Computer Aided Engineering Design
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Lecture - 9

Welcome to NP-TEL video series of CAED lectures. This is lecture 9, Transformation of Solids.

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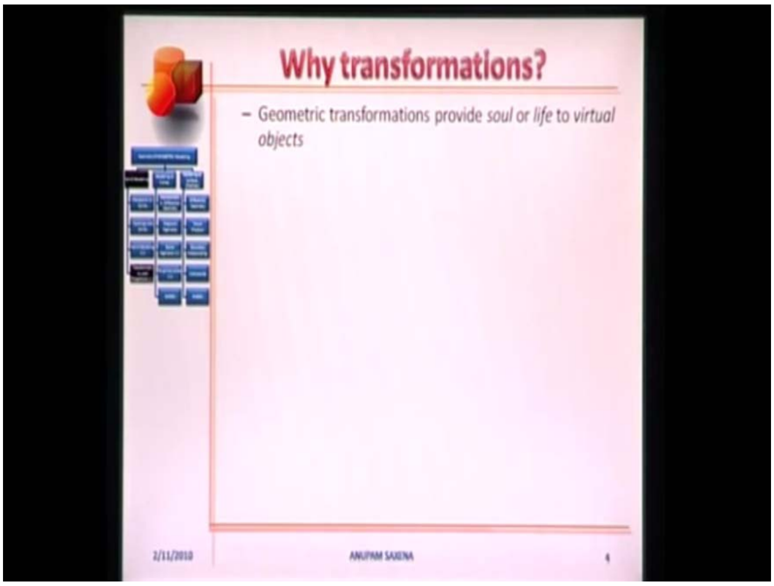

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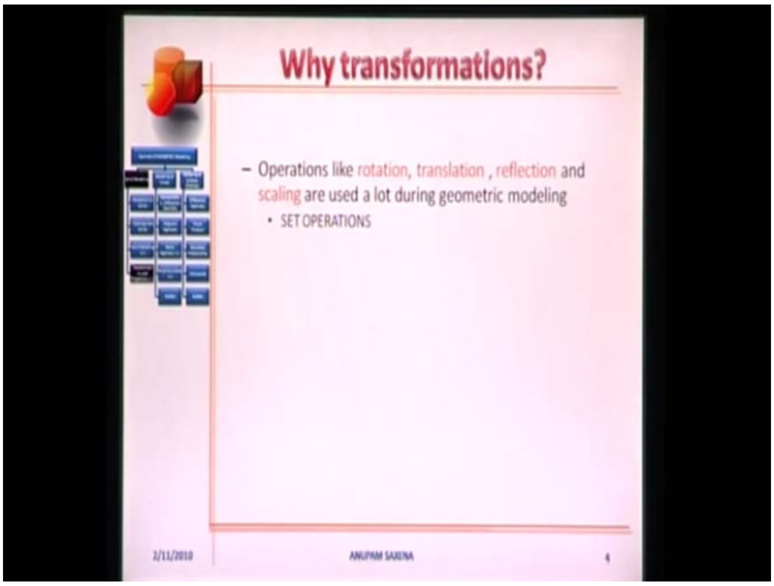

The question that we asked why transformations? Geometric transformations provide soul or life to virtual objects. Operation like rotation, translation, reflection and scaling are used a great deal during geometric modeling. You might want to think about set operations. They have two solids in my hand; solid 1 and solid 2. I need to manowar a manipulate the position and orientation of solid 2 with respect to solid 1 to be able to perform set operation. Unless I do that I will not be able to perform set operations. There has to be a part of solid 1 in dissecting with a part of solid 2 to perform addition, subtraction, intersection, and union like operation.

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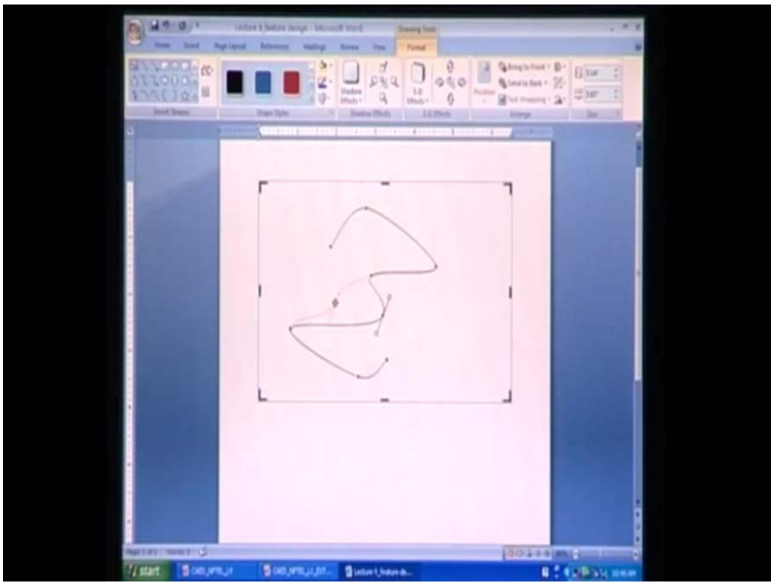

The second example feature design. Let me first draw a general curve here and then finally tends the positions of all these points. What this does is it allows me to change the local shapes of these curves. Many curves construes surface patches, so moving these points or transforming these points allows me to change the local shapes of those surface patches as well. I can go ahead and locally change the slope at a point. I will not be able to do feature design unless I perform transformations.

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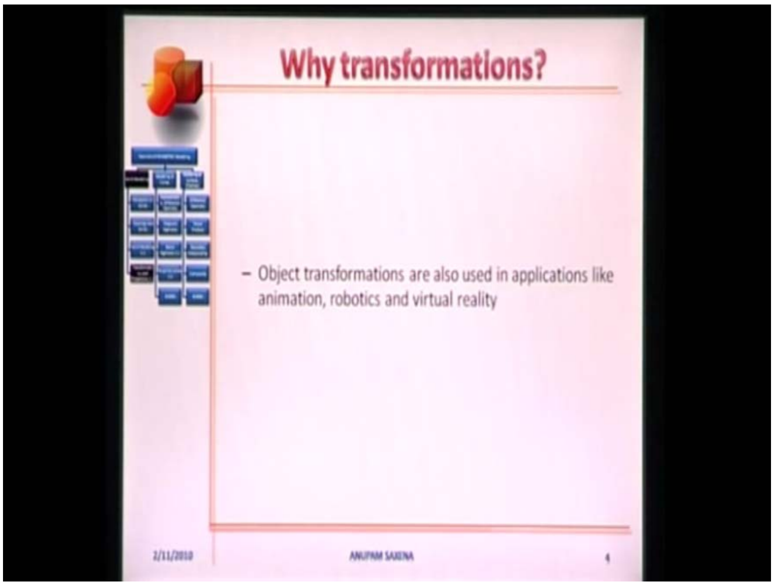

Object transformations are also used in applications like animation, robotics and virtual reality.

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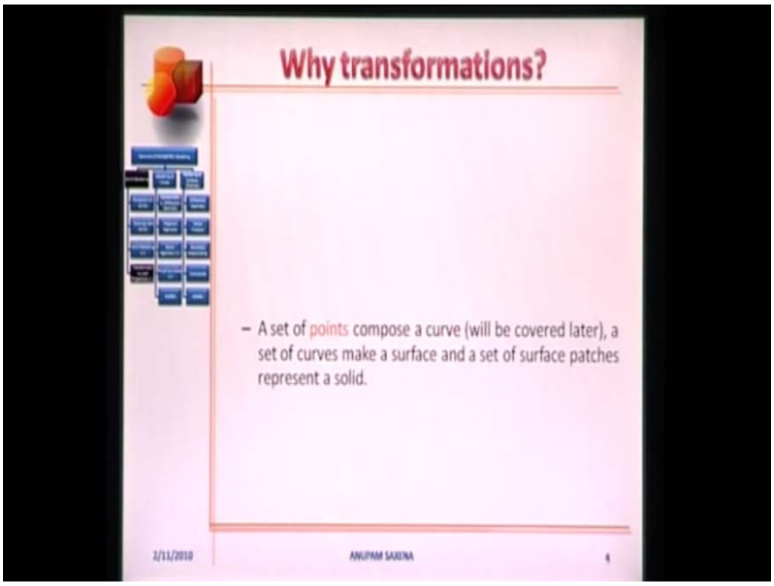

A set of points compose a curve we are going to be discussing curves later in the course a set of curves make a surface and a set of surface patches represent a solid. We have learned this before this is the extended Jordan curve theorem. In a sense what we are doing is to be able to change the shape of a solid locally or the shape of a surface patch locally or a curve locally, all we need to do is reposition the points appropriately. In other words all we need to do is to perform transformations on points and local shapes of surface patches and solids will change automatically.

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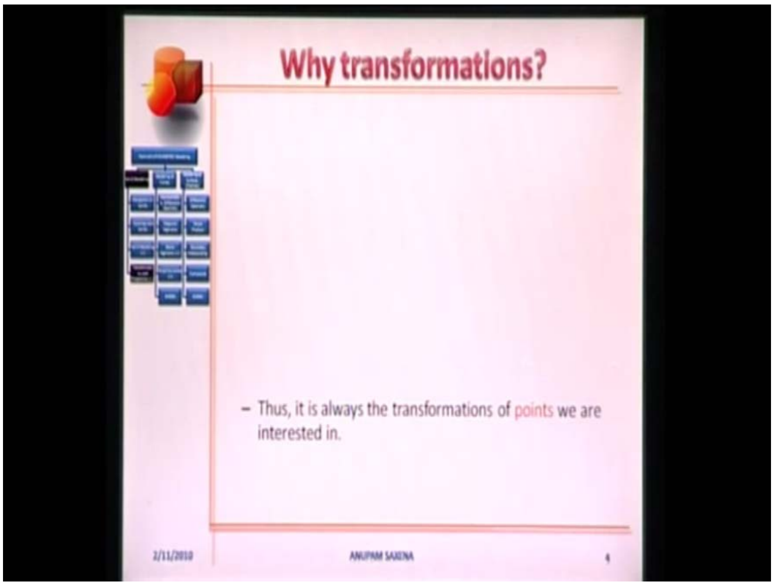

To reemphasize it is always be transformations of points we are interested in.

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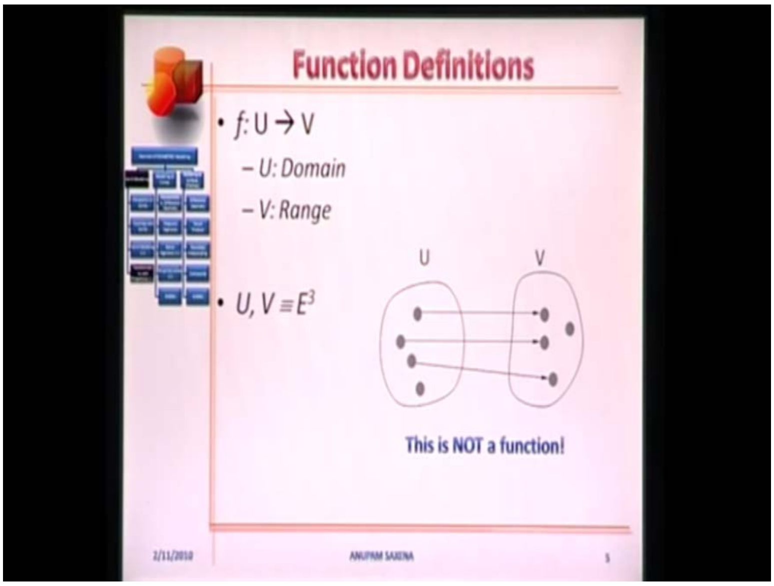

Transformations are essentially function, just in case if you are forgotten what functions are; here is quick recap. A function f is a map from U to V, U is called the domain and V is called the range; U and V are two sets. V is also called the image of U and U is also called the free image of U. In our case we are performing transformations in the Euclidean space E 3 and so U and V are E 3.

Let us look at the definition of function. Say we have a few elements in the domain U and correspondingly, we have a few elements in the domain V, some elements in U are related to the corresponding images in V. Note that one element in U is not related, this is not allow in function definition. Elements in V can exist without any relation this is allow, but elements in U existing without any relation or without any image is not allow. In other words this is not a function.

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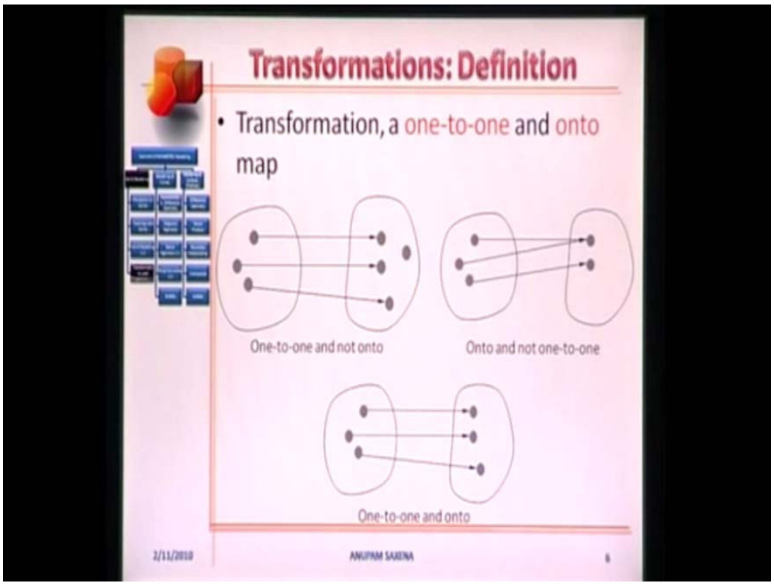

Transformations the definition. A transformation is a one-to-one and an onto map, let see what different map types are. In this figure, this is domain U and this is the range V. All the element of U have corresponding images in V, the images are also unique, but notice this element in V. It does not have any free image, this is a one-to-one map and not an onto map. Let us try to see what an onto map is. We have a few elements in the domain set and then we have a few elements in the range set. Notice here that these two elements have the common image and further there are no elements, in the range which awake. That your function is onto function and not a one-to-one function because two elements in the domain have one image over here for which reason it is not one-to-one function.

Look at this function here all the elements in the domain set have respectively unique images in the range set there is no element in the range set which is making. This is a one-to-one and onto map, which is what transformations are…

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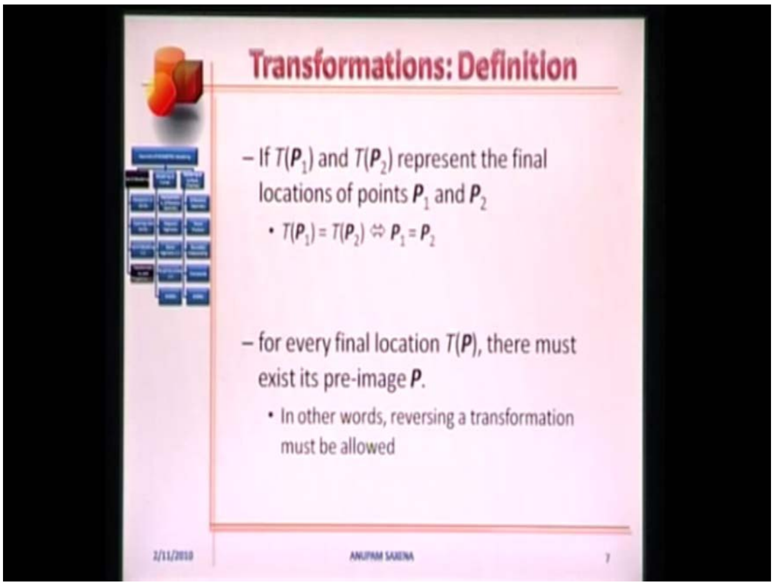

Let us say we have a point P and T of P represents the transformation of that point. If T of P 1 which is the transformed position of P 1 and T of P 2 which is the transformed position of P 2 represent the final locations of points P 1 and p 2. Then if T of P 1 equals T of P 2, then P 1 equals P 2 and the converse is also true that if P 1 equals P 2; then the respective images are the same. In other words for every final location T of P there must exist its pre-image P.

Basically, this is what it means? Look at this solid here it has 8 vertices. If I move this solid from this position to this position I should be able to move it back to the previous position which should be allowed by any valid transformation. If it is not allowed then we are in trouble. In other words reversing a transformation must be allowed.

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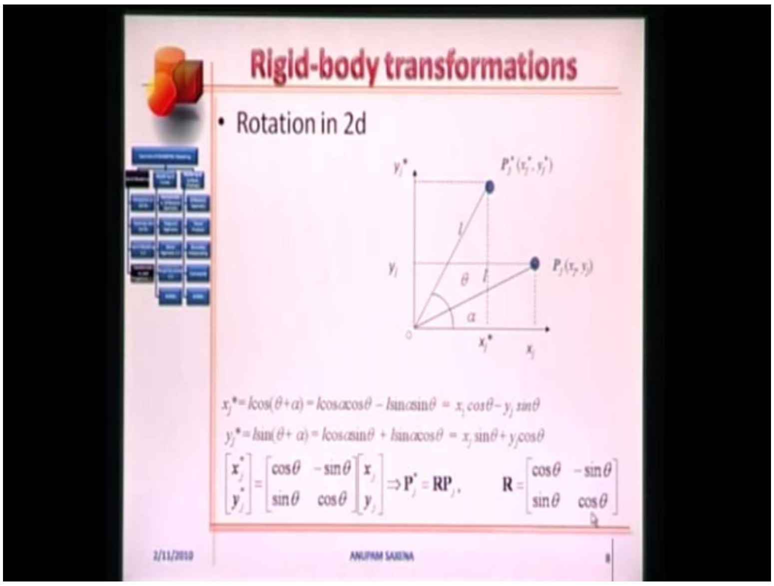

Rigid-body transformations; let us take look at rotation in two dimensions. We have x y plane and we have a point P sub j with coordinates x j and y j. Let me rotate this point about the origin O to a new location P j star with coordinates x j star and y j star. Let me also draw two position vectors joining the two points to the origin, the length of the position vector joining P j and 0 is l. Since we perform rotation the length of the vector that joins the origin to P j star is not change.

Let, alpha be the angle which mean the position vector 0 P j and the x axis and let theta be the angle by which point P j is rotated to P j star. We can determine one of the axis and coordinates of P j and P j star are. This is x j star which is projection of this position vector on the x axis, this is x j which is the projection of P j on the x axis. This distance is y j which is the projection of P j on the y axis and this distance here is y j star which is the projection of P j star on the y axis. We can write relation between x j star theta x j and y j using trigonometry. Clearly x j star is l times cosine of theta plus alpha expanding this gives l times cosine of alpha times cosine of theta minus l times sin of alpha times sin of theta. Notice that l times cosine of alpha is x j and l times sin of alpha is y j.

Likewise, y j star is l time’s sin of theta plus alpha spanning which gives l time’s cosine of alpha time’s sin of theta plus l time’s sin of alpha time’s cosine of theta. Once again l cosine alpha is x j and l sin alpha is y j. I can write these two relations in compact form using matrices. Here we have a column vector that include x j star and y j star. Here we have a two by two matrix cosine of theta minus sin of theta, sin of theta cosine of theta and this is column vector including x j and y j. If I multiply this matrix with this vector x j times cosine theta minus y j times sin theta x j star and x j times sin theta plus y j times cosine theta is y j star. If I treat the column vector as an order pair of coordinates using short form, I can write that position vector P j star equals r the matrix r that pre-multiplies the position vector P j, where r is cosine of theta minus sin of theta sin of theta cosine of theta, we call r as rotation matrix.

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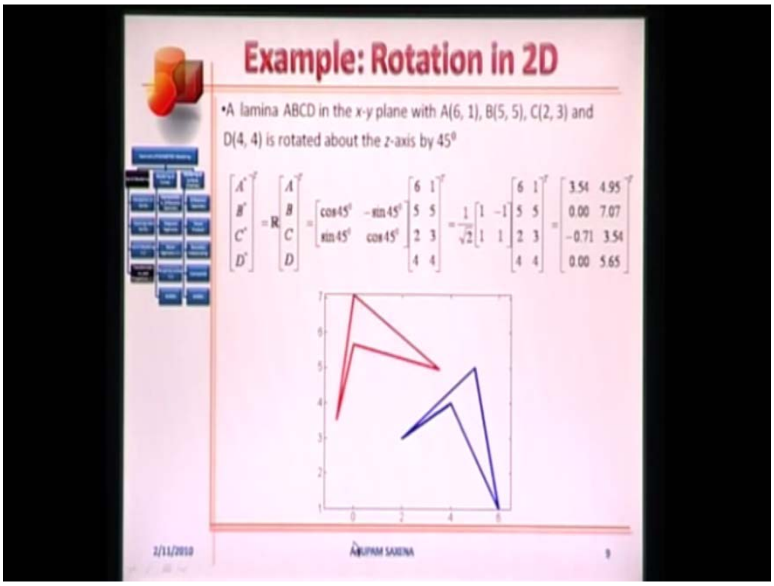

Let us consider an example of rotation 2 dimensions. A lamina A B C D in the x-y plane with coordinates of A as 6, 1 coordinates of B as 5, 5 C as 2, 3and D as 4, 4 is rotated about the z-axis which is coming to you by 45 degree. A star, B star, C star and D star are the new positions of these vertices, A B C D are the old positions. Notice how I am writing this matrix, which is a four by two matrix and the transpose of it. In a sense what I am doing is through a single matrix multiplication operation. I am rotating all the four points. Of course I pre multiply this matrix by the rotation matrix cosine of 45 degrees minus sin of 45 degrees sin of 45 degrees and cosine 45 degrees. It is not difficult for us to work out the math, but it is more important to understand how the final position of the lamina looks. In this figure the lamina in blue is in its original position, lamina in red is in its final position, origin is somewhere here.

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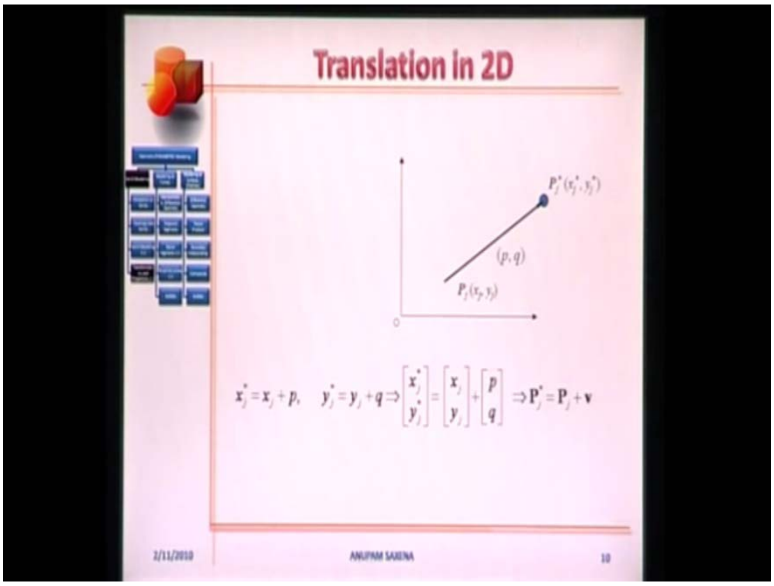

Translation two-dimensional we have the two-dimensional x-y plane we have a point P j coordinates x j and y j. Let us translate this point to this vector to its final location P j star with coordinates x j star and y j star. The translation vector has components p and q, in other words p is the amorphous of translation along the horizontal direction and q is the amorphous translation in the vertical direction. x j star equals x j plus p, y j star equals y j plus q. Once again I can write these two equations in compact matrix form column vector x j star y j star equals column vector x j y j plus column vector p q.

Note that in rotation, we were using a matrix to pre multiply a column vector, but here we are using the addition operation; there is the slight descripance, but we will address that later. In short form, position vector P j star equals position vector P j plus a free vector v.

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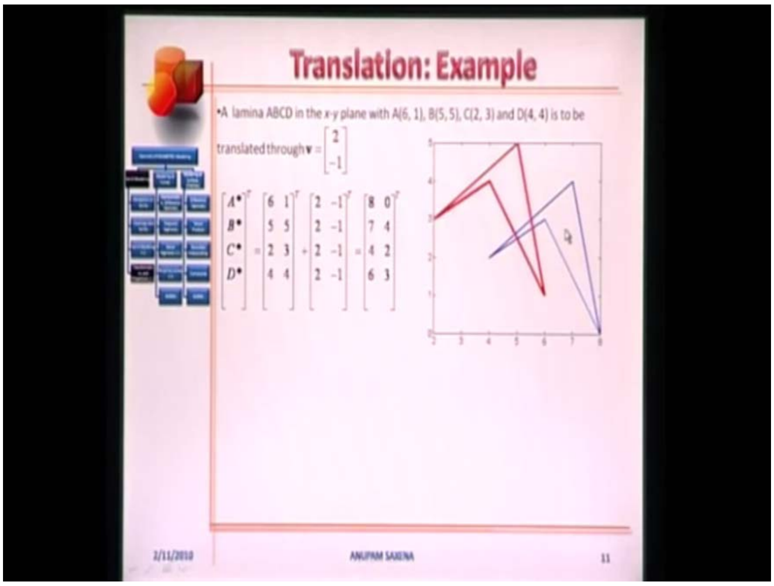

Translation an example the lamina A B C D with the coordinates of A as 6, 1, B as 5, 5, C as 2, 3 and D as 4, 4 is to be translated through a vector v with components 2 and minus 1. Once again we perform all translation operations simultaneously A star, B star, C star and D star are the new positions of A B C D. Here all the coordinates of A B C D respectively I mentioned and they are all translated by 2, minus 1. These are the new positions of the four points; 8 0, 7 4, 4 2 and 6 3 respectively for A star, B star, C star and D star. In the figure the lamina in red is in its initial position and the lamina in blue is in its final position.

To eliminate that discrepancy that I talked to you about earlier, translation may also be expressed as a matrix multiplication operation just like rotation. You would notice later that it will make our lives a lot easier. For that what I need to do is the column vector, I need to introduce an additional entry 1, here again I need to introduce an additional entry 1, you can carely verify by pre multiplying this matrix with the old column vector. The translation matrix is 1, 0, p, 0, 1, q, 0, 0, 1, 1 times x j plus p times 1 is x j plus p which is x j star 1 times y j plus q times 1 is y j plus q which is y j star and 1 times 1 equals 1. Note that without the introduction of the additional entry in the position vectors of both in new and the old points, it would have been difficult for us to express translation as a matrix multiplication operation. In short form P j star is T the translation matrix times P j. Once again translation matrix T is given by 1, 0, p the first row 0, 1, q the second row and 0, 0, 1 the third row.

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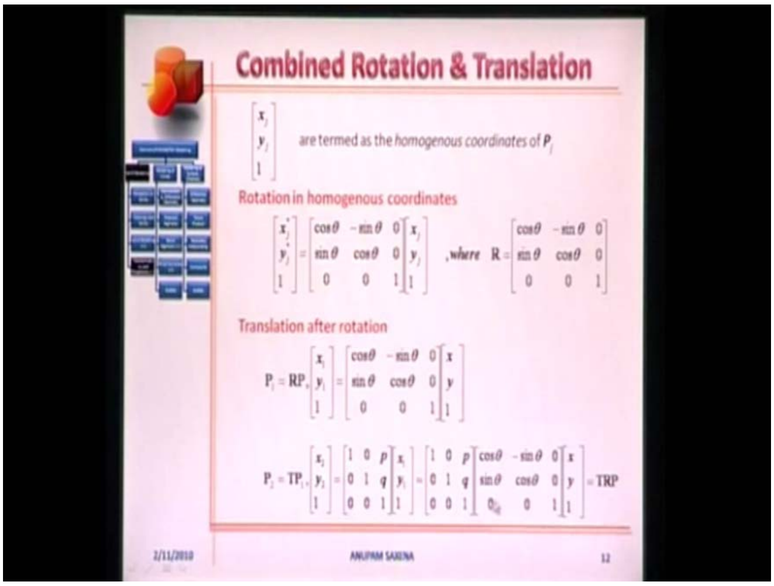

Combined rotation and translation, but a little bit about the additional entry that I had introduced in the position vectors earlier. So this column vector with entries x j, y j and one are now termed as the homogenous coordinates P j. x j, y j would have been the natural coordinates, you would see these coordinates coming in allowed use in areas like robotics and vision.

First rotations in homogenous coordinates; remember that we still have rotation in the natural coordinates. You need to modify the rotation matrix appropriate from its two by two sizes to its three by three sizes. That is not very difficult, to the column vector on the left is x j star y j star 1 cosine of theta minus sin theta sin theta cosine theta was the original two by two rotation matrix. Now we have a third row and a forth column, all the entries in the third row except for the last 1 are 0 and all the entries in the third column again except for the last 1 are 0. The three by three entry in this newly constructed rotation matrix is 1, x j y j 1 are the homogenous coordinates of P j mentioned earlier. R is now my new rotation matrix with the element cosine theta minus sin theta 0 sin theta cosine theta 0, 0, 1, it is of three by three insets.

Let us now perform translation after rotation. So we have the original point P, it is rotated, so point P 1. We know the algebra x 1 y 1 would be the homogenous coordinates of P 1 equals a rotation matrix, pre multiplying the column vector x y 1 which are the homogenous coordinates of P and then P 1 is translated to a new position P 2. The translation vector has components p and q so; we can substitute this expression to replace this column vector. Once we do that, we have the translation matrix pre multiplying the rotation matrix which in term is pre multiplying the original column vector; this is the short form. We can multiply this three by three matrix which is a translation matrix and this three by three matrix which the rotation matrix.

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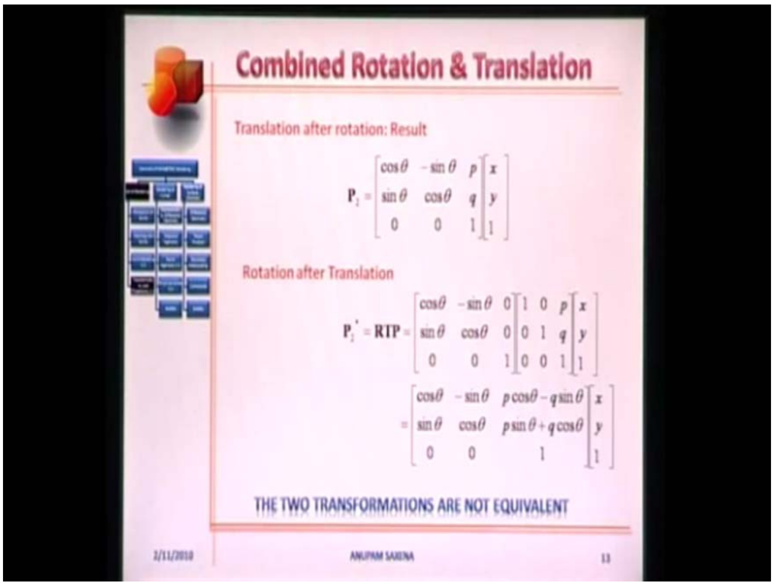

We get cosine of theta minus sin of theta p sin of theta cosine of theta q for the second row and 0 0 1 for the third row. Instead had I perform rotation after translation, thinks what have been slightly different. In this case, the final position of the point is determined by the original position of the point which is pre-multiplied by the translation matrix and this is pre-multiplied by the rotation matrix. These three by three system here; we all know this is rotation matrix and this three by three system here is the translation matrix. If we workout the algebra, we get cosine of theta minus sin of theta p times cosine of theta minus q times sin of theta in the first row, sin theta cosine theta p times sin of theta plus q times cosine theta in the second row and 0, 0, 1 in third row. Appreciate the difference between these three by three translations they are not the same.

Or in other words for specific values of p and q and theta, they would be the same around it and this is what we know from matrix algebra. Two matrixes a and b, if multiply together that is a times p would not gave the same result as p times a; if multiplication between a and b is primitive. Bottom line the two transformations are not equal.

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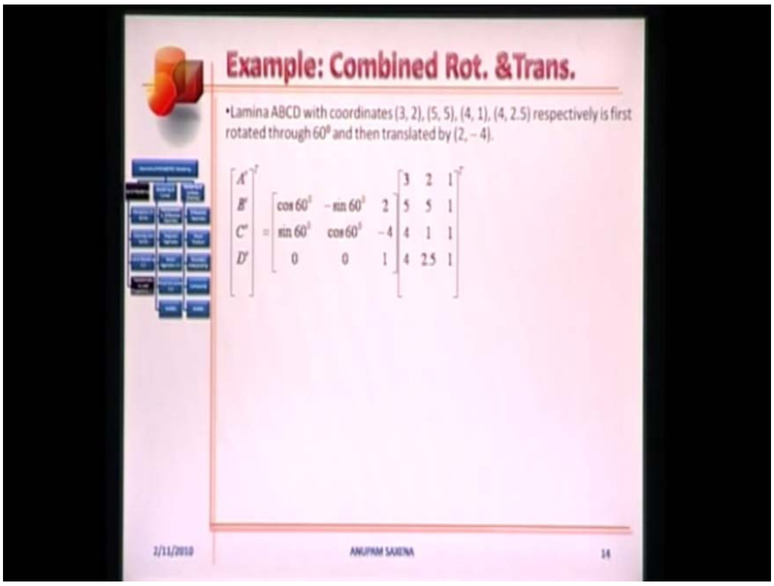

Let us take an example of combined rotation and translation. Lamina A B C D with coordinates of A as 3, 2, B as 5, 5, C as 4, 1, D as 4, 2.5 respectively is first rotated so 60 degrees and then translated by vector 2 minus 4. We can do the math, we know the results from the previous slide, this is the case of rotation and then translation. Once again I am performing translation on all the four points simultaneously here.

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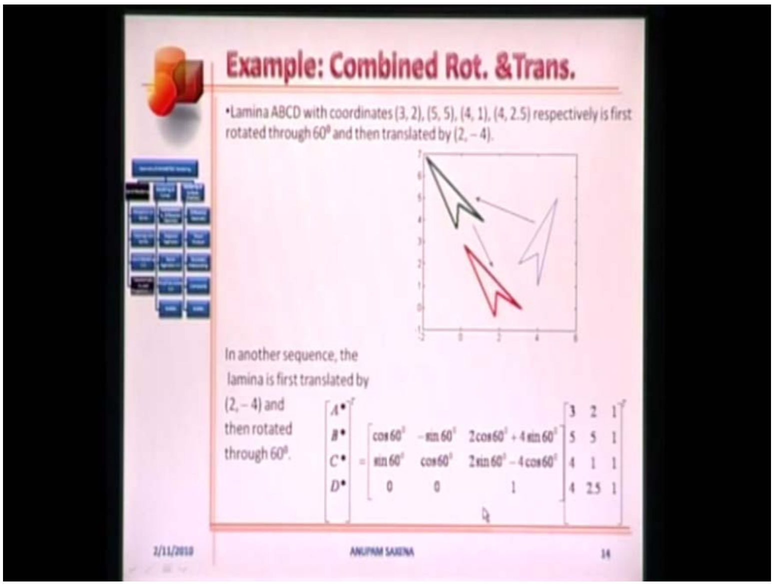

This is the result I see, the lamina in blue is in its original position. This is rotated about the origin by an angle 60 degrees so the green position here and then translated by the vector 2 minus 4 to the red position. Note the position one of the average position of this red lamina, somewhere along the x coordinate of 2 and y coordinate of 1. Had we performed this transformation the other way round? That is first we would have performed translation and then rotation, translation by vector 2 minus 4 and rotation by an angle 60 degrees.

We were have gotten slightly different results, once again we know this result from previous slide all we need to do is substitute for values of a variables appropriately. If we plug four points in deferent position.

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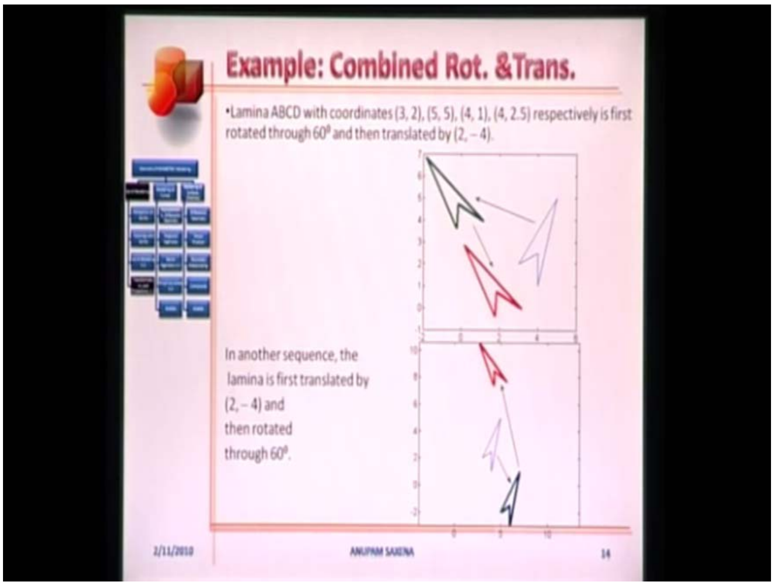

We see this the lamina in blue is the original position, first translated the green position and then rotated to the red position. Note two things here; the orientation of the final position of the lamina here and here are the same, but the positions are slightly different. This is located somewhere around x coordinate 5 and y coordinates 8. What is this lamina? Is located somewhere around the x coordinate of 2 and y coordinate of 1; bottom line the order of transformations is very important, we cannot change the order of transformations can expect the same result.

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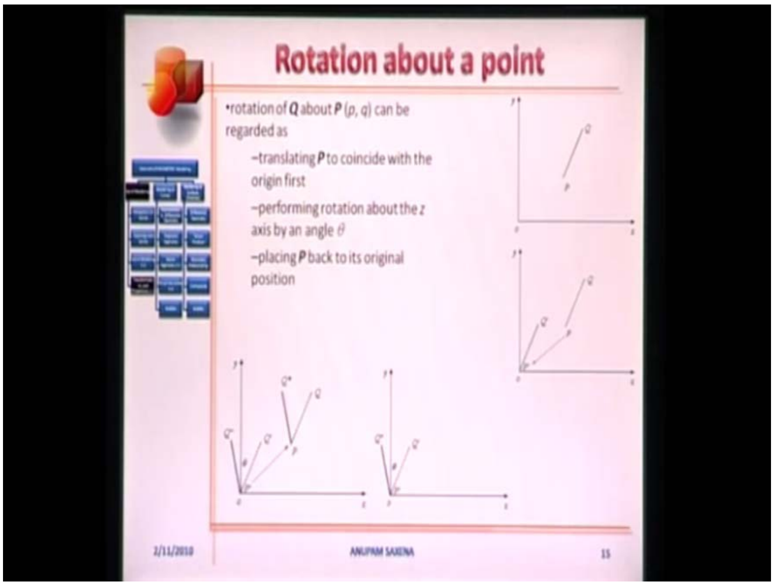

Next rotation about a point; so far we have covered rotation of a point about the origin, for this case is slightly different. We would like to rotate point q about point p this is again in two dimensions what do we do? We can use the knowledge that we have gathered before on both translations and rotations. Rotation of Q about P of which the position vector or the components of the position vector are p and q small case can be performed in the following steps. One can think of translating first point P to coincide with the origin like here, point P can be translated to coincide with the origin.

The new position of P is pre prime; the orientation of the line segment does not change. Step 2: One can then perform rotation about the z axis which again is coming to you by an angle theta like so. This line segment is rotated by an angle theta about the origin and finally, one can place the point P back to its original position like so. The three concatenated translation, rotation and translation operation or inverse translation operation this case, is equivalent to point q getting rotated by an angle theta to a new point q star about the point p. We can work out the math.

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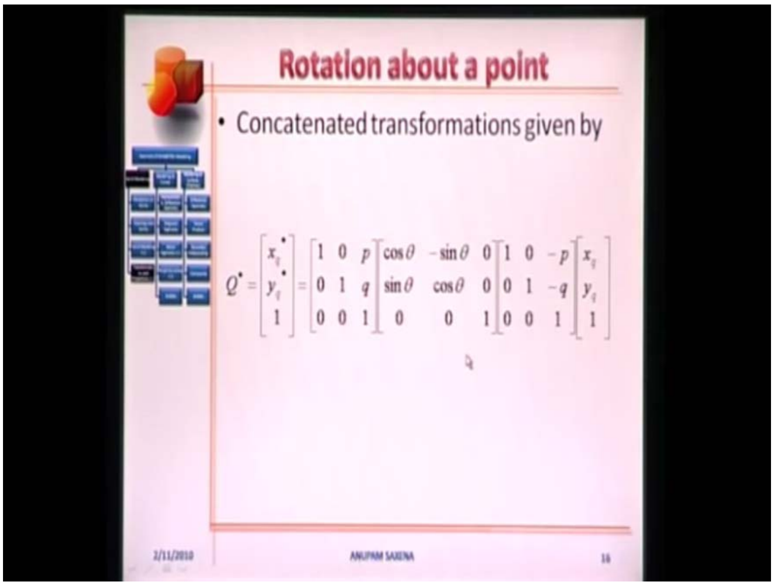

If we think about arranging different transformation matrices, again let me remind you that we are working with homogenous coordinates. x q star, y q star and 1 are the homogenous coordinates of the new location of Q, Q star and x q, y q, 1 are the homogenous coordinates of the previous location of Q. Of course, there are three matrices corresponding to three different transformations.

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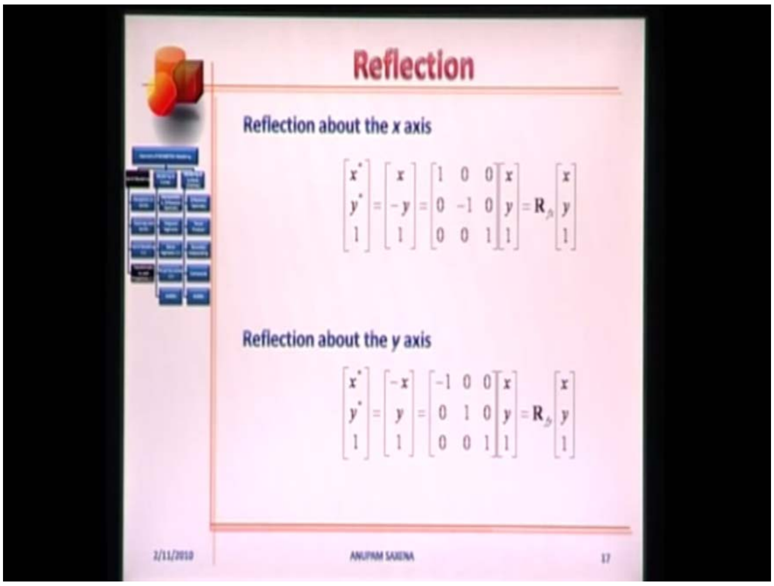

The first one was translation of p to 0, notice the appearance of minus signs over here. The next was rotation of q about the origin which is given this three by three rotation matrix and the third operation was inverse translation that allowed p to move back to its original position. A remark here, notice how all these transformation operations are getting concatenate as matrix multiplication of this. This is what the homogenous coordinates allow us to do to combine translation and rotation transformations as matrix multiplication of rotation.

Next reflection let us see what happens when we reflex a point about the x axis. Well when we reflect about the x axis then the axis a remains the same, the coordinate changes a sign and 1 remains 1; the third come here for now. We can express these relations in a matrix form. We will have a three by three matrix here; the first row is 1, 0, 0, the second row is 0, minus 1, 0 and third row is 0, 0, 1. In short this three by three matrix is represented by R sub f x pre-multiplying the original column vector of the point; likewise reflection about the y axis. In this case the accessor changes in sign, the coordinate remains the same and the dominatrix again remains the same. We can express this using again a three by three matrices with entries minus 1, 0, 0, 0, 1, 0 and 0, 0, 1. This three by three matrix is represented as R sub f y. These are the two reflection transformation matrices about the x and y axis respectively.

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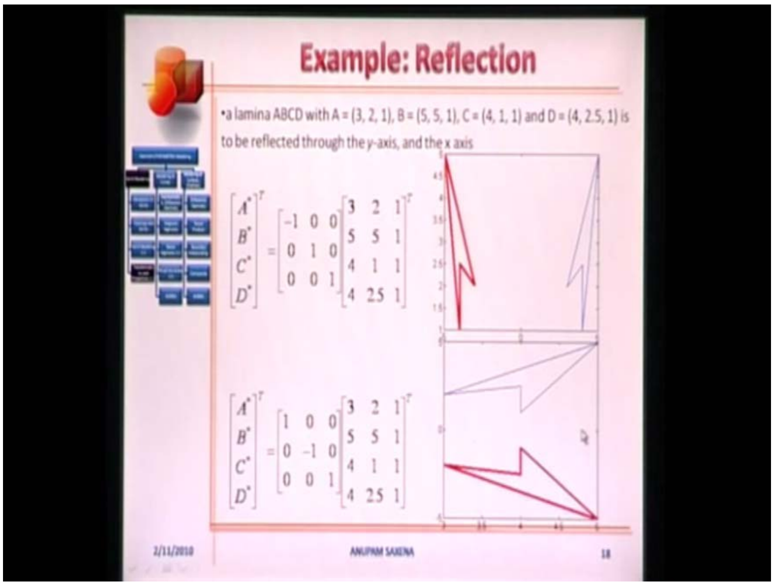

Let us consider an example on reflection. A lamina A B C D with coordinates of A as 3, 2, 1, here we are using the homogenized coordinates, coordinates of B as 5, 5, 1; C as 4, 1, 1 and D as 4, 2.5, 1 is to be reflected through the y-axis and then the x axis; these are two separate reflection operations. I perform the reflection operations simultaneously on all the four points of the lamina. This is as if, I am staking the homogenized coordinates of A B C D column wise and constructing matrix and then pre-multiplying this matrix by a reflection matrices, reflection about the y axis. Working out the math is not difficult for graphically this is for the situation. The lamina in blue is in its original position and the lamina in red is the reflected position; this is the vertical matrix. If I perform reflection about the x axis, then this is how the figure looks; the lamina in blue is the original position and the lamina in red is the final position this is the axis of repletion.

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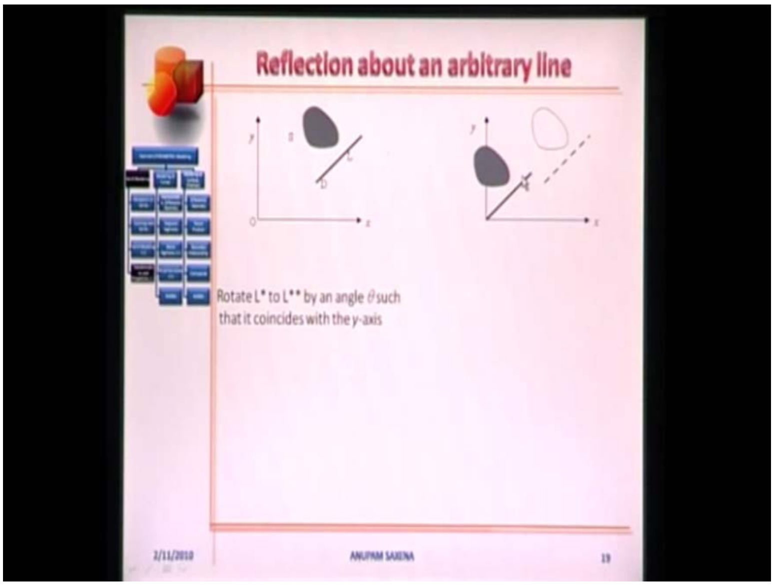

Now, let us consider a generic transformation of reflection about an arbitrary line. Let us see we have a two-dimensional patch S here which is to be reflected about the line L D. How do we go about constructing various intermediate transformation of reflection? Let see or remark here note that these intermediate operations will not be unique. We can choose to work with deferent operations equalize.

Step 1: Translate point D with homogenized position vector p, q, 1 to coincide with the origin, shifting this line L to a new position L star. Line L has gotten translated, so that point D coincides with the origin, along with that S has also gotten translated. Step 2: Rotate L star to L double star by an angle theta such that, this line coincides with the y-axis.

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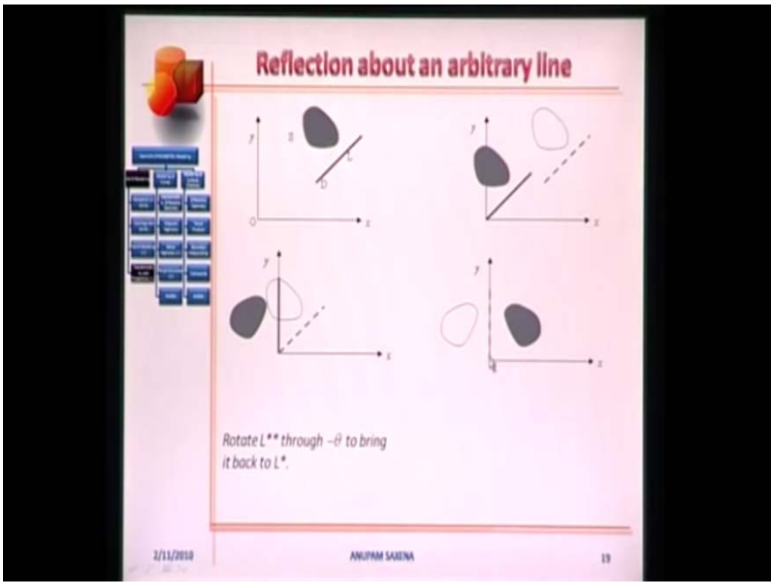

This is the operation; this line is rotated by an angle theta to coincide with the y-axis. Notice that the patch S has also been rotated accordingly. Step 3: Reflect S about the y-axis, graphically this picture defects the reflection. Next step rotate L double star through minus theta to bring it back to L star. In other words, I need to rotate this line to bring it back to this position like so.

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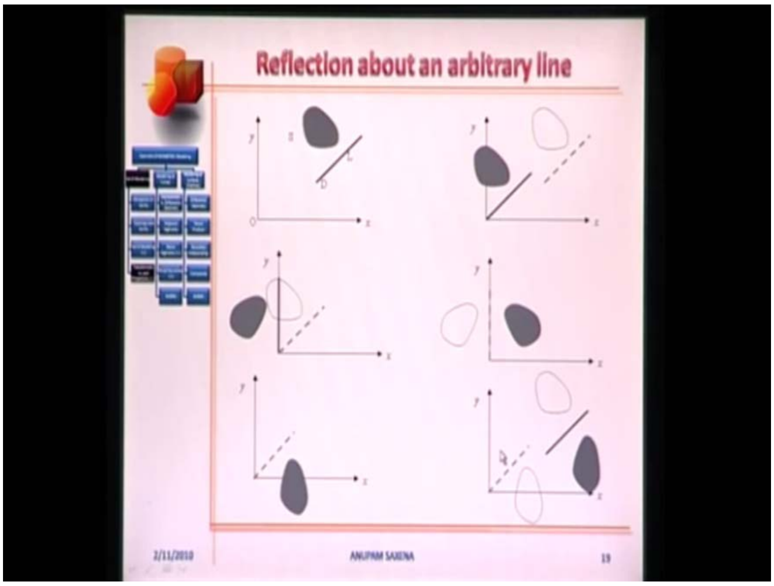

And then finally, translate the L star to coincide with its original position. This is the pictorial representation, line getting translated to its original position here. Compare this figure with this figure and observe how the solid has been reflected from here to here. Transformation algebra, notice that we are performing five transmission steps 1: Translation, 2: Rotation, 3: Reflection, 4: Inverse rotation and 5: Inverse translation.

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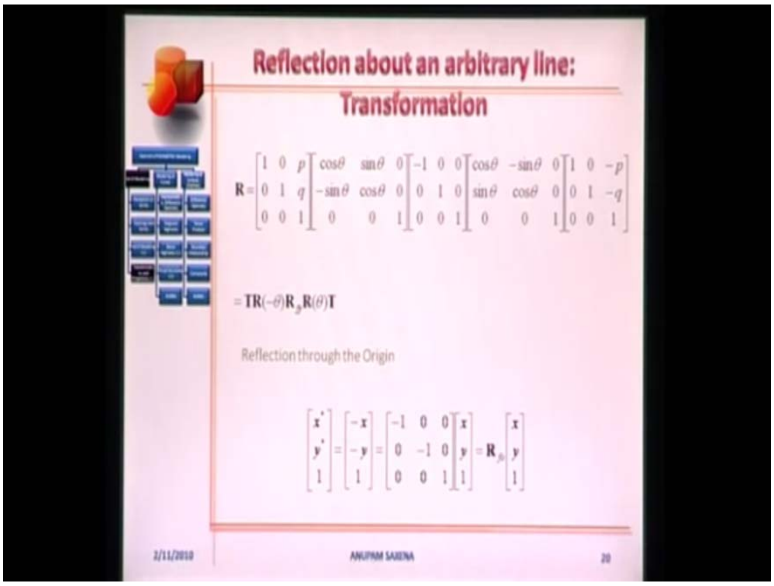

In numbers these transformations appear like so… The first is the translation matrix, the second is the rotation matrix, the third is the reflection matrix about the y-axis, the forth is the inverse rotation matrix. Note here that I have used minus theta instead of theta, so there is a corresponding change in sign in the sin terms and the fifth matrix is the inverse translation matrix. Notice how the transformations have been concatenated. We are concatenating transformations from right to left. In short the transformations are written from right to left as T the translation matrix, R of theta the rotation matrix, R of f y the reflection matrix, R of minus theta the inverse rotation matrix and T inverse translation matrix.

One can as well think of reflecting the geometric entity through the origin. Both the accessor and coordinates would change sign in this case. This transformation can be represented in the three by three matrix form as minus 1, 0, 0, 0, minus 1, 0, 0, 0, 1. In short we call it reflection about the origin r sub f o operating on the original homogenized position vector of a point. One comment an important comment here is realize that we are performing transformations only one points.

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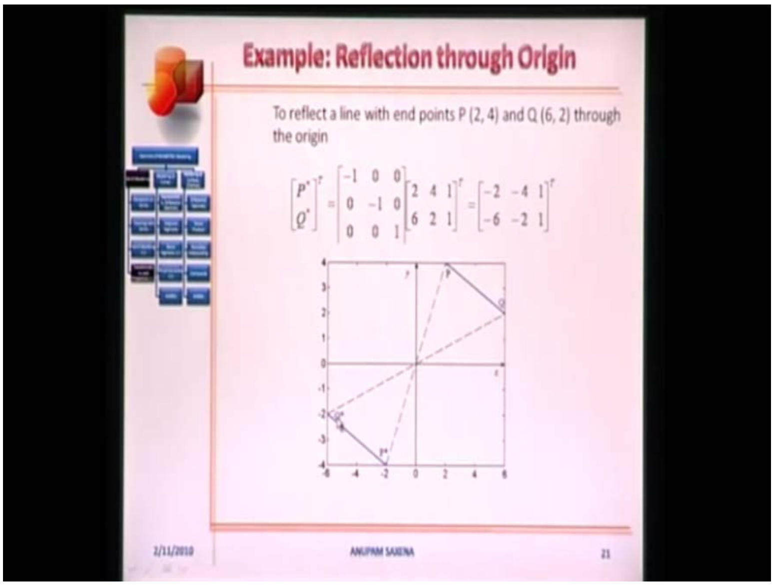

An example of reflection through the origin; a line with end points P and Q with coordinates 2, 4 and 6, 2 respectively, when getting reflected through the origin. The math of that can be worked out look proper looks like this. This is the original line P Q and this the final line P star.

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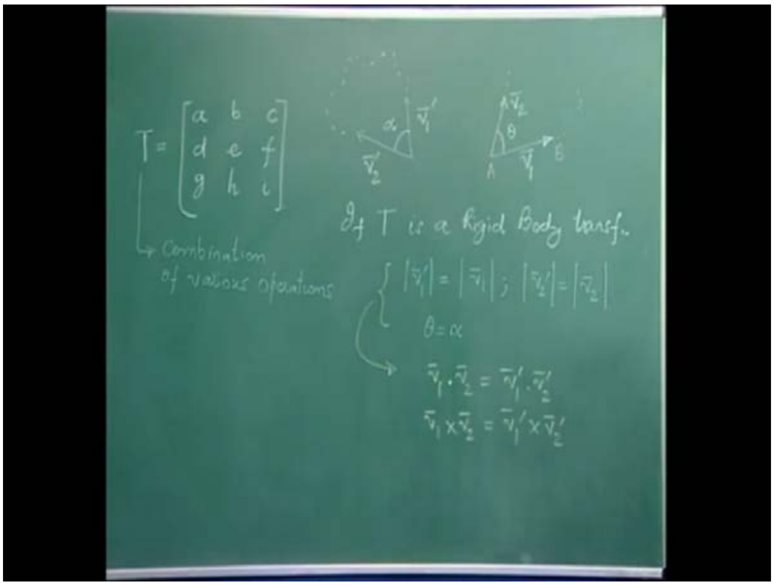

Rigid body transformations have a peculiar property; let me explain this to you on the board. Let T be any generic transformation the components a b c d e f g h i. You can think of T as a combination of various operations, the transformation operation we have studied so far. Let me draw two lines of a polygon, this polygon can be of any shape. I am only interested in 2 contiguous lines of this polygon. Let me represent these two lines in better form, let us say this is v 1, let us say this is v 2 and the included angle is theta. If this polygon undergoes transformation T, the orientation and position of this polygon will change.

Let us say the new orientation is like so this is v 1 prime this is v 2 prime and the rest of the polygon. The included angle let us say is alpha. If T is a rigid body transformation, what do we expect? Well, we expect that the magnitude of v 1 prime is the same as magnitude of v 1, the magnitude of v 2 prime is the same as the magnitude of v 2 and the included angle theta is the same as alpha. If these three conditions for the two vectors and the included angle are not satisfied, then this would not the rigid body transformation. I can replace these conditions slightly different. I can say that v 1 dotted with v 2 is a same as v 1 prime dotted with v 2 prime and v 1 crossed with v 2 is the same as v 1 prime crossed with v 2 prime. A little note about these vectors v 1, v 2, v 1 prime and v 2 prime. Let me represent this vector here by two points A B.

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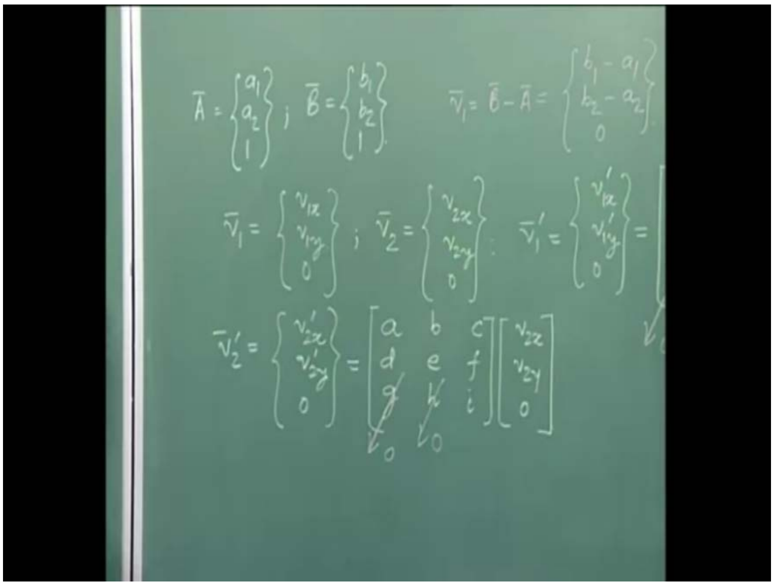

And let us say the coordinates of A, the homogenized coordinates A are a 1, a 2, 1 and those of B are b 1, b 2, 1. Notice that respect to some origin A and B will be position vectors. I vector algebra v 1 equals B minus A and a component form this is b 1 minus a 1, b 2 minus a 2 and 0. Notice this entry here for pre vectors this entry will be 0. Once again for pre vectors like v 1, v 2, v 1 prime and v 2 prime, the third entry will be 0, but for position vectors the third entry will be 1.

With that set let v 1 be equal to v 1 x, v 1 y, 0; v 2 be equal to v 2 x, v 2 y, 0; v 1 prime equals v 1 x prime, v 1 y prime, 0 and note that this is nothing but the transformation a b c d e f g h i operating on v 1 x, v 1 y and 0. Likewise v 2 prime equals v 2 x prime, v 2 y prime and 0 and this is nothing but the transformation a b c d e f g h i operating on v 2. Using this relation, one can get the components v 1 x prime, v 1 y prime or v 1 prime in terms of v 1 x and v 1 y and using this transformation one can get the components of v 2 prime which are v 2 x prime, v 2 y prime in terms of v 2 x and v 2 y. I leave the algebra for you to work on. Once we get these components we can plug in back into these conditions here and compare the deferent terms like v 1 x, v 1 y, v 1 x, v 2 x, v 1 x, v 2 y, v 1 y, v 2 x and so on so forth. You will quickly realize that for these transformations to be valid for any v 1, v 2, v 1 prime and v 2 prime, you will have to have the values of g as 0 and the values of h as 0. Here again g would be 0 and h would be 0.

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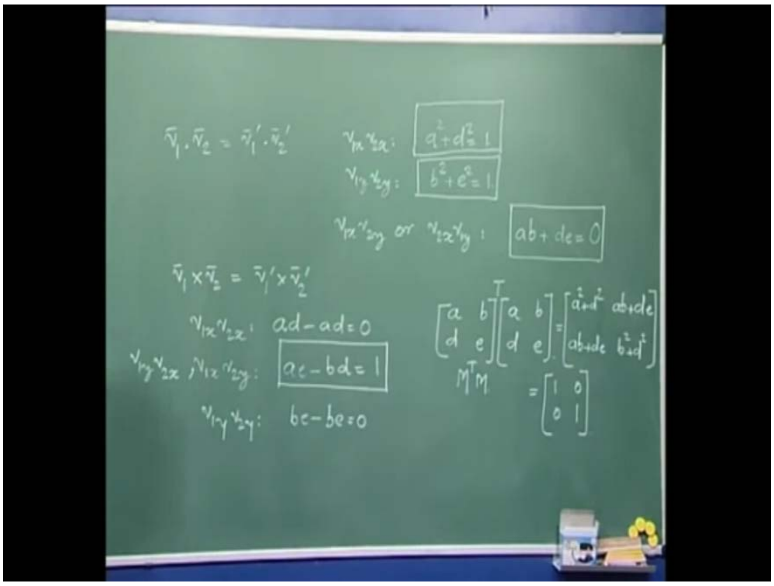

Let us skip the algebra and see what we have. If we use this condition v 1 dotted with v 2 equals v 1 prime dotted with v 2 prime and if we compare the terms v 1 x, v 2 x we get a square plus d square equals 1. If we compare v 1 y, v 2 y terms we get b square plus e square equals 1. If we compare v 1 x, v 2 y or v 2 x, v 1 y terms, then we get a b plus d e equals 0 in both cases. If we use the cross product condition v 1crosse with v 2 equals v

1 prime crossed with v 2 prime. If we compare the v 1 x, v 2 x terms, we get a d minus a d as 0; for the terms v 1x, v 2 y we get a e minus b d equals 1. We get the same relation if we compare v 1 y, v 2 x terms and for v 1 y, v 2 y we get b e minus b e equals 0. So these are four important relations that relate different entries in the transformation matrix. Note that the these are four equations in four unknowns; a b d and e. No where word c f and i come in picture as if now. Let us concentrate on these four variables which form the top left two by two matrix.

And let us perform this operation a b d e transpose a b d e, if I call this matrix m then this is m transpose m. I am working the algebra out this would give me a square plus b square a b plus d e, a b plus d e and b square plus d square. Look at this relation here, it will be not difficult for you to figure that this is nothing but two by two identity matrices and this is precisely one of the properties of transformation. That is two by two sub matrix is orthonormal, orthonormal matrices are those for which m transpose m if m is to be an orthonormal matrix, then m transpose m equals i. How about this fourth condition here a e minus b d equals 1, look at the determine of this two by two sub matrix a e minus b d equals 1. Basically, it is these four conditions that do not allowed the magnitudes of different inter placements between points and the included angles it change are to transformation.