Mathematical Methods in Engineering and Science
Prof. Bhaskar Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur
Selected Topics in Linear Algebra and Calculus

Lecture - 02
Vector Space: Concepts

Welcome. This is the last lecture in the module of linear algebra. This lecture is a little abstract. In this lecture, we try to consolidate all the conceptual ideas on which we have worked till now from a fundamental concepts of vector spaces.

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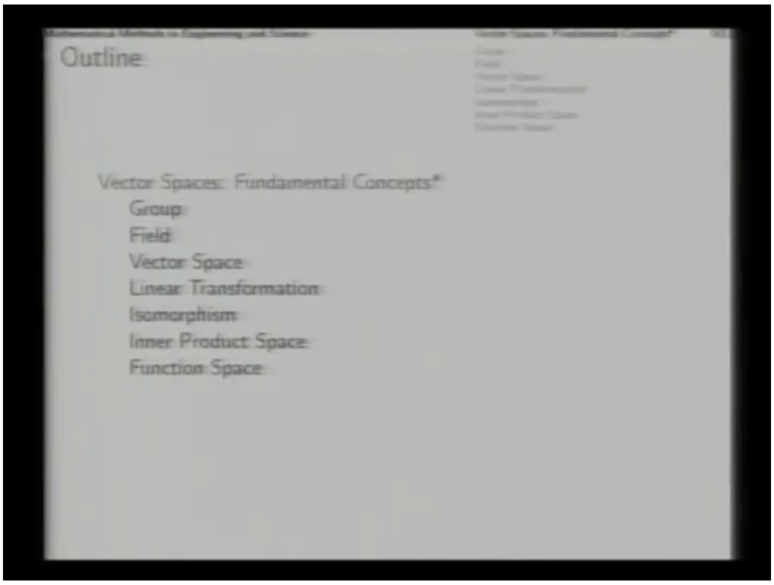

We quickly recapitulate the definitions of group and field and then, continue the discussion on vector, space, linear transformation etcetera in which we find that the mathematical and computational tools with which we have been working till now are all the product of the basic abstract ideas in this area.

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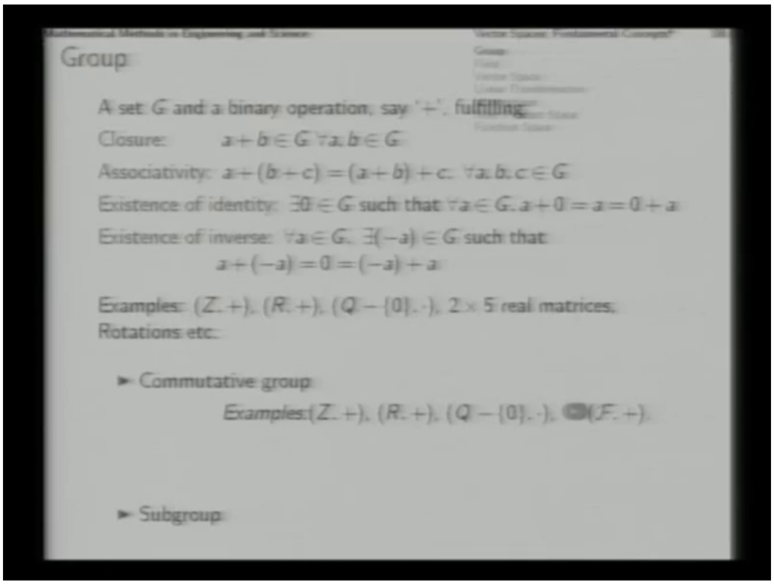

First the group, the mathematical structure of group is defined with the help of A set G and a binary operation say denoted by this sign plus fulfilling these relationship. First relation is the closure. Basically that is the definition of the binary operation that is two members in the set through this binary operation produce the result which is also in the same set G. That means, the binary operation is defined within that set. Other requirements are associativity of the binary operation existence of an identity element that is, there must exist an element in the set which added to any other element from the left side or the right side produces the same element back. So, that identity element we can denote as 0 and finally, the existence of an inverse that is for every element A in the set G, there must be another element can be denoted by minus a which added to a from this side or this side gives the identity element that is 0. If these are fulfilled, then we have what is a group that set G and the operation plus together define the group. You can take these examples of integer with the ordinary addition examples of the set, the members of the set of real numbers with the same edition or the set of rational numbers other than the 0 that is the set of non-zero rational numbers with the modification, then verify real matrices with matrix addition.

So, all these constitute examples of group, the group structure is evident in all of these rotations. Also, rotations in the geometric shape, geometric space is also an example of group that is if you compose two rotations, the resulting complete movement is again a rotation. So, that also fulfills these conditions. Now, if a plus b and b plus a are equal for all a and b, then in particular you have got what is called a commutative group. All these are actually commutative groups, but rotations are not, that is pre-rotations are not commutative. So, this is an example of a non-commutative group.

There is something called sub-group, a subset of G with the same binary operation can constitute a group itself. If it fulfills these requirements, in that case that is called a sub group of the original group. Now, with this definition of the group in the background in the definition of field becomes easier.

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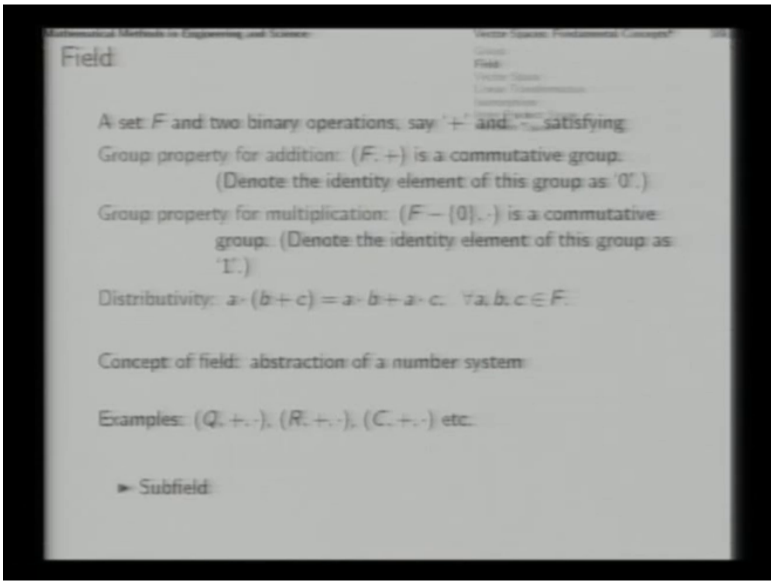

A set F with two binary operations, one of them we can denote as plus resembling the ordinary addition and the other we can denote as dot resembling the ordinary multiplication that satisfy these requirements, we will define a field.

What are these requirements? One is that F set and plus binary operation together is a commutative group for which we will denote the identity element as 0 and then, from the set F if we remove this 0, the relative identity element of the group here then what remains with that the multiplication dot forms. Another commutative group and apart from these two commutative groups, we have the distributive property that is the multiplication is distributive over addition. If this also holds for all abc in the set F, then we call this together, this F with the two binary operations define what is called a field.

Now, this concept of field is actually the abstraction of a number system. So, whatever is defined here formally applies to all the number systems that we use the rational numbers, real numbers conflict numbers. All these full fill these requirements. So, all of these are examples of fields, they are complete number systems. Complete in some sense and here you find that we already have an example of sub field. The set of real numbers is a sub set of the set of complex numbers with the same addition and multiplication rules of complex numbers, you can define real addition and multiplication as well and therefore, this is actually a sub field of this c.

Now, we have got groups and fields defined with the help of these, we can define what is called a vector space.

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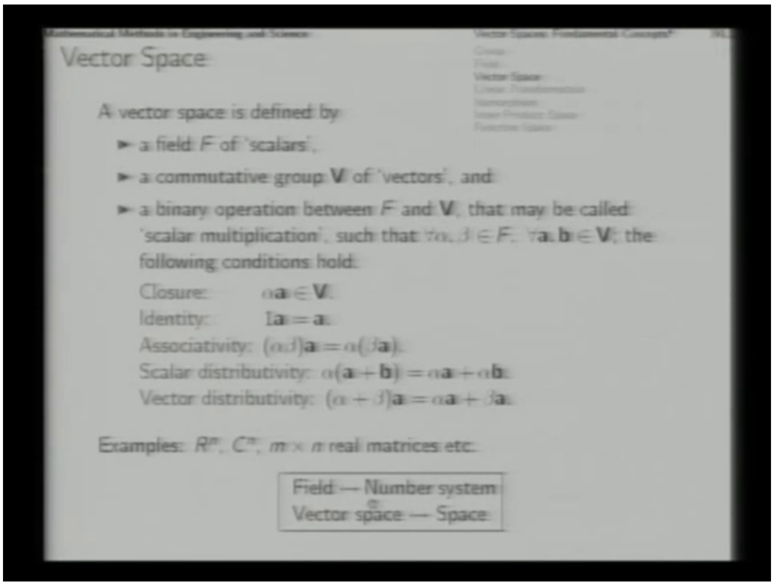

A vector space is defined by first a number system a field F of scalars. Elements of this set F are quite often referred to as scalars, then a commutative group V of vectors. There is a commutative group V of vectors with its own addition rule. Apart from these we have got a binary operation between this field and this commutative group that is a binary operation between a scalar and a vector and that binary operation we will call as scalar multiplication, such that called alpha beta scalars and a b vectors these relationships hold. That means, the first requirement is that a scalar multiplication, a scalar multiple of a vector is also a vector that is you take a vector a from V and an alpha from the number system a scalar alpha from F, then this scalar multiplication result alpha a is again in the set V of vectors.

So, that tells us that the scalar multiplication operation is defined, there is an identity element, there is a reason a scalar that unity that multiplicative identity of F which multiplied to the vector gives the same vector back associativity. If we have to multiply a vector say with 3 first and then, with 2, if the result is same if we multiply in one shot with 6 and if such things happen for all alpha beta in F and all a in V, then we will say that the scalar multiple operation is associative as well and then, there are two distributive properties; scalar distributivity and the vector distributivity.

When all these conditions hold, then we say that what we have got is a vector space V in which lot of vectors are there and all these vectors are defined over the field F of scalars. Now, note that all these conditions here are expressed in only this much, but actually if you open the definitions of commutative group and field and so on, then it is actually much larger here. In one shot we have actually got 11 small conditions here, again another 5, 16 and 5. So, 21 conditions are actually written one side V here in terms of field and group.

Now, R n C n that is n dimensional real coordinate vectors, then n dimensional vectors space with complex coordinates R n C n, these are all examples of vector spaces over the field of real numbers over the field of complex numbers and so on. M by n real matrices will again form a vector space of their own and so on. From here you will note that the way field is an abstraction of the number system, the vector space is actually an instruction of the ordinary geometric space in which we leave.

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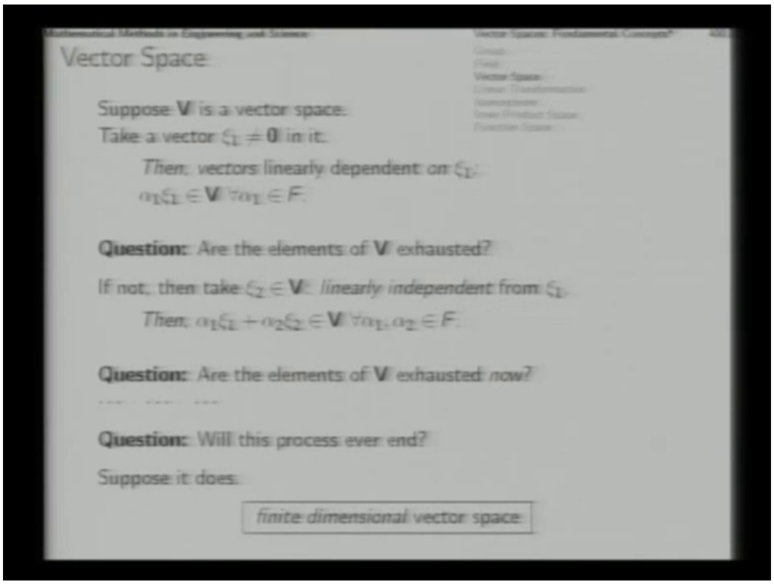

Now, that we have got this formal definition of vector space, let us try to examine its contents. First of all we already know that there must be zero vector in the vector space. That is necessary from these requirements that we set is a commutative group. That means it has its own identity element.

So, there must be a zero element in the vector space, that is essential otherwise it will not be a vector space at all. Other than 0 if possible we take a vector xi 1 in the vector space, then with this xi 1 this vector which we have picked up from the vector space, we get a lot of scalar multiples take a scalar as a 1 and for all such alpha 1 from the underlined field F, we can get several other vectors alpha 1 xi 1. All such vectors we develop and from the definition of the scalar multiple operation, we know that all of these are vectors, that is all of these are in V.

So, these vectors which can be generated from the vector xi 1 through a scalar multiple alpha 1, all these vectors are said to be linearly dependent on xi 1. Now, after we finish all these vectors like this, now we ask the elements if we exhausted, have we finished all vectors, all the elements in V. If not, then we take another vector xi 2 in V which cannot be expressed like this because we have not exhausted it. We have not exhausted V by taking all of these. So, we take an outside element xi 2 which we could not express like this. So, that is linearly independent of xi 1.

Now, alpha 2 xi 2 will give many other vectors and in alpha 1 xi 1 which gave the earlier vectors and alpha 2 xi 2 which gave many other vectors, now they can be added together in all combinations and we get lots and lots of vectors in V. Suppose we pick up all of them. After that we ask the same question again in these two rounds, have we finished all elements of the vector space V, right and like this we keep on asking. Now, before asking again and again and again the same question, let us ask the multiple question as when this process ever end. It may not. On the other hand, it may. Suppose it does. Suppose this process of asking and picking up another, picking up another, suppose this process ends. If this process ends, then we have got what is called a finite dimensional vector space. The dimension of the vector space is finite. So, if this process ends, then we will say that V is a finite dimensional vector space. On the other hand if this process never ends, then what we would have got is called an infinite dimensional vector space.

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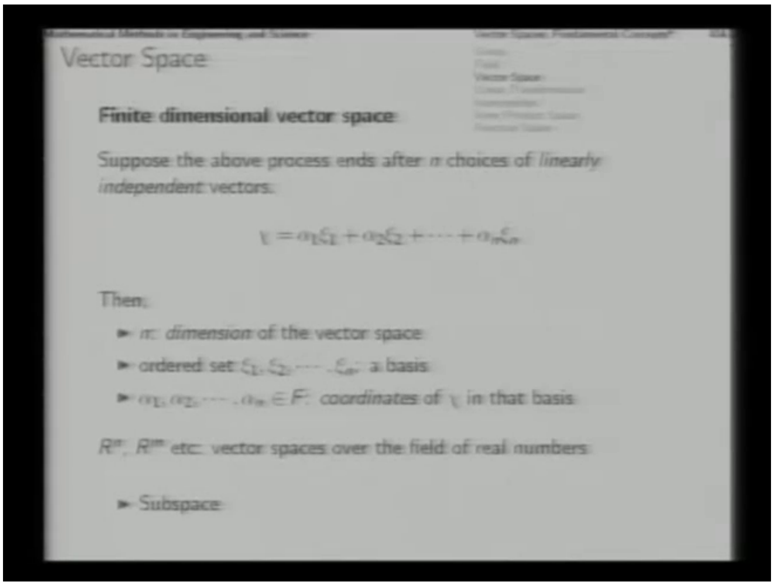

So, in this particular case for the time being suppose we consider finite dimensional vector spaces. So, in our instance suppose this process ends after n, such choices of fresh linearly independent vectors. So, that will mean that all the vectors in V can be expressed in this manner because in n rounds, we have added this n contributions, right and exhausted the contents of V. That means, that all vectors in V can be expressed as this linear combination that is a general vector pi in this vector space has this expression. Nothing else is there in the vector space which cannot be expressed in this manner.

So, then we say that if this process ended with n choices of linearly independent vectors, then we say this n, this number is that dimension of the vector space. So, n lineally independent vectors we could find more than that, we could not find. So, this n is the dimension of the vector space, these vectors chi 1 chi 2 chi 3, sorry xi 1 xi 2 xi 3 etcetera up to xi n. These n choices that we picked up, they are an ordered basis, they are in the ordered set. They formed a basis to represent all vectors in the vector space and for a particular vector chi, the corresponding co-efficiencies alpha 1 alpha 2 alpha 3 etcetera turn out to be coordinates of the vector chi in that basis.

Now, we know that R n R m etcetera vector spaces over the field of real numbers which we have already studied are such finite dimensional vector spaces for a vector space. If a sub set of it forms a vector space is in its own rights with the same underlying operations, then we say that constitutes a sub space. For example, in the three-dimensional space with this frame of a plane passing through the origin will actually define a subspace because zero identity element will be there and all the operations that we could define in this space, we can define the same operations within this plane itself. So, in that space we say that plane actually constitutes a subspace of this three-dimensional vector space which is R3.

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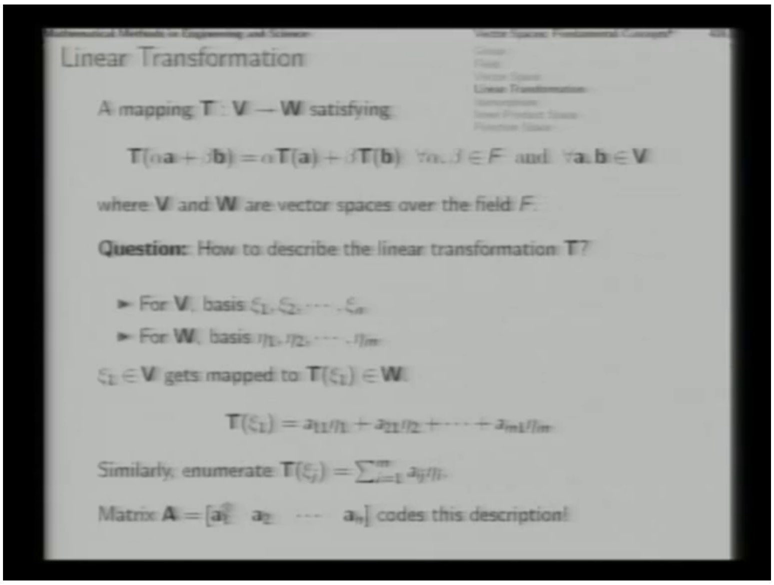

Now, with this understanding of vector spaces, we consider two vector spaces. V is one vector space and W is one vector space and we consider a mapping P from vector space

V to W and then, we will define what is called linear transformation is the mapping has this property. Then, we say that this mapping represents a linear transformation. Now, you see you take two vectors a and b in V, then P of a mapping of a and mapping of b will be vectors in the vector space W. Now, in V if you make a linear combination of a and b in this manner alpha a plus beta b with alpha beta lying in the underlined field F, then this is also be a member in c. So, this can be mapped through the same mapping P. So, that mapping will find now this is essentially is the relationship between the mapping of a linear combination of a and b and the individual mappings of a and b. Now, if the linear combination of a and b gets mapped to the vector W which is exactly the same as the same linear combination of p a and p b on that side, then we say that if this happens for all alpha beta in F and all a b in V, then we say that this mapping is actually linear transformation.

It is as in one side we mix two liquids in a particular proportion and then, boil in other. For instance, we boil two liquids of the same quantities and then, we mix the vapors. If the results in both cases is exactly same, then the whole process is behaving something like a linear transformation. So, this is the underlined requirement for a linear transformation in which V and W are vector spaces over the same field F because this alpha beta which is used here in composing a vector in the space V is the same alpha beta here which is also used to compose a vector here in the space W. So, the two vector spaces must be over the same field.

Now, that we have defined a linear transformation like this with this requirements, now if we want to describe the linear transformation, how do we describe it? One way to describe the linear transformation is to describe how several vectors get mapped. If we have a big bunch of vectors in V and for each of them if we can establish the mapping ,if for each of them we can say where these vectors can get mapped and through that if we can find out the mappings of all vectors in V, then you would say that we have described that linear transformation space, but then in the vector space V, there are infinite element. We are not going to numerate the mappings of each of them.

So, we want a description which is complete, but not that detailed. So, for that we again take the help of the basis that we have defined for vector space V, there is a basis say xi 1 xi 2 xi 3 up to xi n say it n dimensional vector space for W. Again similarly there will be a basis say eta 1 eta 2 eta 3 up to eta n, then for xi 1 in V which is a vector in V, xi 1 is vector in V gets mapped to T, xi 1 which is in W, now how do we describe that T xi 1 in W. So, that description is as a linear combination of the basis members of W. Suppose it is like this where a 11 a 21 a 31 etcetera are the scalars in F, then this is a description of how xi gets mapped and that will immediately gives a description of how all alpha 1 xi 1 type of vectors in V get mapped, right. Similarly if we can describe how xi 2 xi 3 xi 4 get mapped to the vector space W, then up to xi n if we define like this, then in effect we have described the complete mapping, complete transformation because all other vectors in V are actually linear combination of these only and we can map them individually and workout the same linear combination in the target domain target space W. So, now you find that a 11 a 21 a 31 etcetera a m 1 describe how the image of xi 1 is described in the target space W and such other elements a 12 a 22 etcetera, all these kinds of scalars a i j will similarly describe how all the basis members of V get mapped to W and how they are described in terms of eta 1 eta 2 eta 3 etcetera.

So, that tells us that these coefficients of here are actually collected to whether in vector a 1 coefficients of other ones, other mappings are similarly collected in the other columns in this matrix such that we find that this matrix a which we have been working, all these files is actually the cord of the description of the linear transformation from these vector space W to the vector space from the vector space V to the vector space W. So, this matrix essentially has the elements scalar elements m n of them from F which encodes the description of a linear transformation.

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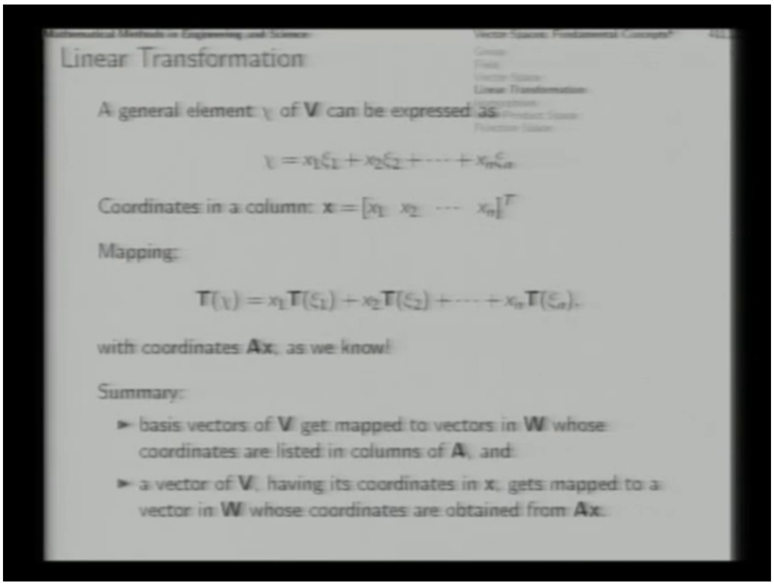

As we have earlier discussed, there we wrote this coordinates as alpha 1 alpha 2, now if we write them as x 1 x 2, then we say that a general element chi of V can be expressed as a linear combination of the basis members of V, right and that in our ordinary representational tool we represent as a column vector x 1 x 2 up to x n transpose, right. So, this column vector is actually a listing of the coordinates of a vector chi in the vector space in terms of the basis members, right.

Now, similarly the mapping T of chi will be this and consulting T xi 1, T xi 2 which we just now worked out, we will find that the coordinates of this will be the same as what we have as the elements of a x. So, the mapping will be found two coordinates which are elements of a x, right. So, thus we find that the basis vectors of V, the domain of the mapping domain of the linear transformation get mapped to vectors in W whose coordinates are listed in columns of the matrix and a vector V having its coordinates in x will get mapped to a vector there in W whose coordinate will be obtained from the product multivariate product A x.

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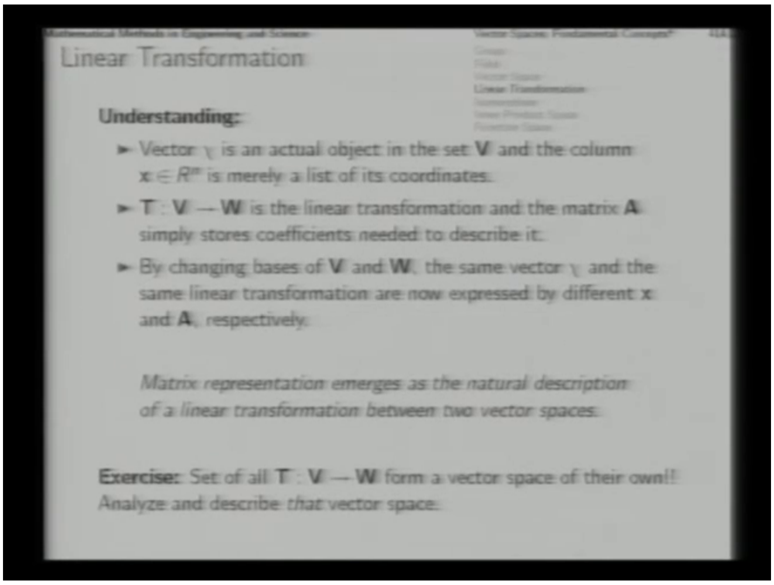

So, the understanding here in this whole discussion is that vector chi is an actual mathematical object in the set V. It is a vector and the column xn dimensional column vector in R, n is nearly a least of its coordinates and T from V to W is a linear transformation which is an event which is a situation. The description of it is actually stored in the rectangular area of numbers which is the matrix A. Therefore, by changing the basis of V and W if you find that the coordinates x gets changed in order to describe the same object which is the vector, which is a geometrical entity, which is a geometrical object and similarly the linear transformation which is a geometrical event that remains same, but with the change of basis of the vector spaces, the corresponding matrix encoding or the matrix representation changes as we have seen earlier in the context of basis change.

Now, in this entire scheme, we find that the matrix representation emerges as the natural description of a linear transformation from one vector space to another. So, this unguent of writing the matrix in the form of a rectangular array is something that is a natural outcome of the way we think of linear transformations from one vector space to another which has a deep geometric meaning. Now, as an exercise you can consider this all linear transformation that you can define from one vector space V to W, they also kind of collected together to form a set. So, all of these T s from one surface space V to W, you can define several linear transformations.

So, all of these linear transformations if you collect and then, the collection of these linear transformations that itself forms a set of linear transformations, it forms a set and you can verify that this set in itself actually defines a vector space forms. A vector space you can analyze and describe that vector space in the context of its dimension, its element, the way its elements get added and so on. So, that I am leaving for you as an exercise.

So, all linear transformations from one vector space to another, they together actually form a vector space of its own.

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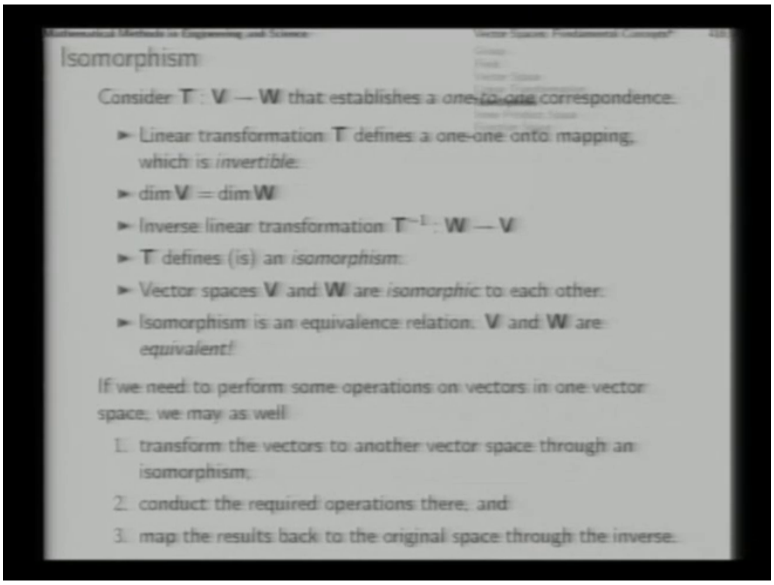

Now, let us continue this discussion into other very important point that is Isomorphism. Consider a linear transformation T from with respect V to W and a transformation of the kind which establishes a 1 to one correspondence, that is for every vector in V, you find a vector in W and for every vector in W, you find a vector in V. That is one to one correspondence. One element of here is directly related and linked to exactly one element from there. So, in that case you will find that the linear transformation T will define a 11 12 kind of mapping and this mapping is invertible and for that the dimension of the two vector spaces must be equal.

If from here for whichever vector you take, you get exactly one there and for every vector there, you get exactly one here. That is a mapping which is invertible and so, you can represent, you can denote the inverse linear transformation like this. In this type of equation, this in this kind of a situation we say that T defines or T is an isomorphism equally organized, similarly organized. So, that means that V and W are two vector spaces which are similarly organized isomorphism.

They define T defines a an isomorphism and you say V and W are two vector spaces which are isomorphism to each other and from the definition of equalization relation, you can show that isomorphism turns out to be an equivalence relation and therefore, we can call V and W was not in practice in the ordinary sense of the term equivalent of V and W to be equivalent to each other and they are equivalent in practice in the ordinary sense of the transform equivalent. Also in the sense that if we want to perform certain linear operations among vectors in V, it will be equivalent, it will be same.

If we first map these vectors to W and then, conduct the same operations in W and the result we map back through the inverse mapping, so in that sense it will be actually equivalent whether we conduct our actual operation here or there as long as we have two way communications through the isomorphism.

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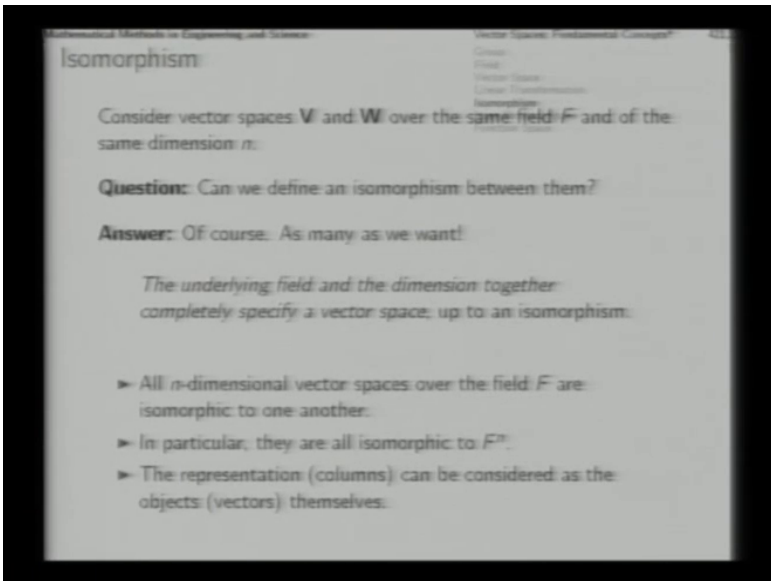

Now, consider two vector spaces V and W over the same field and of the same dimension. Then, can we define an isomorphism between them? Answer is of course we can. In fact, we can define as many isomorphism as we want. You will find upon the little reflection that any square non-singular matrix will actually give you one such isomorphism which connects. This is the element of this vector space with element of that vector space in a 11 12 passion in a one to one correspondence. So, as many as we want, we can define isomorphism.

So, any non-singular matrix we will actually define one such isomorphism. The simplest one is identity where n basis members from V get mapped exactly to the n basis members in W in the same order that is the identity transformation which in the matrix terminology is the identity matrix. So, you find that the underlying field and the dimension together actually completely specify the vector field because other than that even if you define two vector fields which are vector spaces, which are defined over the same filed F and same dimension n that means whatever you can do in one, you can do in the other. So, in all practical terms, they are actually the same vector space. So, that is why we say that the underlying filed and the dimension together completely specify the vector space for all practical purposes and one is say for all practical purposes that is another way of saying up to isomorphism that is other than that whatever difference is there, that is basically only in the details. From one of the vector space, you can always go to the other vector space and come back through that isomorphism one to one correspondence.

So, you find that all n dimensional vector spaces over the field F are actually equivalent. So, they can be considered as same in particular. The vectors with which we have been dealing the representation, the column vectors in which the coordinates are the just listed, so in particular all of these n dimensional vector space are isomorphic to F n itself. The column vector, the listing of coordinates that representation is also a vector field to that is also a vector space. So, with that it will be equivalent and therefore, we find that the representations, the column vectors, the listing of the coordinates themselves can be taken as the objects. So, for practical purposes there will be actually no difference in between and that is why we after studying one vector space of n dimensions over a scalar field, we do not have to study another such vector space of the same dimensional over the same field again.

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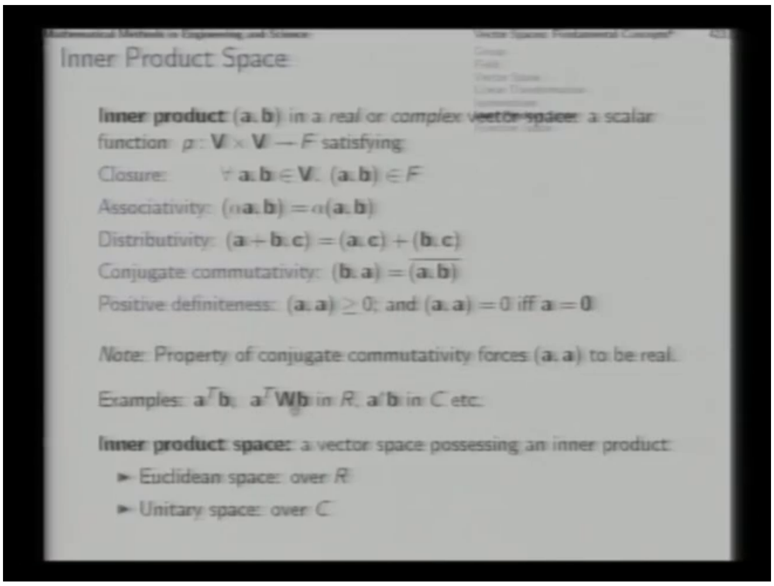

We have actually studied all of them in one shot. So, till now we have found a lot of geometric ideas in the algebraic description of the vector space. Now, we bring in another idea from geometry into the algebraic representation and that is the idea of directions and angles and that we get from the definition of inner product in a vector space over the set of real numbers, over the field of real numbers or complex numbers. We can define a linear product which is denoted with this sign a, b in parenthesis and the definition of that is this, that is it is a function which takes two vectors from the vector space V and as a result produces a scalar in the field F.

It can be R, real or complex numbers, such that it is defined for all vectors a b and it has the property of associativity. If you multiply one of the components with alpha, then the product also gets multiplied with alpha. There is associativity, it has the distributivity and it has conjugate commutativity. This is this operation is not just commutative, it is conjugate commutative that is for real field. It will be commutative. The inner product b and a will be the same as inner product of a and b for the complex field. It will be bar of a b, b a will be the bar of a b, that is will be a conjugate of a b. Now, this essentially means that if you take a as the two vectors and try to work out this inner product, then that can be a, a as well as a, a conjugate and these two have to be equal.

So, this conjugate commutativity forces this a, a inner product to be real, then you can talk of its being positive or real negative real number and this is another requirement which makes sense in that context that it has to have positive definitions that is this must be positive or 0 and it will be 0 only if a is equal to 0. So, a product satisfying all these requirements is defined as the inner product. These are all examples of inner products. In this you will make note of this particularly a transpose W b while defining an inner product as a transpose W b in the field of, over the field of real numbers.

One must be very careful about ensuring that this weight metrics W is positive and symmetric and positive definite. This is another point which we have discussed earlier and this is the reason of that, that is if the matrix W is not positive definite, then this condition may get violated for all a. This will not be correct and that is why for defining a weighted inner product, one must ensure that the weight matrix W is positive definite. Now, a vector space processing an inner product is called an inner product space called the field of real number. We call that space as Euclidean space and over the field of complex number, we call the space as unitary space. Most of the time we have been talking about actually Euclidean spaces of several dimensions.

Now, I make this point that for the rest of course also most of the time our discussions of multi-dimensional vector spaces will be mostly associated with the Euclidean space. So, R n R m etcetera are actually all n dimensional m dimensional Euclidean spaces. So, we find that inner product bring in ideas of angles and length in the geometry of the vector spaces.

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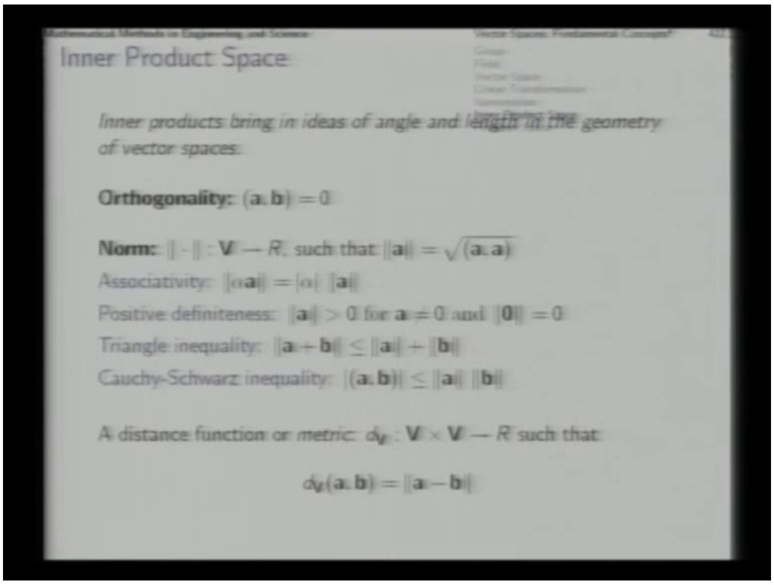

You know that the inner product a, b will have if you consider your ordinary definition of that product that will have size of a into size of b into the cosine of the angle between the two vectors. So, the idea of angle something to pictures in particular, we say that the two vectors a and b are at right angles if their inner product is 0. So, the question of orthogonality comes into picture, then the size of a vector norm comes into picture that is norm is again a function from the vector space to the set of real numbers, such that the norm is actually equal to the square root of the inner product of the vector with itself and see it must be positive and you will find these are some properties of the inner product and norm taken together associativity. That is the norm of alpha times a vector is model for times the norm of the original vector and so on.

What is the difference which we have already seen? These are two important inequalities. Triangular inequality a plus b norm is less than equal to norm of a plus norm of b and this is Cauchy Schwarz inequality. The inner product will have an associate value which is the less a which is less than or equal to the product of the sizes of the two vectors. This is known as Cauchy Schwarz inequality. Based on these you can also work out a distance function or a matric. So, if you have two vectors, then you can work out a distance function between the two vectors in the sense of joining the arrowheads and working out the size of that vector, that is with this much on the vector spaces of finite dimensions.

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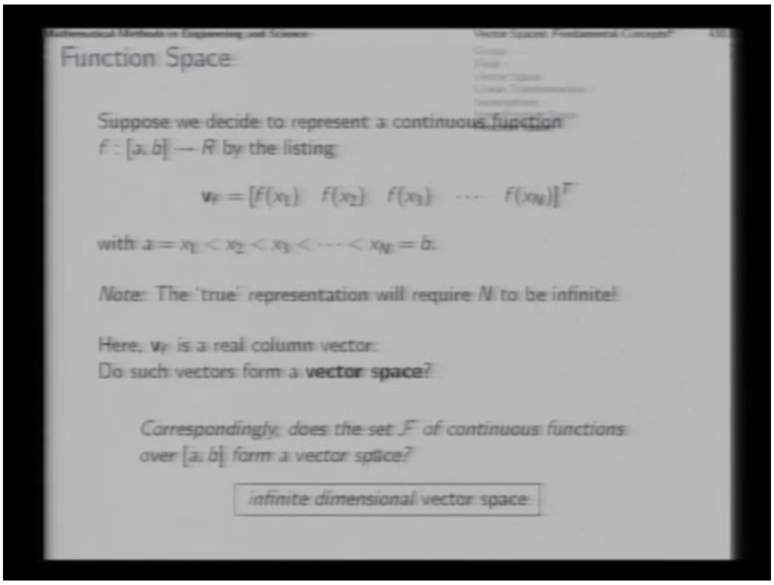

Now, let us go little into the discussion of infinite dimensional vector spaces. The set of continuous functions over an interval provides such a vector space of infinite dimensions and that is known as the function space. Suppose we are working with a lot of continuous functions and we want to represent a function like that real valued continuous function on an interval by the listing of its values over several values of x like this from a to b, right. The true presentation of the function will require this capital N to be infinite because the function is continuous function over the entire continuous domain of a to b for the interval. Now, this vector that we written here and n dimensional vector capital N column vector, so if we take several functions and for each functions, we work out its values at these n points and get such column vectors, then will all these possible column vectors together form a vector space.

Answer is yes, because they are column vectors and all sort of continuous functions over that interval we can keep in the discussion. So, they will form a vector space of dimension capital N and for more and more precise true representation as we keep on increasing capital N and try to take it infinity, then we will have an infinite dimensional vector space to set this capital F of continuous function over continuous real value functions over a b form a vector space which is infinite dimensional.

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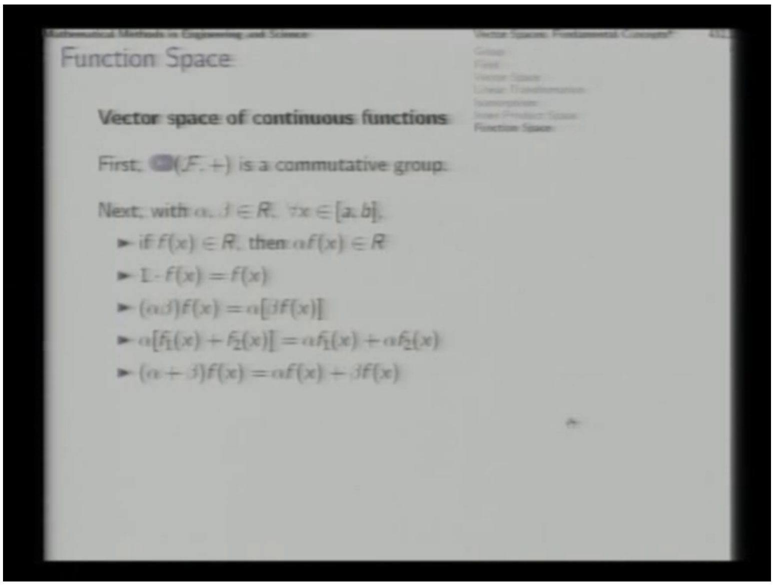

You can verify that infinite dimensional vector space if you can just check whether these forms a commutative group and whether the vector space conditions are met. So, that is a basic verification with the definitions of group and vector space with it is interesting to conduct once.

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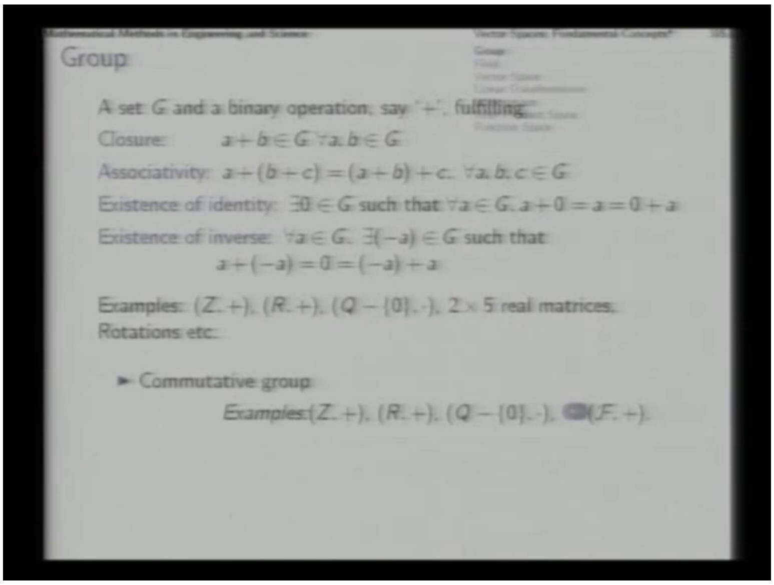

So, whereas commutative group is concerned suppose a and b in place of a and b you think of f1 x f 2 x. Now, f 1 x f 1 and f 22 continuous real value functions, so there is another such continuous real valued function defined over the interval a b and similarly, they fulfill these requirements and there is a zero function and for every function F, you can define a function minus F and then, f 1 plus f 2 will be the same as f 2 plus f 1. So, they will form a commutative group and apart from that for being vector space, these are exactly the vector space conditions which we have in a way copied from there. So, you can verify all of these and find that all of these conditions hold. So, that way mean that the set capital F of all such real valued continuous functions over the interval a b among themselves form a vector space of infinite dimensions and listing of values at selected points is actually just a basis to describe all such functions.

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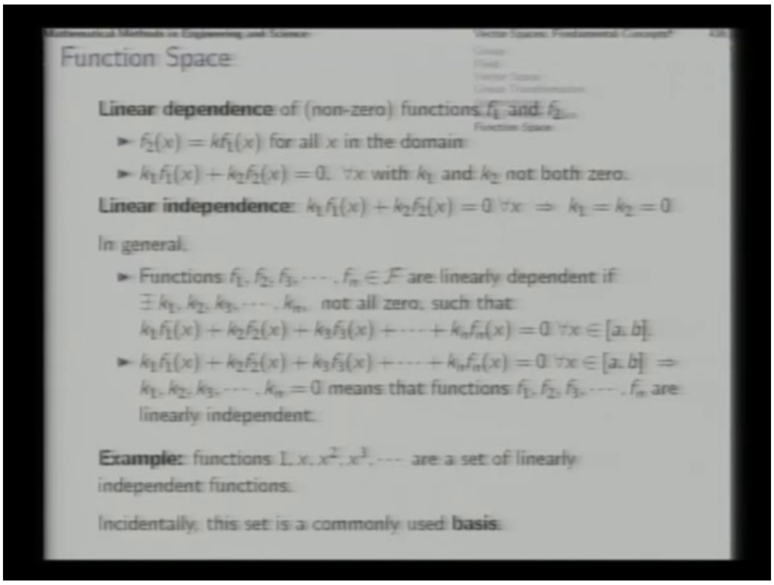

We can talk of linear dependence and linear independence of these functions. So, two functions f 1 and f 2 if they have this definition or this definition which is equivalent actually, then we say that they are linearly dependent and if it happens that this linear combination equal to 0 necessarily implies that all k 1 and k 2, this k 1 and k 2, both are individually 0, then you say that f 1 and f 2 are linearly independent from each other. In general, among n such functions you can say that if you can find k 1 k 2 k 3 k 4 up to k n, not all zero together such that you can make their linear combination 0, then you say that these functions among themselves are linearly dependent.

On the other hand, if you cannot find such non-zero set k 1 to k 2 that is not all zero together with that understanding that is if you find that f 1 and f 2, this linear combination is 0 essentially implies that all of them have to be individually 0. You cannot find any non-zero set making this linear combination as 0. Then, you will say that these functions are linearly independent. So, these notions will be using later in detail as tools when we study differential equations. So, you see 1 x square x cube etcetera are actually a set of linearly independent functions and quite often this is used as a basis.

To describe such function for example, when you say that I have taken a function F and we want to describe f x as a 0 plus a 1 x plus a 2 x square plus a 3 x square x cube etcetera, basically you are using this set of functions as a basis.

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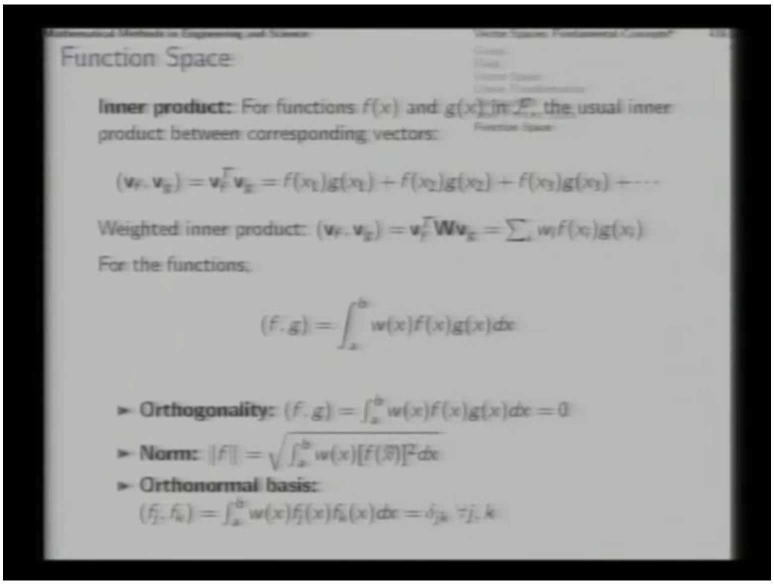

You can define inner product between vectors a between functions. So, suppose f and g are two functions and then, for f and g we can work out those large column vectors v f and v g which among themselves have this inner product. Then, as v f and v g are the function values, then this will be same as this f x 1 plus f x 2 g x 2 and so on. Like this you can work out a weighted inner product also which will be something like this if this weight matrix W is a diagonal matrix which is w 1 w 2 w 3 etcetera existing on the diagonal positions.

Now, as the number n, the number of atoms become 16, this large and then, this large number of terms getting some dot gets replaced it an integral and you say that the inner product in the function space is defined in this manner f x g x and this w i gets replaced with w x d x x varying from a to b. So, summation gets replaced with an integral and this turns out to be the definition of the inner product in the function space. You can similarly define norm, you can similarly talk about orthogonality. So, you will say that two functions f and g are orthogonal when their inner product turns out to be 0. In that case, you say f and g are orthogonal functions. Orthogonal with respect to the weight function W, you can talk about norm. So, if f and g in both places if you put f, then you will have W into F square, right and that integral value evaluate and take the square root of it that is the norm, right.

You can talk of orthonormal basis if you have taken the basis as functions f 1 f 2 f 3 f 4 etcetera each of which has a unit norm in this sense and every pair which is orthogonal in this sense, then you say that we have got an orthonormal basis for describing functions in that set f. So, for ortho normality of a set of functions, you require this condition that is the inner product between each pair f j s k in that basis must be delta j k if j and k is same. Then, this should be 1 and if del and if j and k are different, they should be 0 for all j k, right. That way you will get an orthonormal basis for that function space.

Now, how many such f 1 f 2 f 3 f 4, you will ask for? Since the dimension of the vector is infinite, you will need infinite members in the basis. That means that basis you will require a familiar functions which is of infinite members.

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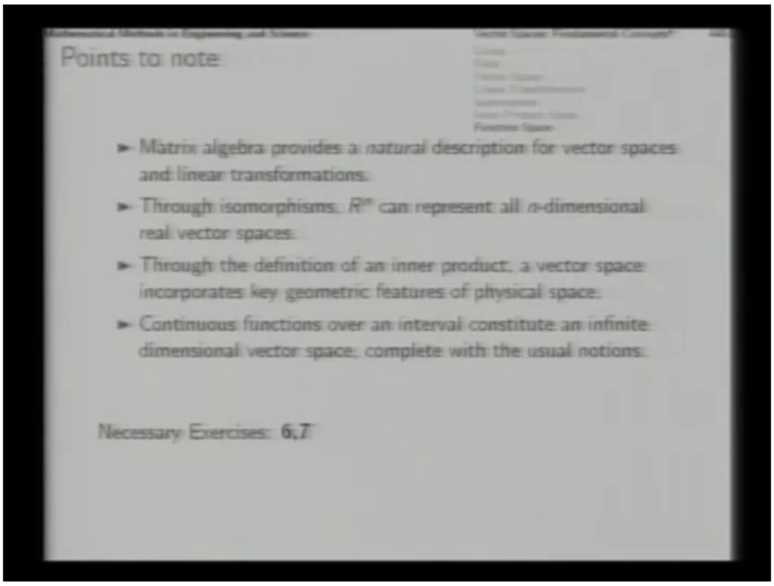

Now, from this discussion we have these important points to note. First is that matrix algebra provides a natural description for describing vectors, spaces and linear transformations and whatever R n till now we have studied is actually the complete representation or a norm representation for all n dimensional vector spaces over the field of real numbers. The third important issues that we have seen in this lesson is that through the definition of an inner product, the key ideas of angle and length from ordinary geometry are brought into the discussion of vector spaces. So, these incorporate the key geometric features of physical space. Another important issue discussed in this lesson is the topic of continuous functions forming a vector space of their own and we can talk of the function space of infinite dimensions later when we study differential equations. We will also see how linear operators or linear transformations get a meaning in this kind of a function space.

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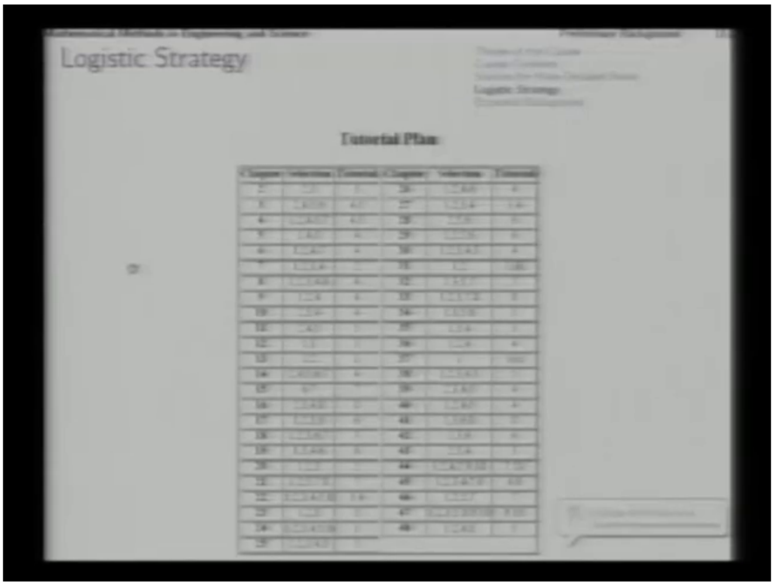

So, with this our module on Linear Algebra gets over and again I will quickly remind you that it is important to go through these lectures along with the exercises because many of the ideas are actually left from the discussions in the lectures because we are trying to squeeze in a lot of topics into a single course and therefore, a lot of issues, a lot of conceptual details will become clear when you try to work out the exercises and consult the solutions there off.

So, till now we have completed 15 chapters of the book and if you find it little too hectic to complete all examples, then a selection is given here in the tutorial plan which appears in the slides of the first chapter. So, if you complete this much, then you would have got sufficient background to continue with the rest of the lectures that we take up later and you will find that since one of the essential features of this course is the interconnections among several areas, so you will find that though this module on linear algebra is formally over, but the ideas developed here will be used throughout the other modules of the course.

So, after this in the next lecture on wards, we will be handling the module, small module on vector calculus and multivariate calculus and vector calculus.

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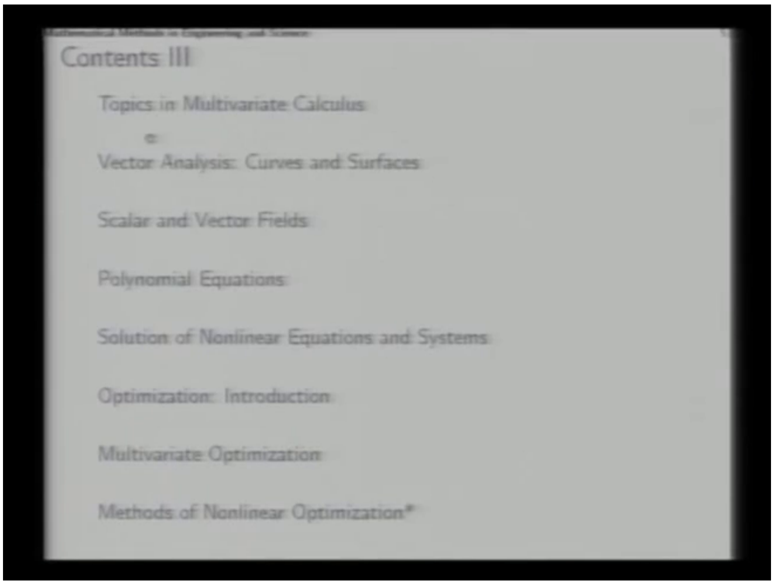

That will constitute of these 3 chapters of the book chapter 16, 17 and 18.

Thank you.