have the starting point of the curve r 1 here and its u b p Serret Frenet frame oriented in this manner and for the second curve also, we have got the corresponding pieces of information r 2 at the starting point and u 2 p 2 b 2 at the starting point and the Serret Frenet frame. Now, if the 2 represent, if the 2 parametric equations, these 2 represent the same curve, then through a rigid body motion a displacement and a rotation if we can bring this r 2 and this u 2 p 2 b 2 in the location and orientation of the first curve r 1, then the complete curve should match. So, let us try to do that. So, what rotation transforms these three vectors in the direction of these three vectors, this is the question we ask.

So, suppose that rotation matrix 3 by 3 rotation matrix is r. So, r transforms u 1 p 1 b 1 to u 2 p 2 b 2. We could do the other way also that is what rotation would transform u 2 p 2 b 2 to here, but they are not very different because the 2 rotation matrices transposes each other. So, if we try to do this, we are going to do it like this because the matrix that will appear here will be easier to invert the matrix that would come from, there will be more complicated to invert and we must invert one of the 2 matrices. So, r into u 1 what is the column representation of this vector u 1. It is u 1 is j. That means, 0 1 0 0 i plus 1 j plus 0 k. So, u 1 is 0 1 0 0 1 0 p is minus 1 0 0, that is minus i minus 1 0 0 and b is the z vector k, that is 0 0 1. We find that we want the rotation matrix which transforms u 1 to u 2, p 1 to p 2 and b 1 to b 2, right.

So, u 1 to u 2 r into first column of this should be here that is the vector u 2. Similarly r into p 1 should be p 2 and finally, r into b 1 should be b 2 that is this. Now, we want to determine r. So, we need to post multiply the matrix on the right side with the inverse of this which is quite simple. So, let me give you the expression for that r which you can verify later. Now, it is actually quite simple what we need to do is to interchange these 2 columns on both sides. So, as we interchange these 2 columns, we get 1 here and minus 1 here. Then, the first column we should make negative because this is minus 1, ok.

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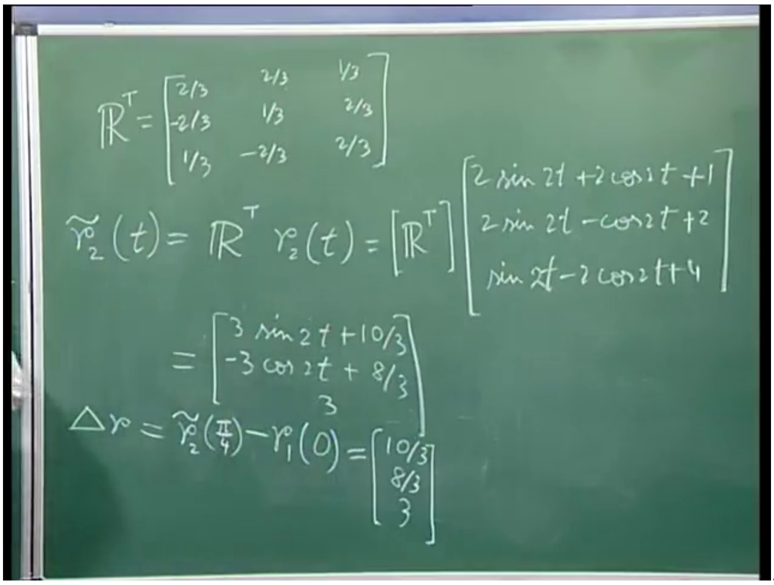

So, that will immediately make this identity matrix and whatever is there will be the product of this with its inverse from the right side, right.

So, this tells us that the matrix r is the negative of this column comes first, this column goes next, right and the third column remain in its own place because this is 0 0 1. So, those column operations which will make it identity the corresponding column of operations done here give the value of the rotation matrix r. Now, this is the matrix r which operated over u 1 p 1 b 1 takes it to u 2 p 2 b 2 and what we are looking for is actually the inverse transformation because we want the second curve to come and merge here, right if they turn out to be the same curve, right. So, we need the inverse of this rotation matrix and inverse of rotation matrix is just transpose. So, it transpose this matrix. So, these 2 exchange their places. In this case, we do not have to make any change because they are same and these 2 exchange their places.

So, this transformation applied on the second curve will bring it to the first curve if they turn out to be the same curve. So, that is the check that we have to perform. So, the second curve transformed through now first orientation till now we ave not given the position transformation, right. So, the orientation of the second curve is changed in order to match the orientations at the starting point. So, let us call it r 2 tilde that will be r transpose into what is r 2 given. So, that means this matrix multiplied to the vector. Now, this i j k the way it is written, the first entry will be here, the second entry the j entry will come here and k entry will come here. If we multiply this matrix with this completely, then we get r 2 t r 2 tilde t this one as 3 sin 2 t plus 10 by 3 minus 3 cos 2 t plus 8 by 3. This is after turning it.

So, now the orientation of the second curve at its starting point is the same as the orientation of the first curve at its own starting point. That means, now if they are the same curve, then this curve and that curve now have taken parallel position. Now, to get the second curve here we need to apply a displacement. So, what displacement we will need? The displacement that is needed the delta r will be this, r 2 tilde at its starting point which is pi by 4 minus this r 1 at its own starting point and earlier we saw the starting point of this r 2 at pi by 4 and that was 3 i plus 4 j plus 5 k from that we subtract this which is 3 i. So, the rest of it we get 4 j plus 5 k. So, r 2 has been transformed. So, better we evaluate this. We evaluate r 2 tilde at pi by 4 from here fresh because during rotation its starting point has gone to some other location. So, put pi by 4 here. So, that means sin pi by 2 is 1. So, 3 plus 10 by 3 from which we subtract this 3 i. So, only 10 by 3 will remain and then, minus 3 into cos pi is 0.

So, this 8 by 3 minus 0, so 8 by 3 remains and finally, that k part which is 3 minus 0. So, we have got 3. So, this is the displacement which we have to give in a negative direction to r 2 tilde to bring it here, right. So, now let us call this; now this r 2 tilde was a rotated version of r 2.

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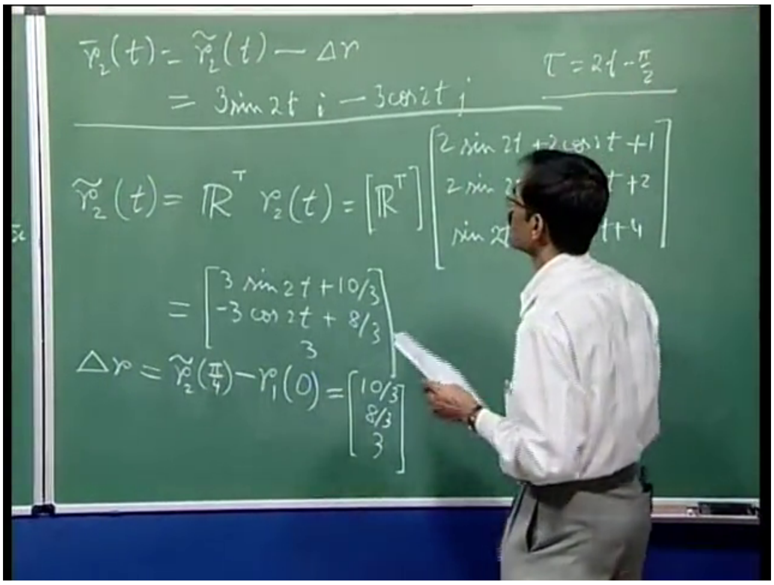

Now, r 2 bar will be the translated, further translated. So, this r 2 bar will be this r 2 tilde minus this delta r. So, what will be find from here? If we subtract this, then we will get 3 sin 2 t i minus 3 cos 2 t j and 0 0 k. So, now we find that this is also a circle in the x y plane starting from here. No question about all these, but still you find that the equation of this and the equation of this do not exactly match because there is a change of the parameterization. If you now try to re parameterize this same curve with a little different parameter say with tau equal to 2 t minus pi by 2, then you will find that the equation of this transforms exactly to the equation of this in terms of tau.

So, whatever is this equation in terms of t, the same equation here you will get in terms of tau. You will get 3 sin 3 cos tau 2 t minus pi by 2, you will get 3 cos tau i plus 3 sin tau j and that will also transform the starting point in the first curve. The starting point was 0 in the second curve till this point the starting point is pi by 4. When you put that pi by 4 here, you will find that the starting point of tau will be 0. So, that way you will find that through these changes rotation to make the Serret Frenet frames of the 2 curves parallel, then translation to make the merge at the initial point and then, reparametrization we will show that the 2 curves exactly have the same equation. So, up to this stage the curve has actually come here, but its parametrization is different. So, at different values of the parameter, it is going to different points rather than reaching the same curve at the same diameter values. So, this reparametrization will convince that the 2 equations are found to be exactly same. So far in the discussion on vector calculus and next lecture onwards, we will start our next module on numerical analysis starting from polynomial equations.

Thank you.