Mathematical Methods in Engineering and Science

Prof. Bhaskar Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Module - III
Selected Topics in Linear Algebra and Calculus
Lecture – 04
Vector Calculus in Geometry

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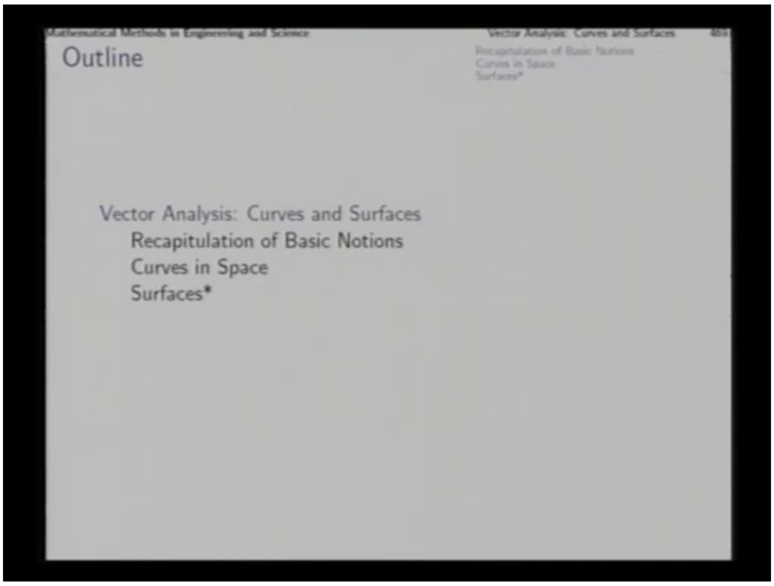

Good morning. So, in this lecture, we will study the phase curves and that will be our first lesson on vector analysis; vector calculus, in this I would advise that before going through the lecture, you should ensure that the basic notion of vector algebra are firmly in your mind. So, that there is no difficulty in following through the discussions that we do in vector calculus.

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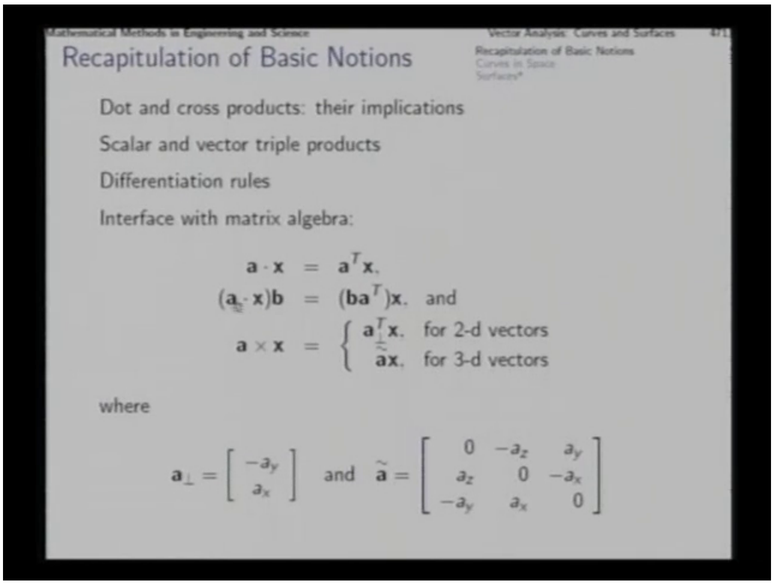

So, the text book the first section of this chapter of the textbook recapitulates the basic notions of the vector algebraic relationships, we will not go through the details of that, but I advise that you go through those things before following this particular lecture. It should be clear the what are the implications of dot and cross products in vector algebra and scalar and vector triple products because they will be very frequently used in this lesson and the next.

Further the differentiation rules of a vector functions, it should be clear in following the analysis that we take up now a few points are special which I would like to remind you, one is that when we try to interface the typical vector algebraic notions with the matrix algebra, then we find that the dot product in between 2 vectors is equivalent to this a transpose x which is the inner product in vectors as understood in the linear algebra terminology next quite often you will find that you come across expressions of this kind a dot x b in which a b and x all 3 are vectors. So, in that situation if you take the expressions of a dot x as it is and put that in this place you will find that the components of x that is x 1, x 2, x 3, they get somewhere lost in the interior of this expression and sometimes it happens that you want to treat x as the right side of a linear algebra expression in such situations you can consider b.

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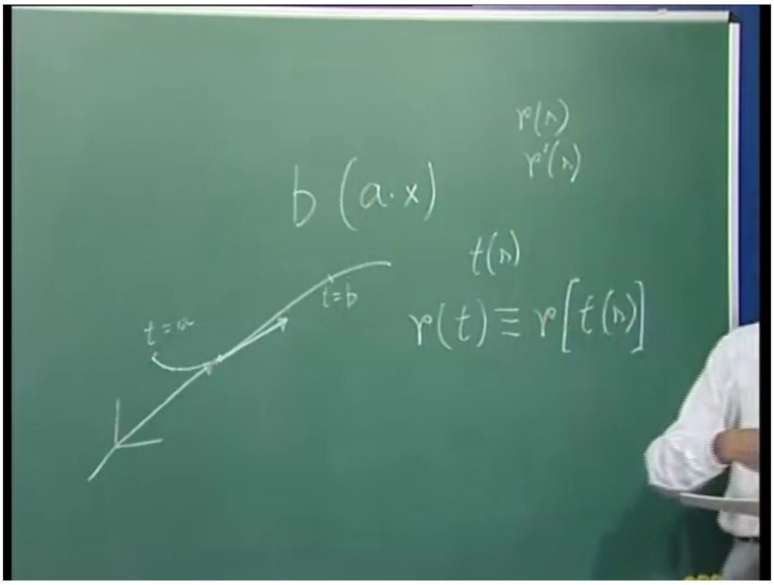

So, we kept on this side and it does not matter because a dot x is a scalar and if you do that if you keep b on this side and then you have an expression which looks like b a dot x and here in this a dot x, if you use this relationship that is a transpose x then you will find that then after that you have all 3 linear algebraic notions b is a vector column vector a transpose is a row vector and then x is again a column vector and then using the associativity of matrix multiplications you come in to this relationship in which b a transpose multiplied together gives you a 3 by 3 matrix and x is a c by 1 vector and you find that you have got the vector x on the right side and that may be helpful in situations where you want to treat x as the unknown or variable vector with a and b as constant and known vectors.

Another important interfacing between the vector algebra and matrix algebra is in the matter of cross product the cross product is something which unlike the dot or inner product is a matter which is very special to 3 dimensional vectors for general n dimensional vectors you do not have anything called a cross product now in the case of 3 dimensional vectors which are sensible in geometry ordinary Euclidean spatial geometry you have got the notion of a cross product between a 2 vectors and in 3 d space it makes direct sense and the result is a vector in 2 d space also in certain circumstances there is an interpretation which makes sense.

So, for example, in the x y plane, if you have 2 vectors a and x then their cross product is a vector which is actually perpendicular to the plane of a and x and; that means, perpendicular to the plane under consideration and that way if you take the straight away expression from a cross x, then you get a scalar value. So, in the vector algebra sense whatever you get as a cross x, if you want put it in the linear algebra sense then you can do it by this relationship. In the case of 2 d vectors, you get a perpendicular transpose x now this particular notation a perpendicular is a vector which is a rotated by a right angle counter clock wise.

So, if the vector a is a x a y, then the rotated vector rotated by ninety degree is this vector and inner product of this vector with x gives you the same result as you would get as a cross x in the case of 3 d vectors if a and x are 3 d vectors then a cross x can be represented through this matrix vector multiplication where a tilde is a c by c matrix which is a skew symmetric matrix with elements of a that is a x a y as it arranged in this manner. So, quite often in simplifying complicated expressions involving vector algebra notions into linear algebra formulation these relationships will help you now with this background of the basic notions of vector algebra rules we proceed to the study of parametric curves or curves in their Parametric representations.

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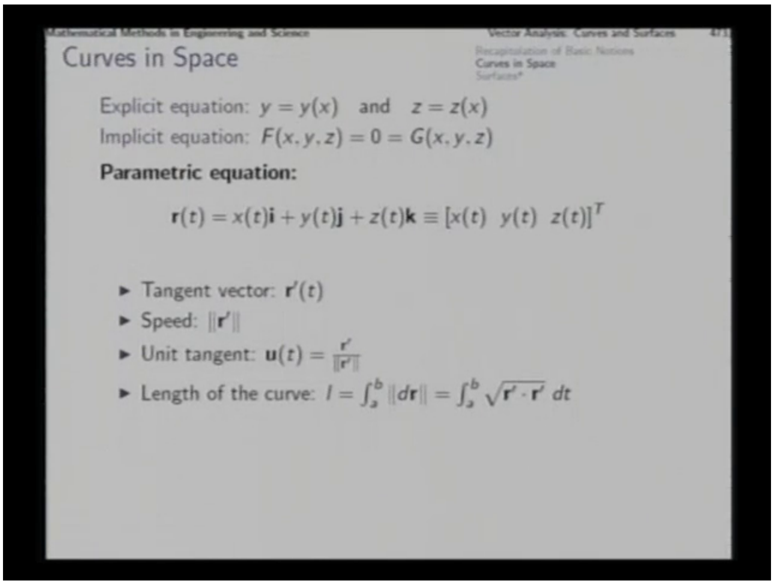

When we try to describe a curve in space we can talk of several kinds of representations this is called the explicit equation of a curve in which y and z coordinates are expressed as functions of x coordinate as you know that for all curve this may not be possible that is for a general cure it may not be possible, it will be possible only when for any given x the values of y and z are unique. So, if a curve goes like this then with as given value of x there can be several vectors of y and z which will be on the curve and such a curve will not be expressible in this kind of a representation this representation through 2 equations in x, y, z like F of x, y, z equal to 0 and G of x, y, z equal to 0 is another representation of a curve and this is called the implicit equation in this case the idea is to represent a curve as an as an intersection of 2 surfaces F of x, y, z equal to 0 is the equation of one surface and G of x, y, z equal to 0 is the representation of another surface and the intersection of these 2 surfaces is a curve and that is a curve which is expressed through this relationship.

Now, this is a representation which is actually unnecessarily complicated for the description of a curve the prior description of 2 surfaces is going to be much more complicated than the original issue in combination to these 2 notions these 2 representations you will find that the parametric representation is much more straight forward and completely general in that the position vector of a general point in a curve is represented as a vector function of the of a parameter in this case t and in terms of t you can express all the 3 x, y and z components as x of t y of t x of t and when you combine them together you get this as the representation of the curve.

So, as the parameter t changes you find that the xyz coordinates of the point on the curve changes and accordingly if you try to plot you get the curve you can represent this curve in this manner or in this manner the 2 representations are equivalent now if you represent a curve in this kind of a parametric representation then a few notions appear immediately if you differentiate this vector function of the scalar variable then you get this derivative and this is called the tangent vector.

So, if this is the curve at this point suppose for a value of t value of the parameter t you get this point say origin is here then by the standard definitions of derivative you get the derivative of this position vector at this point as a vector like this; this is a this is the tangent vector to the curve at this point and this is r prime. Now if you take the modulus the magnitude of this vector, then what you get is called the speed of the curve speed in what sense as if the parameter t is time, then you will find that the r prime gives you the velocity of a particle moving along this curve as with time t as the parameter. So, if you consider the parameter t to the time then what you get as speed is indeed the physical speed that we know of if you consider the unit vector along this direction then you get the unit tangent as this vector divided by its magnitude.

In this manner, if you find out the derivative then you can use it to find a length of a curve from one point to another suppose this is a point corresponding to the value of t as a and this is another point which corresponds to the value of t equal to b now from here to here if you want to find out the length of the curve then what you can do the speed at which it moves with respect to t at this point. So, you take that and multiply that with a little small change d t, then you get this segment similarly you get lots of segment right.

So, the some of these segments will give you the length of the curve as the length of each of the segment tends to 0 and there are infinite such segments very large number of segments and each segment extremely small; that means, it is the sum of a large number of small parts that is integral where you get the length of the curve from t equal to a to t equal to b as this in which you have d r mod here which means d r by d t and d t separately. So, d t is here and d r by d t modulus is this. So, this gives you the length of the curve now if you keep the initial point.

For calculating the length as constant and the final point if you do not specify as a particular value, but if you keep that as variable then through that exercise you can define a very important function which is called the arc length function. So, if you keep b variable then you get this arc length function which is the integral from a to t; that means, starting from this point for every value of t you get an arc length which is the length of the curve starting from this point to the current point.

So, this is a function of t and this is called the arc length function in which you get d s is equal to the same mod of d r which is this and from here if you represent them as d x by d t whole square plus d y by d t whole square plus d z by d t whole square, then a term d t comes outside and here what you are left with is simply d s by d t that is this now a curve r of t is called a regular if the tangent vector never vanishes that is at every if a curve is called regular if at every point of it there is a non zero tangent vectors the direction is clearly representable from the derivative from the first derivative and in such situations you can affect a reparametrization of the same curve with respect to some other parameter say t star which must be a strictly increasing function of the current parameter which is t.

Now if you check the arc length function for a possible parameter then first you note that arc length s of t is; obviously, a monotonically increasing function.

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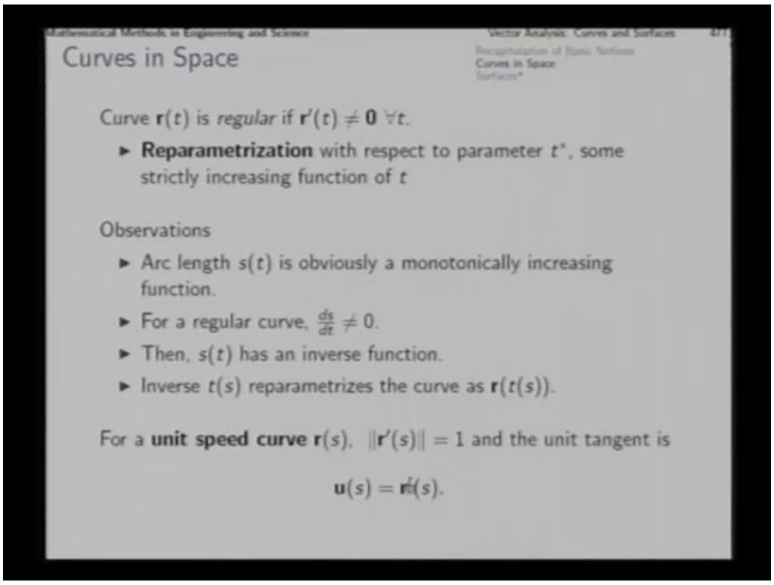

That is as t changes as t increases along the curve you keep on getting the values of s which is continuously increasing. So, arc length is monotonically increasing function next for a regular curve if r prime is not 0 then d s by d t will not be 0 and in that case the function arc length function s of t has an inverse that is its inverse can be expressed as t of s and if we can do that then whenever we had the representation of the function as r of t we can revise it as r of t of s that is by virtue of the fact that the arc length function has an inverse that s from s of t; we can drive the inverse t of s and if we can derive the inverse then we can insert that inverse here and then finally, this becomes a function of s. So, in this manner we can define the curve as with s as the parameter with the arc length itself as the parameter and in fact, the arc length s gives you a natural parameter to describe the curve.

If we parameterize a curve with the arc length as its parameter then we get something interesting what we get in that case is that if we represent it as r s then as we try to find out the derivative for the tangent then we get r prime s and this will be always of unit length this will be always of unit length and that is why a curve parameterize with arc length as the parameter is called a unit speed curve because for every small distance covered you find that the length of the small segment is same as d s. So, this derivative will be always a unit length. So, the tangent that you will get will automatically have unit length. So, you do not have to really divide it with r prime norm r prime norm is one.

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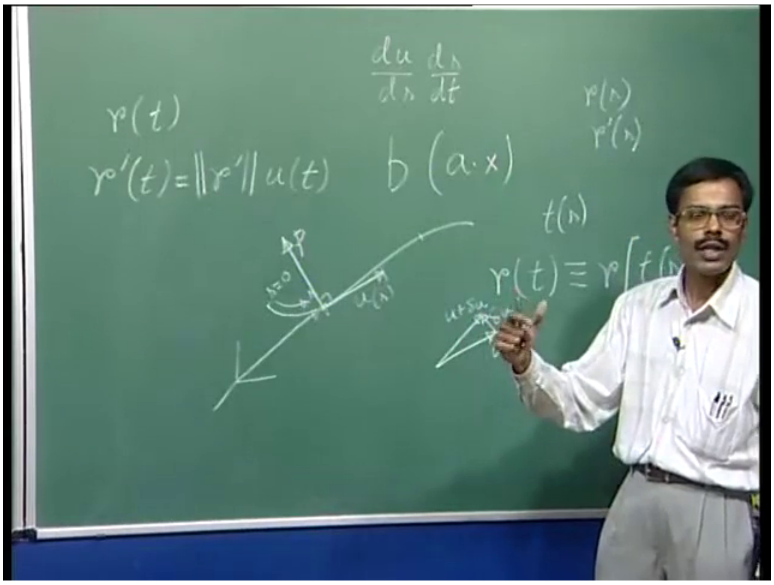

So, in that case that unit tangent you get directly as u of s as r prime s. Now, we come to the most important property of a curve that is a curvature after defining the tangent from the first derivative you try to take the second derivative and then you can see how this tangent changes how this tangent turns and now with the arc length parameterization you have got the unit tangent that ordinary tangent turns out to be a unit tangent directly and in that case all that you need to see is that how this unit vector changes as you proceed along the curve and that gives you the notion of curvature in undergraduate calculus for planar curve you might have studied.

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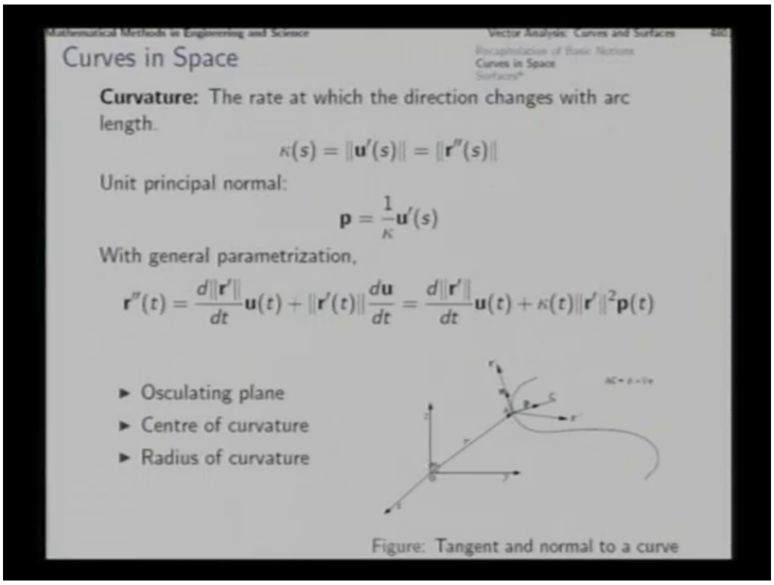

This notion of curvature that is curvature is d psi by d s in which psi is the angle that the tangent makes with the x axis right. So, here also you find the notion is same that is as the as a particle proceeds along the curve, then with respect to distance covered along the curve how the fast direction changes direction is represented by psi; the same thing is here in which we say that with respect to s; how fast this tangent vector u is turning. So, which try to take the derivative of u with respect to s.

So, d u by d F and that gives you the measure of the curvature and the way this curvature function is defined it is going to be non negative always because we are defining the curvature function with a norm which will be never negative. So, you can say a cupper of a that is curvature function with respect to the arc length is norm of u prime or norm of r double prime. So, from here as you do that you note that the unit tangent if this is the u u of s in which case we will be defining the curve not with respect to t, but with respect to s then if this is u s, then note that this is a unit vector and with respect to the arc length the rate of change of the unit vector will be perpendicular to this vector itself because for any small segment here it is a unit vector here also at the next point also, it is be a unit vector.

So, if you take this as u then at the next point also it will be a unit vector tell u plus del u and then this is the change delta u and as s tends to 0 as this change becomes very small then you will find that this change of u that is delta u turns out to be perpendicular to u.

So, any unit vector you take and you consider its derivative with respect to arc length itself then you find that the derivative is perpendicular to the unit vector.

So, in this case that perpendicular vector will be in this direction right perpendicular the way it is defined the vector like this will be always in the interior that is in the direction along way in which the turn is taken by the curve it could be 0 if around that place around that location temporarily the curve was going straight then it would be s 0 vector otherwise if there is a turn then it will be in towards the interior of the turning. So, this vector is a normal; normal vector to the curve at this point and as we know that its magnitude is defined as kappa. So, its direction the unit vector along this direction which is found by dividing u prime with kappa that is called the unit principle normal unit we can understand because this is going to be a unit vector because we are dividing u prime with its own magnitude.

So, it will be unit normal we can see because it is at right angles to the tangent vector. So, that is normal to the curve at this point, but why do we call it unit principle normal why principle because at this point if this direction gives us the tangent then perpendicular to this line at this point we can actually draw a plane and every vector in that plane passing through this point is actually a normal and this is the principle normal other vectors in this plane are also normal, but they are not the principle normal this vector is called the principle normal because it this vector along with the tangent defines the plane which can be taken as the plane of the curve.

At least locally at this at least around this point that is why the normal which falls in the plane of the curve in this local neighborhood that is the principle normal if you take the cross product of u and p if you take the cross product of the u and p then you get another normal out of the backboard out of the plane of the blackboard and that is called the by normal now this is easy for a representation with respect to arc length as the parameter in the case of general parametrization in which we use the parameter t not the arc length for this curve you will find that it will not be so easy to derive the expressions for the principle normal and the curvature function.

So, for that what you can do is you can consider it in this manner that r prime t has a magnitude which is not unity in general. So, you can say that r prime is its magnitude not necessarily one into the direction unit vector along the tangent now if you differentiate this for developing r double prime, then you will find that it will have 2 parts one because of the variation of this and the second because of the variation of this in the case of r of s this part was missing because it was constant.

So, if you consider a general parameterization like this, then you get r prime which can be represented in this manner and then if you formally differentiate it using the product rule, then you will get the second derivative in this manner that is the derivative of r prime norm into u t plus r prime norm into du by dt and if you differentiate it like this then you will find that in the derivative this part du by dt will be du by d s du by d t will be d u by d s into d s by d t now d s by d t is again this r prime norm itself and d y by d r is for to e and countered earlier that is kappa into p. So, when you insert those things then you get this kappa into p and d s by d t gives you one r prime norm and another r prime norm this is already sitting here. So, in the second term you get this first term has remained unchanged.

Now, you find that this second derivative is not a vector in the direction of the unit principle normal, but it is a combination of u and p the reason is here the speed is not constant speed is changing. So, you have this. So, because of the change of speed you get this is exploration actually you get one component along the tangent that is because of the change of speed change of the magnitude of the speed and this is the representation of the part which is the result of the change in direction. So, now, from here if you want to find out the curvature then what you need to do from this second derivative you need to subtract this part out what remains you examine that and from there you get the direction vector d t and the curvature because you have already r prime.

So, when you consider r double prime minus this term whatever is its component along the tangent that part you subtract, then whatever is left from that you get the direction and by inserting the values of r prime you get from the magnitude you get the curvature kappa now this is the situation if you try to see it symmetrically, then at point a this is the position vector r; r of t; that means, at every point r of t changes and at this point this is say r prime r prime of t unit vector size u size one; this is the unit vector u; u of t and then as you calculate the second derivative of r it is not necessary that it is normal to the curve, it may be in some direction like this out of that when you subtract away the tangential component, then what is left is in this direction and in this direction you get this vector the unit vector along that direction is shown as p of t and whatever is kappa that is the curvature.

If you take one by kappa then you get a length and in this direction if you mark out that length then you get a point denoted by c here in this figure and that is called the center of curvature why center of curvature because locally around this point in that case you can represent the very close neighborhood of the curve by a circular arc the center of which is at c. So, in that sense this is the this is called the center of curvature and the plane that is defined by these 2 vectors u and p that is the plane in which the curve resides in the immediate neighborhood of this point and that plane is called the osculating plane as we have already discussed that the 2 principle unit vectors u and p if you take from the osculating plane and define this first product then you get the third mutually perpendicular vector be like this perpendicular out of the plane of the backboard then that vector is called the bi normal.

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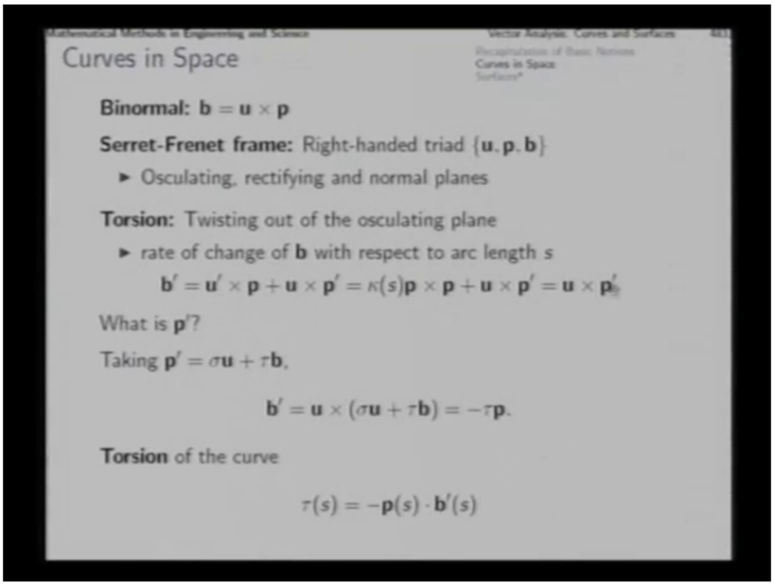

Why bi normal because it is normal to the curve certainly and it is normal to both the important vectors that we have earlier defined u and p that is why this is called the bi normal now you find that through derivatives we have defined 3 unit vectors at this point which are mutually perpendicular u p b and u p b together defined a right handed triad the kind of 3 reference vectors that we typically use to represent vectors to get a frame of get a frame of to describe 3 d vectors. So, this particular right handed triad formed by that point as the origin this particular point as the origin and these 3 vectors these 3 unit vectors u p b as the 3 reference vectors that frame of reference is called the Serret Frenet frame. Now in this frame you can directly describe the osculating plane which is the plane represent formed by u and p and then if you consider the plane formed by u and b that is called the rectifying plane and the plane formed by p and b the 2 normal that is called the normal plane because that entire plane formed by p and b is normal to the curve.

After this we get into the discussion of the next important property of a curve which will be there in the case of spatial curve 3 dimensional curves in a in the 3 dimensional curve the tendency of the curve to twist out of this local plane that osculating plane to the tendency to twist out of this plane and become a spatial curve go out of this plane is measured by the quantity torsion. So, how would you describe it you find that the plane osculating plane is the plane found by u and p and the direction of the plane is given by the direction of it normal which is the bi normal b in this case right now as the curve tends to come out of this plane, then at the next point the osculating plane will be somewhat different which means the bi normal will be somewhat different. So, then the way the curve tends to twist out of the osculating plane will be measured by the way this by normal changes that is from this unit vector if it changes to this unit vector then the manner in which this change take place will give you a measure of the twisting tendency of the curve or torsion of the curve that is why we measure torsion through the rate of change of b with respect to r plane.

So, try to differentiate this to get the rate of change of b, then you get b prime as u prime cross p plus u cross p prime now here u prime we already know as kappa of s into p right. So, we insert that and then we find that the first term has p cross p which is 0. So, this goes out right what remains you have u cross p prime now we know u p b.

Now, we ask what is p prime now what is p prime till now we have not come across the expression for p prime, but one issue is very clear p is a unit vector and therefore, its derivative with respect to the arc length must be perpendicular to itself. So, if there is a vector which is perpendicular to p; that means, it will be in the plan of u and b right. So, we can consider p prime as a linear combination of u and b. So, let us take it like this and try to put it here. So, this kind of n expression with a linear combination of u and b sigma u plus tau b we insert here as we do that we find u cross this now the first term in this product within was u cross u which is 0 and the second term will be tau into u cross b.

Now, u cross b will be minus b because this is a right hand triad. So, we get this see we have got b prime the rate of change of the bi normal in terms of the known vector. So, it is a vector in the direction of the principle normal p and the magnitude tau that is defined here is call that torsion of the curve and how do you find it you just take inner product of this equation with p and then you get p dot b prime negative of that as tau. So, from here we get first the torsion of the curve second the derivative of the bi normal which is in this manner a vector in the direction of p itself now you see that earlier we defined u prime and we got that is kappa into p when the parameterization with respect to the r plane now we have defined and got an expression for b prime now out of the 3 vectors u p and b for 2 of them for u and v, we know we have described, how they change that is we have evaluated their rate of change u prime and b prime.

Now, what about p prime here we assumed that p prime is this linear combination of u and b because we knew that it must be in the plane of u and b, but this sigma and tau we did not determine at that time tau turns out to be the torsion of the curve and that is how it is defined now sigma is left if we can determine the value of sigma then we would have completed the description of p prime that is the rate of change of p that is how this principle normal changes. So, we can do that now completely because we have got now both u prime and b prime and from this traid, we find that p is b cross u.

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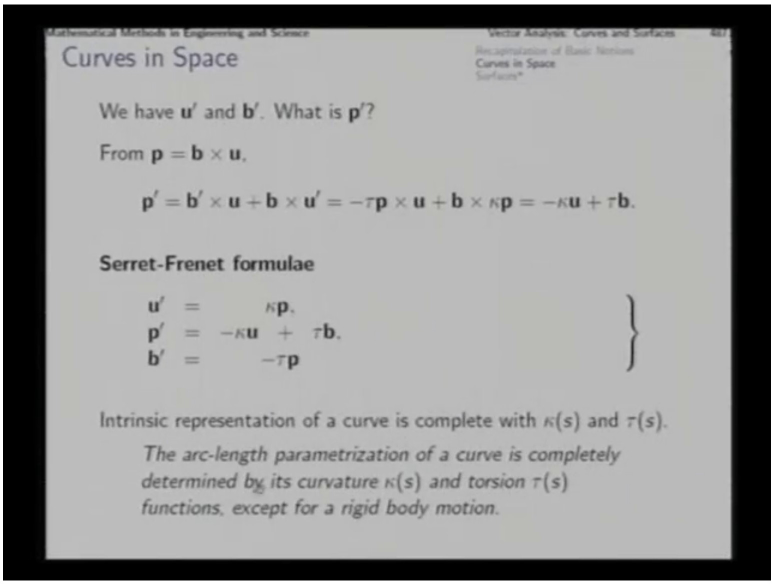

So, using that we can try to determine p prime from this expression itself. So, if p is b cross u, then its derivative will be b prime cross u plus b cross u prime right and if we use the expression for b prime as we derived just now that is b prime is minus tau p if it we use that here then we find minus tau p cross u plus b cross u prime which is kappa p.

Now, if we simplify this we find that p cross u is minus b. So, we get tau b and b cross p is minus u. So, we get from here minus kappa u. So, the sigma that we assumed here turns out to be minus kappa and with this now we have got p prime also this complete set of 3 formulae that give the rate of change of the 3 unit vectors u p b in terms of those vectors themselves are called the Serret Frenet formulae now with these formulae if we know the curvature and torsion as functions of the arc length then given the initial vectors upb at the starting point of the curve you can determine the curve for all subsequent values of p; how do you do that if you know not t, but F the arc length function.

If you know kappa and tau as functions of s then with the given value of u p and b at s equal to 0 you know the complete right hand side; that means, you know the rate of change of these 3 vectors upb. So, the initial traid is given and from these formulae you can find out the rate at which the change that is for a very small distance ds traveled along the curve you know how this will change the d u d p and d b you can work out and with that with that knowledge you can effect that little changes in the in u p and b and proceed to the next point along the curve as you know that next point along the curve and the u p b values u p b vectors at that point and again you know kappa and tau as function of s when that new point you can again completely determined the right hand side and keep on determining the next point and the next traid upb next frame the next select on that frame and this way you can continue along the entire curve not only that if you have 2 different parameterizations of a of a curve starting with different points then also, but if you find that curve but it separated one curve here starting from this point and going like this.

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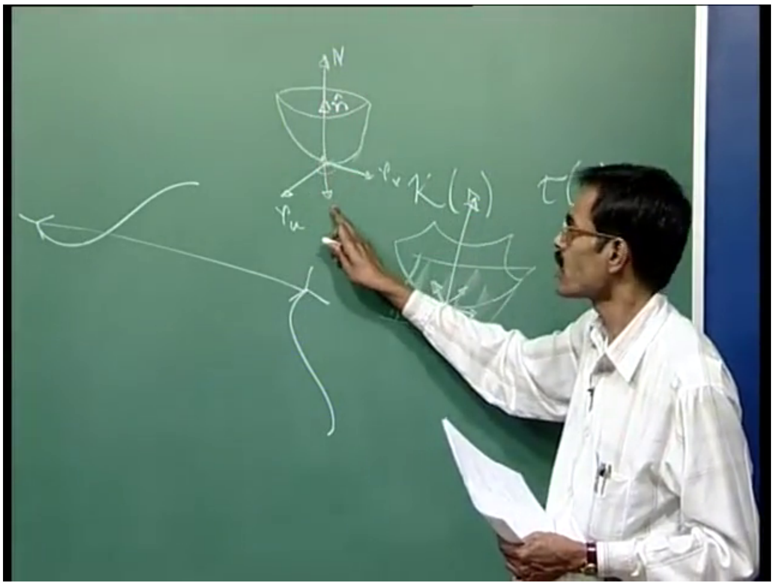

And another curve starting from here going like this if they are the same curve then with these 2 different starting points and different directions in which you proceed you can find out that these are actually the same curve moved and rotated by this distance and some rotation. So, this you can establish if you can find out what are the upb vectors at this point and what are the corresponding upb value vectors at this point and if you can give a compensatory motion to this a compensatory rigid body motion a rotation and a translation, then you find that the entire curve comes back at this location

So; that means, that if the curvature and torsion kappa s and tau s are 2 given function that is curvature and torsion as functions of arc length parameterization as long as these 2 functions are same for 2 curves the 2 curves will turn out to be the exactly the same curve except that they may be rotated and displaced through some rotation and a displacement. So, that is why we can say that the Serret Frenet frame and the curvature and torsion functions give you an intensity representation of a curve intensive representation in the sense that they do not depend upon and external frame of reference.

The upb traid itself is its own most natural frame of reference in which to describe the curve and that frame keeps on moving along the curve as you change the parameter value that is the arc length. So, this result you can establish very easily that is the arc length parameterization of a curve is completely determined by its curvature function and torsion function except for a rigid body motion some of the exercises in the chap in the corresponding chapter of this of the text book you will find 2 different looking functions 2 different looking vector functions representing the same curve and through the reduction to the standard form to arc length parameterization you can establish the equivalence of the 2 different parametric representation for the same curve looked at the different locations with this much discussion on curve.

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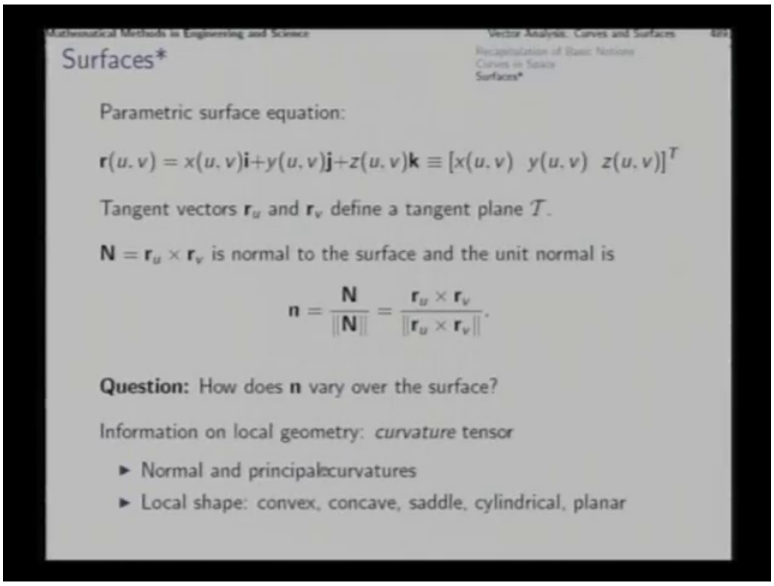

We conclude this lesson with a little discussion on parametric surfaces the way, we describe curves by the help of a parameter we can also present a surface by the help of 2 parameters say u and v, if you can represent the position vector of a point on the surface with the help of 2 parameters as a function of 2 parameters then the representation looks like this x of u V i plus y of u V j plus z of u V k you can also represent it as a column vector with x, y, z functions of u and V being the members of being the components in the vector now for a function for a surface which is a 2 dimensional entity at every point you can find out 2 independent tangent vectors that is you can define a complete tangent plane and on that tangent plane you can define infinite tangent vectors 2 of them will be linearly independent in terms of which as linear combinations of which you can represent all the other tangent vectors. So, for example, suppose this is a surface then at every point of it you will find that you can describe a complete plane which is tangential to it. So, from this point you can determine several tangent vectors tangent vectors like this.

So, in that plane all vectors in that plane which are touching the surface for example, if you draw a cylinder like this and then at this point you will find that you can find out a complete plane every point of it is a tangent. So, the line like this is also a tangent and in that plane whatever in the plane perpendicular to the board you can draw this kind of a tangent this kind of a tangent and so on. So, in that tangent plane 2 of the tangents are linearly independent as a linear combination of these 2 you can construct any other tangent in that same plane.

Now, if you change y through a small amount u if you change u through a small amount parameter then you move along a particular tangent on the surface and that tangent is known as r u that is the derivative of r with respect to u that gives you one tangent vector and derivative of r with respect to V gives you another tangent vector now if the parameterization is not degenerate at that particular point then you will find that r u and r V are linearly independent there maybe a point there maybe some points where the 2 parametric tangent vectors turns out to be in the same direction and that will be a regional point for this particular parameterization.

Now, considering a situation in which the point that we are considering is not degenerate you can find 2 linearly independent tangent vectors r u and r V by straight forward derivate. So, as you find 2 such vectors 2 such tangent in the tangent plane then the cross product of these 2 tangent vectors will give you the normal to the surface like this which is normal to the entire tangent plane right. So, that normal you can represent like this and quite often you are interested in the unit normal vector which you can find out like this now the local shape of the surface along this point can be explored can be analyzed through the investigation of how this particular normal this unit normal changes as you move along the surface along this tangent or that tangent or any other tangent.

So, if you have got say a surface like this at this point you have got this as r u and this is a tangent vector and this is another tangent vector this is r V cross product of them is this normal then at this point if you say that now along r u if I try to move on the surface then I get a point which is arbitrarily closed to each closed to this and a little in this in the direction of r u along the tangent plane then you move to this point which is very close to the original point here and what is the normal at this point.

So, that normal you can work out similarly if you move along r v, then you can get another normal and then you try to see the projection of these normal on the tangent plane again. So, that way for every move along a tangent vector you can get a new normal and you work out the rate of change of this unit normal unit normal n and through the those rates of changes you can work out the local shape of the surface around this point. So, how does you asked the question how does n vary over the surface. So, this gives you the information about the local shape local geometry of the surface.

So, you can determine a curvature which is called normal curvature at every point along every direction. So, in the tangent plane if you take another direction another independent direction neither r u nor r V some any other direction arbitrary direction in the tangent plane which is a linear combination of r u and r V then you can cut the surface through a plane which includes n and this vector the normal and this vector. So, you can make a normal cut on the surface through the normal plane of n and this vector and that will define a planar curve and the curvature of that planar curve is called the normal curvature of the surface along this direction at this point and then that is another candidate for a direction for a tangent.

And then if you can find try to find out those directions along which you can determine the normal curvature in such a manner that the curvature tensor that you get you see that curvature is actually a tensor because the moment you decide on a direction in the tangent plane finally, you get another vector in the tangent plane itself. So, from tangent plane to tangent plane that is a mapping. So, that way the curvature at every point is actually a sensor quantity and as you diagonal sensor quantity the curvature tensor you can find out those directions in which the entire curve curvature is contained within the normal plane. So, those directions are called principle directions and the corresponding normal curvatures are called principle curvatures.

So, you can diagonalize this curvature tensor and find out 2 principle directions and the corresponding principle curvatures depending upon the signs of these principle curvatures you can determine you can classify the local shape as convex concave saddle and. So, on. So, if both the principle curvatures turn out to be positive then you say that locally the surface is concave convex if both of them are negative then locally the surface is concave if one of the principle curvature is positive and the other is negative then that point is termed as at that point surface geometry is termed as a saddle a saddle point that is a saddle point if one of the principle curvature turns out to be 0 and the other is positive or negative then it is cylindrical on the other and if both the curvatures both the curvatures at point turn out to be 0 then locally the curvature the surface is a planar surface. So, this you can work out the local geometry of a surface there is a term the way the normal way is in the immediate neighborhood.

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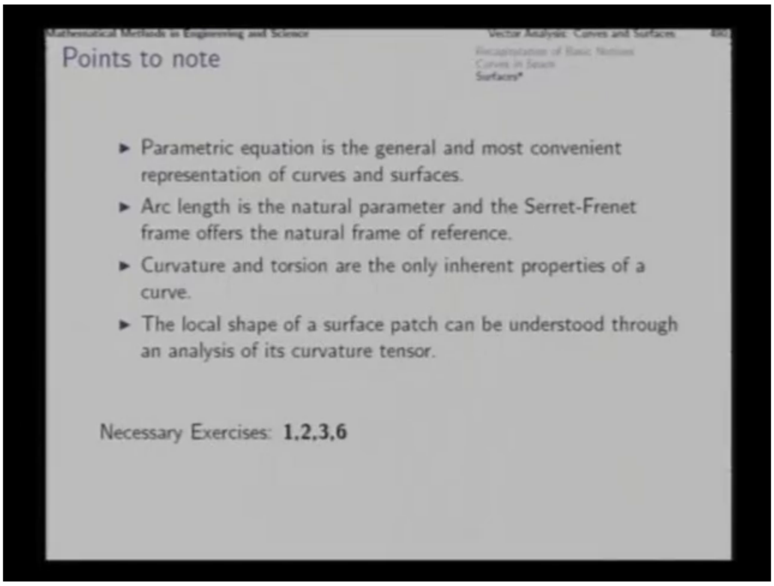

So, let us summaries the important points in this lesson, first point is that parametric equation is the general and most convenient representation of curves and surfaces in 3 dimensions and second for curve for parametric curves arc length turn out to be the most natural parameter and the Serret Frenet frame gives the most natural description of the curve which is intranet to the curve which does not depend on any external reference third important point is that the curvature and torsion s are the only inherent properties of a curve the tangents, etcetera are themes or properties which are dependent on frame of reference also parameterization also and so on curvature and torsion as functions of the arc length parameter are the only intrusive properties of a curve the rest of which is due to parameterization and frame of reference chosen finally, for a surface patch for a surface patch the local shape can be understood through an analysis of the curvature tensor which is found by exploration of how the surface normal changes as we move in the immediate vicinity of the surface around a given point.

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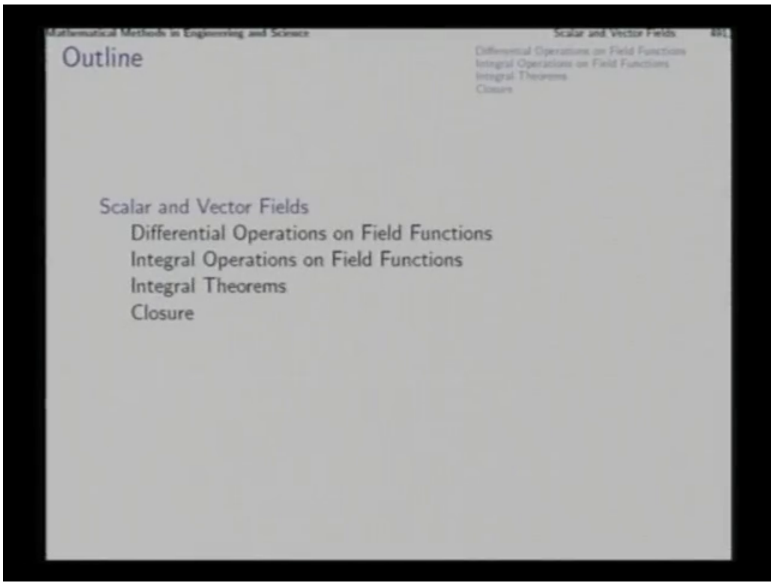

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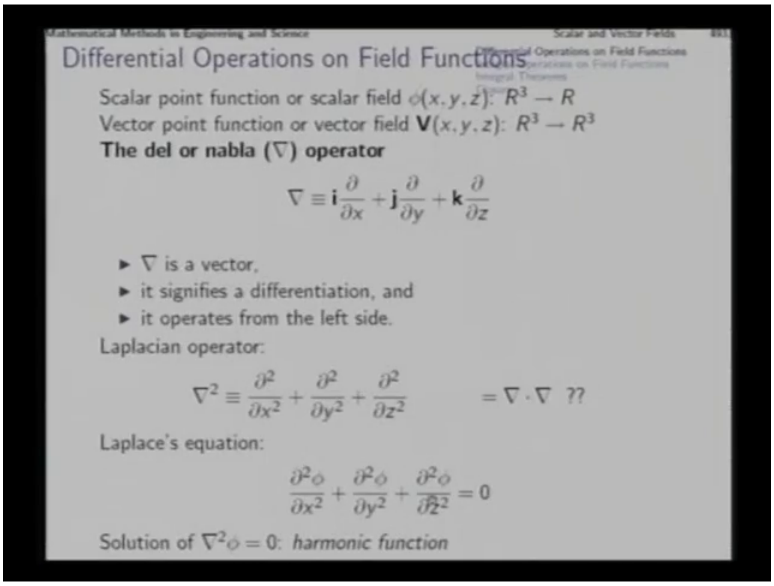

Now, in the next lesson we continue, the vector calculus topic and in this lecture, we will see some of the differential operators unfilled functions and in the next lecture, we will continue into a discussion of the integral operations and integral theorems first the differential operators a function of x, y, z which is a scalar function that is a mapping from 3 dimensional space to the real line is called a scalar point function or scalar filed similarly if you have a mapping from 3 dimensional space to a 3 dimensional space that is a vector field or a vector field function you can represent it with this in this manner now on both of the scalar and vector fields.

You can apply an operator the differential operator first order differential operator which is denoted as del or nebula and the meaning of which is this note that particular way this particular this operator is written normally when we write a vector which components along ijk, we write a x i plus a y j plus a z k, but here it is written as i into something plus j into something plus k into something the reason is that this something is not a quantity, but an operators.

So, when you carry out the algebra of this del or nabla operator there may be situations where slight changes will mean different things. So, to save one self from confusion one must remember these 3 things one is that del is a vector quantity second it signifies a differentiation and third it operates from the left side you cannot multiply this thing to a quantity from the right side and expect that you will get the quantity back you always apply del from the left side.

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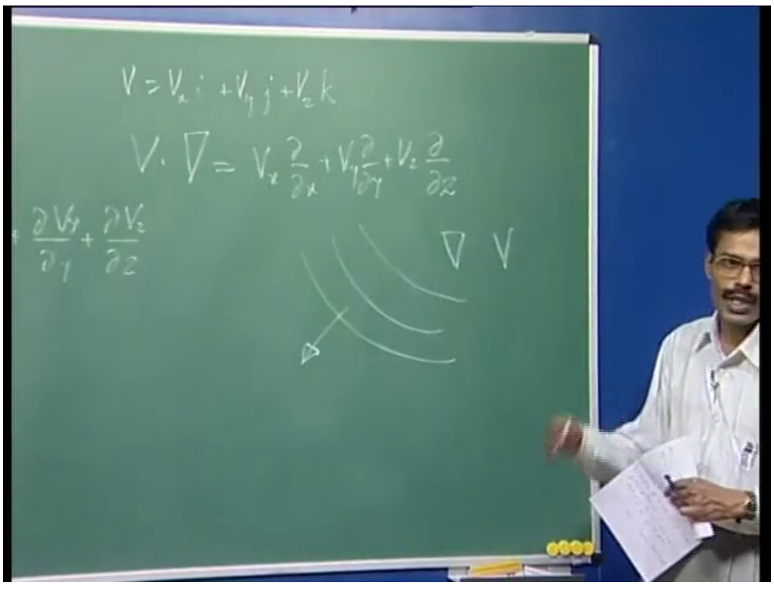

So, that is why, 2 notations del dot V and V dot del mean 2 completely different things you see V is a vector which is V x I plus V y j plus V z k. Now V is a vector quantity this del is a vector operator this is also vector.

So, you can think of dot product between them you can think of this and you can think of this, but both of them are meaningful expressions, but they mean 2 completely different thing normal vectors for example, V vector like that w another vector like that in that you know that V dot w and w dot V are same that is dot product is commutative, but not when one of them turns out to be an operator a differential operator like this del what is this if you apply the del operator through a dot product then as we apply that on this on V and you work out all the dot products as usual, then you will get something which will be del V x by del x plus del V y plus del y plus del V z by del z which is a quantity which is scalar quantity and this is; this has a meaning this called the divergence of V as we will shortly see on the other hand this one we mean V x del by del x plus V y del by del y plus V z del by del z. Now, this is also scalar, but this is not a quantity this still as an operator waiting to operate on some function on this side.

So, these 2 mean 2 completely different things because this del is an operator and it operates from the left side. So, when on the right of del there s nothing given. So, it remains still an operator after we put something on the right side of it then this operator will operate from the left side and produce a quantity the another important operator which is a second order operator for that matter is this del square which is this; this is called the Laplacian operator because when this operator is operated on a function phi then you get this equation which is the famous Laplacian equation and a solution of this kind of an equation that is a function phi which satisfy this equation is called a harmonic function which we will come across quite often from the course particularly in this chapter also.

Now, when you apply the first order linear differential operator del on a field function the scalar field or vector field you can develop 3 different notions of vector calculus first is the gradient.

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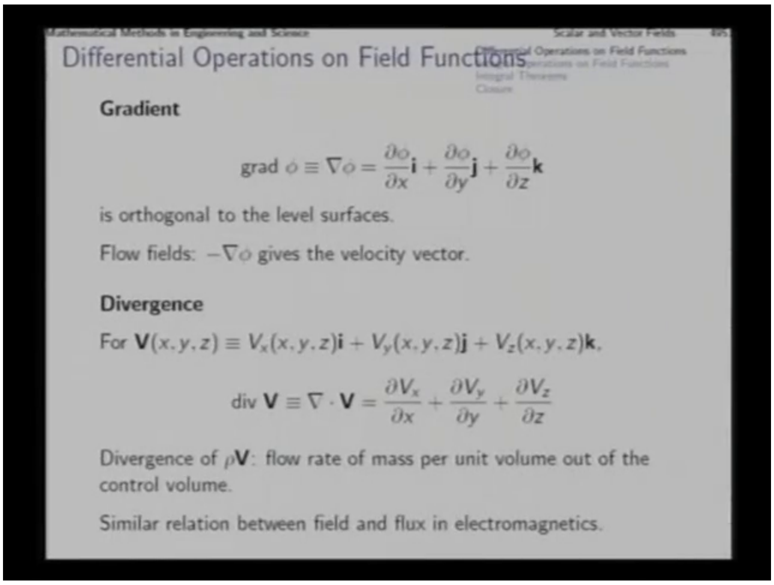

Which is the result when you apply the del operator on a scalar function a potential function. So, that is the gradient quite often denoted as grad phi or simply del phi like this is if you apply that del operator term by term on phi, then you get this and this is the gradient vector this gradient has the same meaning as the gradient which we discussed in the previous lecture that was in the in that case the dimension was free in this case dimension is just see. So, at that in that in the other case in the linear algebra sense dimension could be anything it was free here it is only for 3 dimensional vector that is why 3 components i j k are there.

So, now if phi of x, y, z is a scalar field then phi of x, y, z equal to constant will give the level surfaces or equipotential surfaces of that potential function and at any point there is corresponding gradient vector that you can determine like this turns out to be normal to that orthogonal to the level surfaces that is gradient now in the scientific problems in same problems this is very important because if in a potential flow you have the potential described by a function phi then the negative of the gradient phi gives you the velocity vector now on a vector when you try to apply the del operator then del itself being a vector on a vector function try to operate it you can operate in 2 manners one is by a dot product and the other is by a cross product. So, through the application of a dot product between the del operator and V you get this kind of thing and which is called the divergence of the vector p or the divergence of the vector point function.

So, the vector field is given like this V x, V y, V z by its components along x, y, z directions and the divergence is given by this expression now this also has a direct meaning in the context of fluid flow that is divergence of rho V where rho is the density and V is the velocity vector gives us the flow rate of mass for unit volume out of a given control volume there are similar relationships between field vector and the flux in the case of electricity and magnetism.

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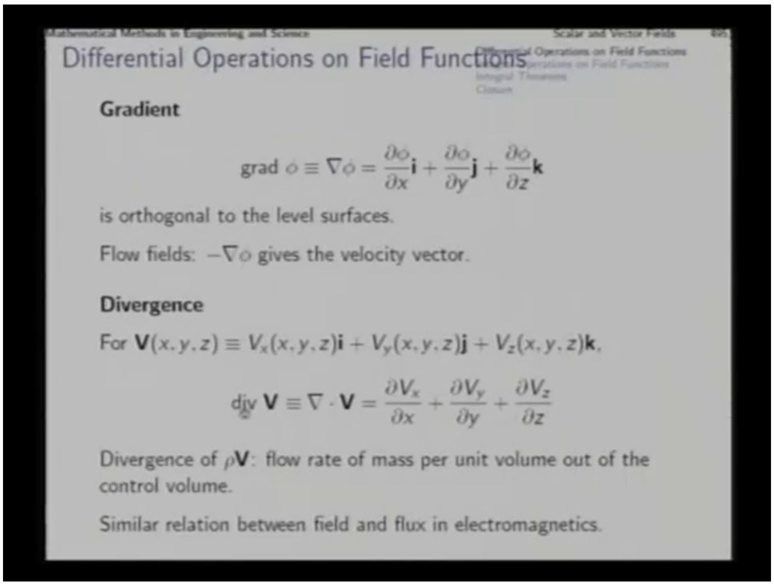

Now, if rather than dot product between the operator del and the vector function V if you have a cross product then as a result you get a vector function because cross product between 2 vectors is a vector and that is called the curl and that is defined in this manner curl of V is like this cross product which you can evaluate like this and get the 3 e i j k components in this manner. So, that word curl literally means rotation turning and that is the precise meaning when you try to explore the situation in the case of a fluid flow for example, if V equal to omega cross r represents the velocity field then the curl of the velocity vector gives twice the angular velocity that is twice the rotationally so; that means, if curl of a of the velocity vector is 0; that means, omega is 0 and; that means, it is an irrotational flow. So, curl represents the rotationally in the flow in electro magnetism the relationships between electric field and magnetic field is also established through the curl of the corresponding filed beyond this we will consider a few further differential operators composite operators and second order operators and integral operators in the next lecture.

Thank you.