Computer Aided Engineering Design
Prof. Anupam Saxena
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 5

Hello and welcome, this is lecture 5 of the NPTEL video series lectures on CAD. There are certain properties in solids, which are invariant under homeomorphic transmission. Today, we are going to be studying those.

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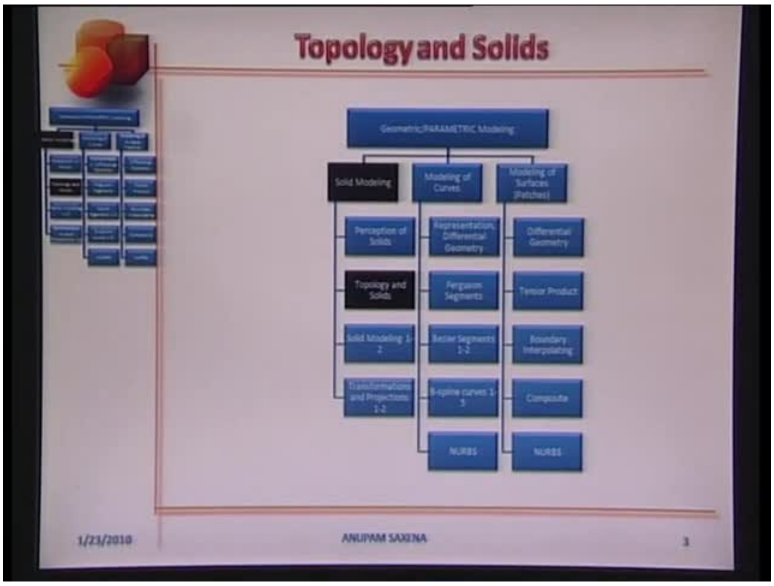

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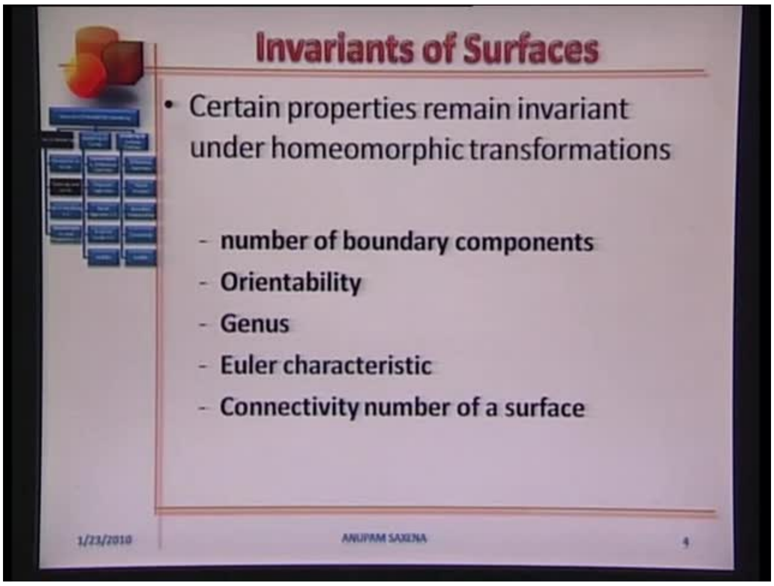

We are further going to tell into topology and solid invariants of surfaces. There are certain properties that remains invariant under homeomorphic transformations. They are number of boundary components, orientability. I am going to talking about orientable and nonorientable surfaces. Here genus, Euler characteristics, this could be an important point and finally, connectivity number of a surface.

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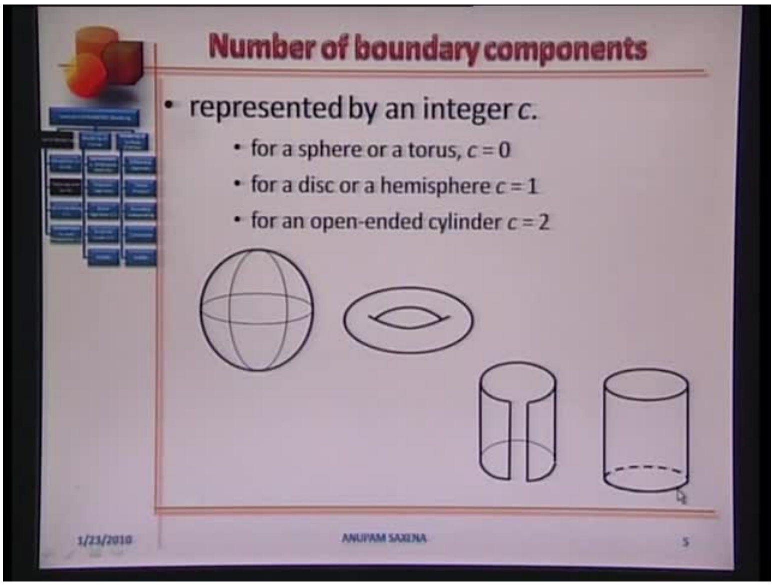

First the number of boundary components. The number of boundary components is represented by an integer c. For a sphere or a torus, there are no boundaries and therefore, c equals 0. For a disc or a hemisphere, we call that hemisphere is a homeomorphic transmission of a disc.

The number of boundary components or the number of boundaries is 1 and therefore, c equals 1. The figure here is a piece of paper bent into a cylindrical shape. If I start from this point here travels leftwards and then travel rightward and go up to the same point, I find that this surface has a single boundary for this. Therefore, c equals 1. For a open ended cylinder, there would be two boundaries. A circular shape boundary on the top and a circular shape boundary at the bottom, for this surface c equals 2.

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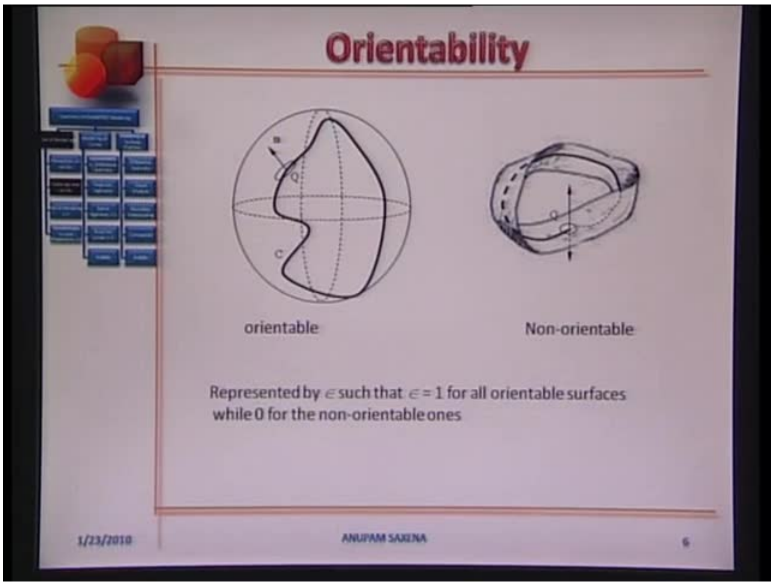

The next property is orientability of a circle. Consider a sphere here, imagine that you are standing at point Q with your head pointing upwards. Therefore, represent a normal component at point Q. If you take a close path and travels along that path from point Q and come back, you would find that you are still standing with your head pointing upwards. That means that the direction of the normal would still be pointing upwards. If that is the case, we call this surface an orientable surface. A torus also would be consider an orientable surface.

Let me explain this concept to you with the physical example. What I have in my hand is a paper weight. It is not very much like a sphere, but I think it got as an nice example. Consider this paper weight to be like a (( )) and say you are standing somewhere here, in the northern hemisphere, say New York. So, the pencil represents you and the tip of the pencil pointing towards the sky. If you starting moving around from this point and go around the world and say 80 days or may be 8 Dollars, go anywhere on the world and of you come back to the same point, your head still be pointing towards the sky, this is an orientable surface.

The next example is of a mobies curve. Imagine now, that your standing at the point q with the head pointing upward. You start travelling leftwards, goes inside the strip, keep going, here you are getting below the strip and comeback to the same point. What you realize is now, that your standing with your head pointing downwards. In case of a

Mobius strip, if you take any point on the surface and if you choose any closed path, it always happened that when come back to the same point, if the directions of the two normals will be pointing in the opposite directions.

This is an non orientale surfaces. Let me explain this on a physical Mobius strip. This is a paper Mobius strip, imagine your pencil and your head is the tip of the pencil and you are standing here. Start travelling in one direction, keep observing the tip of a pencil, if Icome back to the same point, the pencil is now pointing downwards. Let us see this again, once again the pencil pointing downwards. Now, you can always start from this point here, travels the strip come back, not come back to the same point here, pass it, once again travel and then come back to the same point.

You will notice that you are now standing upwards and you would argue, now you have a closed path that retains the orientation. This argument is okay, but note that concept orientablity should be true for every points on the surface and for a every closed path chosen. Orientablity is represented by apslon, such that apslon equal to 1 for all orientable surfaces like sphere or torus, while apslon 0 for all non orientable ones, such as in Mobius.

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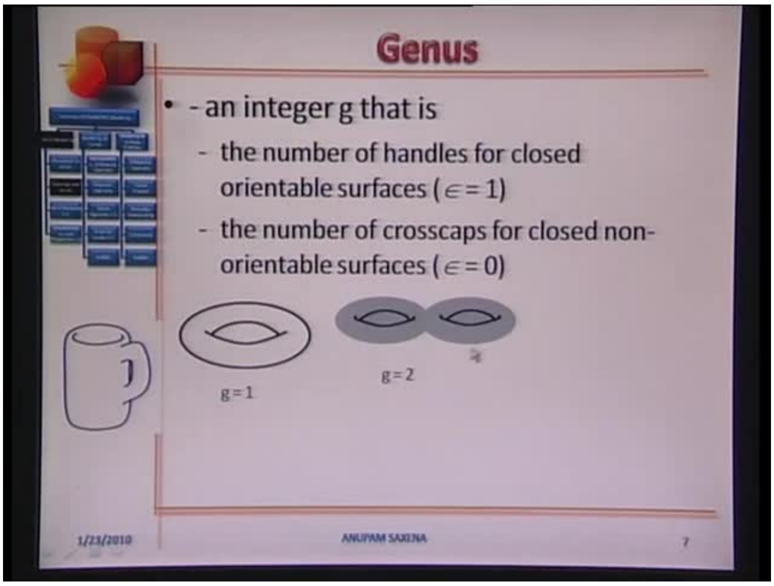

The third property is the genus, is it represented by an integer g. That is number of handles for closed orientable surfaces, for which apslon is 1. If we recall this animation, a torus getting transformed into a coffee mug. This is what I mean by a handle or genus is equal to the number of cross-caps for closed non orientable surfaces, for which the apslon values is 0. Let us consider few examples, this is the torus and this is w connected of a torus or connected some to torus; for torus the genus value is 1, for a double torus since there are two handles each torus can be transformed into a coffe mug, the genus value for the number of handles is 2.

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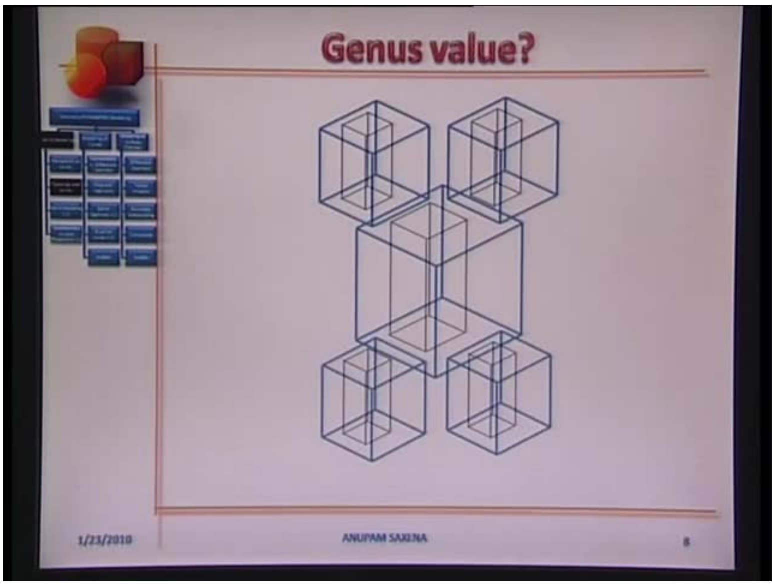

Question, what do you think would be the genus value for such a solid? A figure looks little complex to start with, but let me explain. This solid composed of five blocks, each block has a through hole and the blocks are attached to each other in such a way that apart of each phase of the block is common with that of the contiguous block. So, imagine that these are five blocks with through holes, all of them attached to each other. Once again, what is the genus value for this?

If you consider any block, we can transform it using topological operations like bending, twisting, stretching into an torus, all five blocks. Therefore, can be transformed into respective toride, your guess is correct, this solid has five handles and therefore, the genus value is 5. Euler characteristics, represented by a Greek letter by chi, this is an important topic.

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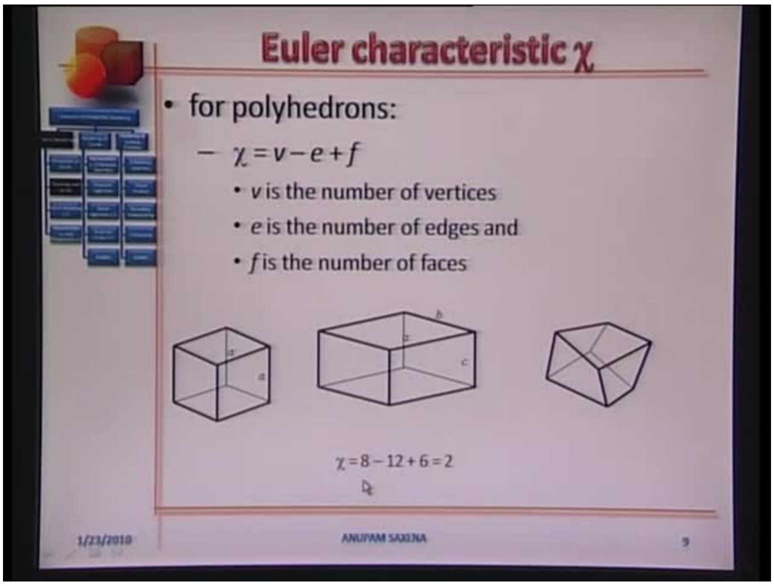

For polyhedrons, it is given by v minus e plus f. v is the number of the vertices in the polyhedron, e is the number of edges in that solid and f is the number of faces. Let us consider an example of a cube, a block and an distorted block. We have seen earlier that each of these solids can be homeomorphically transformed into any other, via bending, stretching etcetera. We all know that these blocks would have 8 vertices, 12 edges and 6 faces. Therefore, chi would be 8 minus 12 plus 6 equals 2. Let us first test this result and try to prove it later. For a tetrahedron, the number of vertices are 4, the number of edges 6 and the number of faces are 4 chi is 2 for a tetrahedron solid as well.

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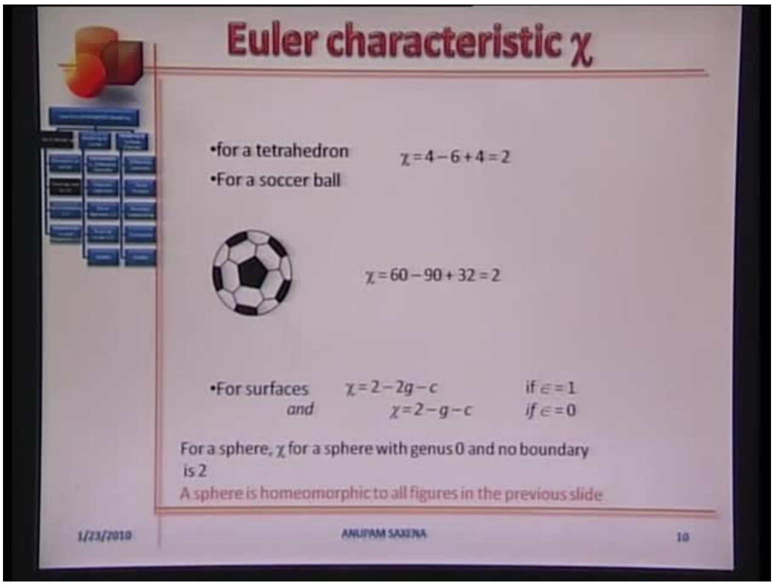

For a soccer ball for this one in particular the number of vertices is 60, the number of edges is 90 and the number of faces 32, chi therefore, is 2 for a soccer ball as well. If you notice this soccer ball has bunch of hexagons and a few pentagons. The question why can be answered using the concept of Euler characteristics. For surfaces, which are not polyhedral surfaces, the Euler characteristics chi is given by 2 minus 2 times g minus g for all orientable surfaces.

Recall again that g is the genus value in a solid and c is the number of boundary components for nonorientale surfaces, for which apslon is 0, chi is given by 2 minus Q minus c, that is 2 minus the genus value minus the number of boundary components. For sphere the Euler characteristics will be 2 because the sphere has no genus value and does not also have any boundary. g is 0 and c is 0 and we all know that a sphere is homeomorphic to all the figures in the previous slide. This in fact the sphere is homeomorphic to all polyhedral solids without any holes.

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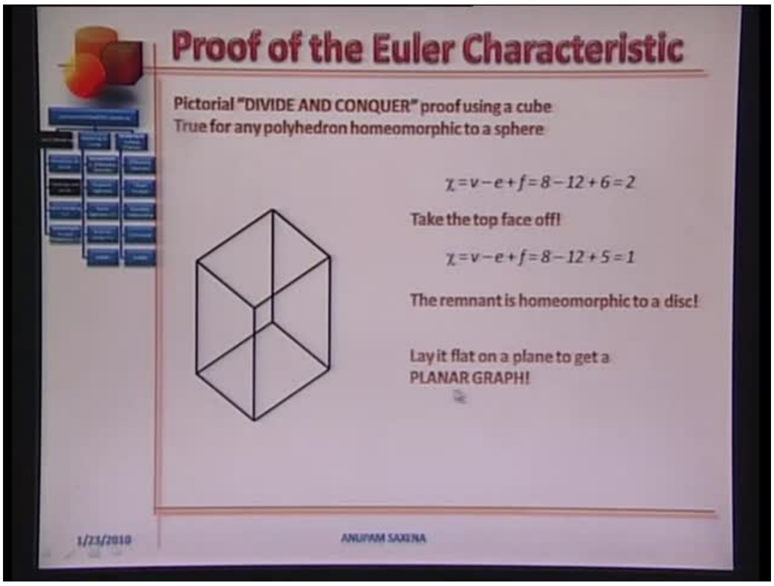

Let us now try to prove Euler characteristics. What I do is, I will provide you a pictorial divide and conquer rule using the example of a cube. A cube would be true for any polyhedron, which is homeomorphic to a sphere. Here is the cube or a block, for this you now know that the Euler characteristics is 2, as the number of vertices is 8, the number of edges is 12 and the number of faces is 6.

Now, imagine that you are taking the faces up. If you do that, the Euler characteristics changes to 1, because number of faces reduces from 6 to 5. You can also note that this surface line, which is not founded any more is homeomorphic to a disc. Therefore, it can be laid flat on a plane, when you do that you get a planargraph.

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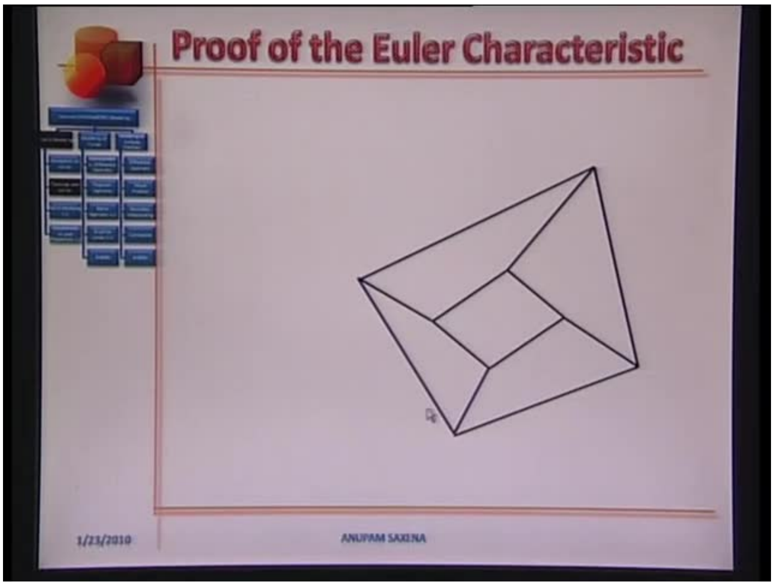

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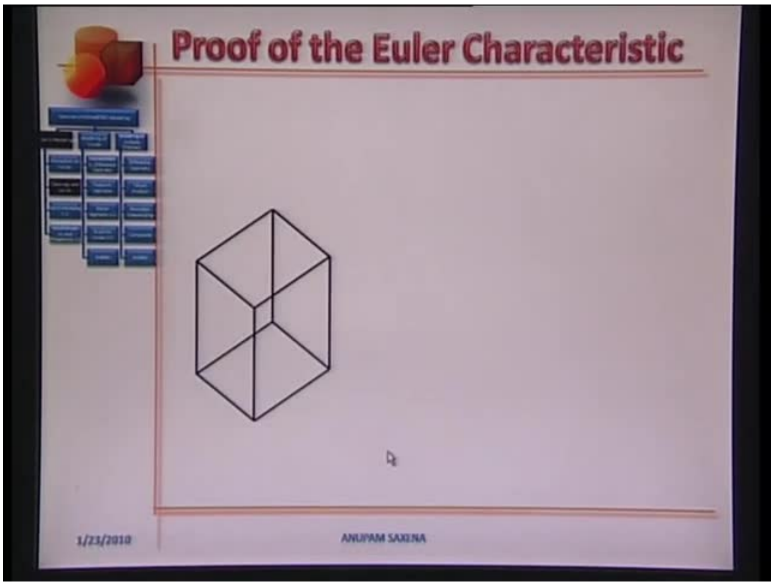

This is how once again stretches these edges outward and stretch this face upward. You squeeze it back on a plane like this.

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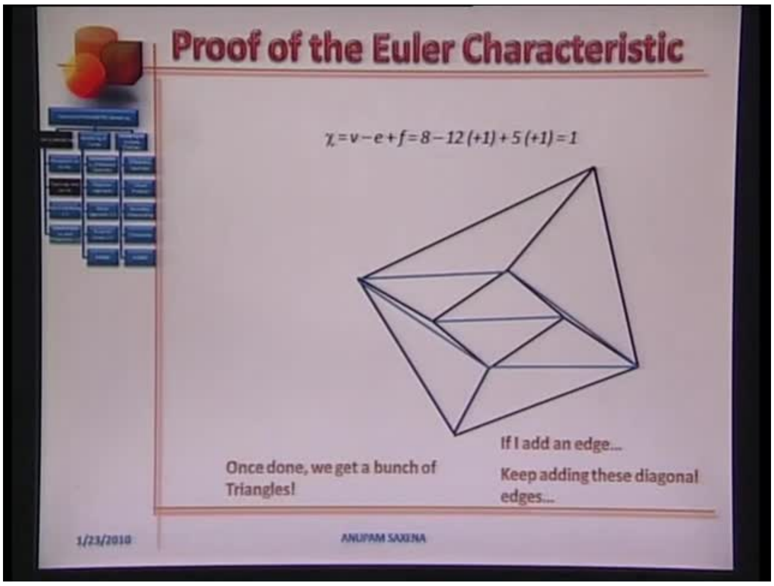

This is Euler characteristics noted before is 1. If I add an edge like this, what I have done is, I have increased e value by 1 and what I have also done is I have divide this face into 2 faces. So, effectively I have increased the f value as well by 1. This is an increase in the number of edges and this is an increase in the number of faces. Even then the chi value does not change. I can keep adding these diagonal edges and will store the values of Euler characteristics still to 1. Let us see, how?

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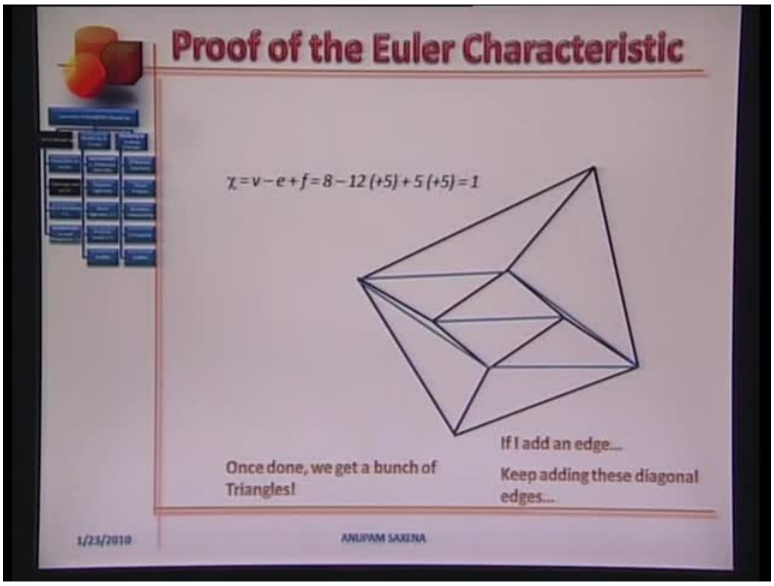

This is another one, leading into new faces. This is the third one, again dividing the face into two parts. This is the fourth one, well what I get here little cubes face well that will work for us. Finally, that divides the bottom face of the block into two faces. You would notice that we have bunch of triangles now in a planargraph. If you are confused, I will repeat what I said once again; you start with the cube of a block, take the top face off. We know that hemispherical surface is homeomorphic to a disc and can be laid flat on a plane.

Now, start adding edges such that, such we start making triangles, each edge that is added would divide an original face into two parts. We do this ensuring that the chi value does not change. I keep adding the edges and keep dividing quadrilateral faces into triangular faces. Note that while doing so, the value of chi of Euler characteristics has always remains 1.

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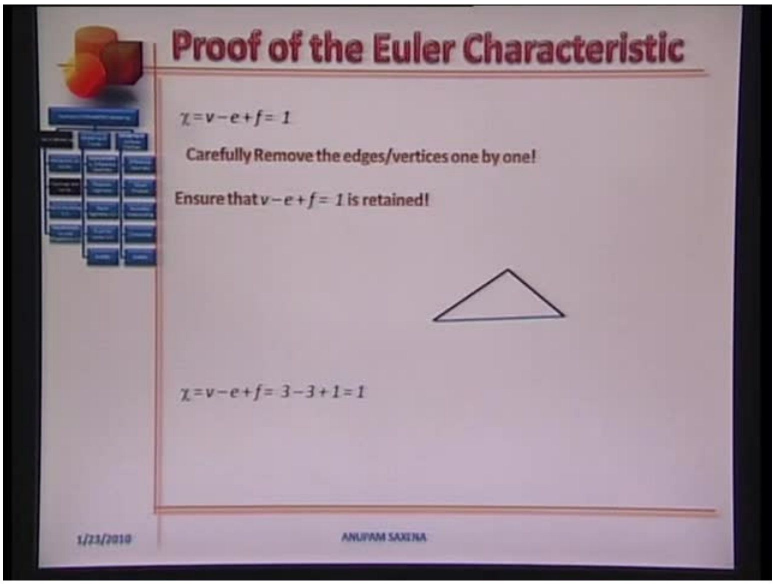

The next step now, we carefully move edges of vertices one by one and while doing so ensure that v minus e plus f equals 1 is retained. Let us see how? If I let go of the edge, what is happened, there is decrement in the e value and the face is also gone. What remains is still a set of triangles. So, this removable of edge has not cause any change in this relation. Likewise, if I remove this edge, once again the chi value remains the same and I still have bunch of triangles.

Now, if I remove this vertex and on this vertex and the two edges together, there is a decrement of 2 in the e value, decrement of 1 in the v value and the decrement of 1 in the f value. Once again chi equals the number of vertices minus the number of edges plus the number of faces, equals 1. I remove this triangle now, again there relation isn't change. I keep on moving the edges and the triangles such that, the relation retained. Please follow this carefully now.

Remove an edge, still retain all triangles to move an edge. Still I have all triangles. Remove a vertex into edges, remove a vertex into edges and remove vertex into edges again. What remains finally, is the triangle for which the number of vertices is 3, the number of edges is 3 and the number of faces is 1, so the chi equal 1 remains. This is QED for a pictorial proof for a Euler characteristics.

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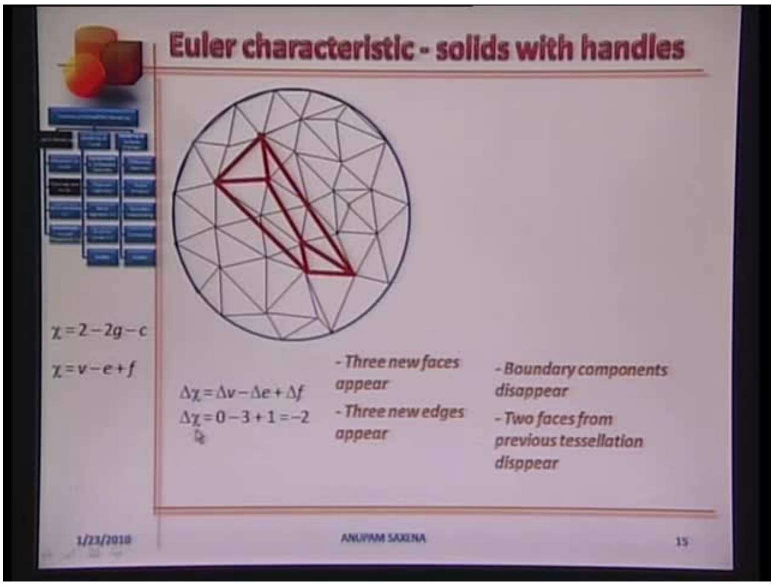

We will now try to relate these two results chi equals 2 minus 2 times the number of genus value minus the number of boundary components. This is true for all non polyhedral solids or smooth solids like a sphere or torus. Chi equals the number of vertices minus the number of edges plus the number of faces, is true for all polyhedron solids. Note that this result will be valid if both polyhedron solids are homeomorphic to their corresponding smooth solids. The proof is again pictorial in nature and slightly in form. I just give you the concept, consider a spherical surfaces a smooth surface for instance, a smooth surface.

Since, this smooth surface is homeomorphic to any of the closed polyhedral surface without any holes, the chi value is 2. The first thing we will want to do is triangulate to the surface of the sphere. To render it the polyhedral approximation, so in a way what we have now is the polyhedral spherical surface. Really, since the two surfaces are homeomorphic each other, the Euler characteristics remains the same, 2. What we do now is we take of any triangle from the sphere or from the surface. What you have done? We have taken away the face and introduced an boundary components.

The number of vertices and the number of edges remains the same. The result the Euler characteristics decreases by 1 to 1. Let us do this again. Let us introduce another boundary component by taking away another triangle here. We have ensure that none of the vertices or none of the edges in the two boundary components are common. Once we introduce another boundary components, once again the number vertices remains the same, the number of edges remains the same, but the number of faces decreases by 1. Now, the chi value Euler characteristics is 0.

We can generalized this result, we can keep on adding this boundary components and we can decrementing the Euler characteristics. In general chi will be 2, which is the characteristic for solids surface without any boundaries minus the number of boundary components, just in case, if we happen to introduce boundary contours by these. Here in this case the c value is 2, therefore chi 0. Let us connect the two boundary components. This is the first edge connecting the two respecting vertices, this is the first edge and this is the second edge, and this is the third edge by connecting these two boundary components by these two edges.

We have introduce through hole in a sphere. When doing so, note that these boundary components are no longer present, they disappear and few new edges appear. Two faces from the previous specialization, they disappear and three new faces appear. So, this is the first face, the rare side is the second face and the righted side face it the third face. Now, let us see how the Euler characteristics changes? The change in the Euler characteristics is the same as the change in the number of vertices minus the change in the number of edges.

Thus the change in the number of faces, there is no change in the number of vertices. Therefore, we have 0 here, the number of edges increased by 3, 1, 2 and 3, and the number of faces has effectively increased by 1. Note that these three edges have generated three faces and from previous operations. These two faces were already gone. So, effectively the increment in the number of faces is 1, therefore the change in the Euler characteristics is minus 2.

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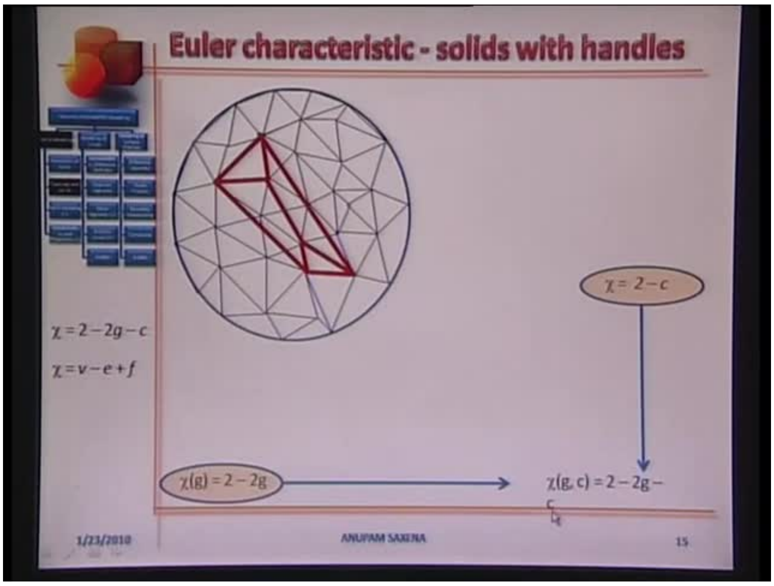

The change that the chi is nothing but the Euler characteristics when there is a through hole is generated. This is also like a handle, if you think about a sphere with a through hole is homeomorphic to the torus. The chi for Euler characteristics for this new solid minus chi for the sphere is equal to minus 2, that we have just observed. This implies that the Euler characteristics for the new solid which is a sphere with a through hole equals 2 minus 2, it is 0.

We can keep performing operations very similar to what we have done here. Keep on introduce new through holes, new handles and we can see the results can be generalized. Each time I generate a hole like, so I Euler characteristics by 2. Effectively for g number of through hole and g number of handles that I have introduce, I would decrement the Euler characteristics by 2 times g. This 2 here is the characteristics for a spherical surface of all polyhedral surfaces, which are homeomorphic to the sphere.

If we combine this result with that we obtained previously, this was when we were introducing the number of boundary components on sphere, we can combine both these together. We can say that the Euler characteristics chi which is functional g and c equals

2 minus 2 times the number of genus values or the number of handles minus the number of a boundary components.

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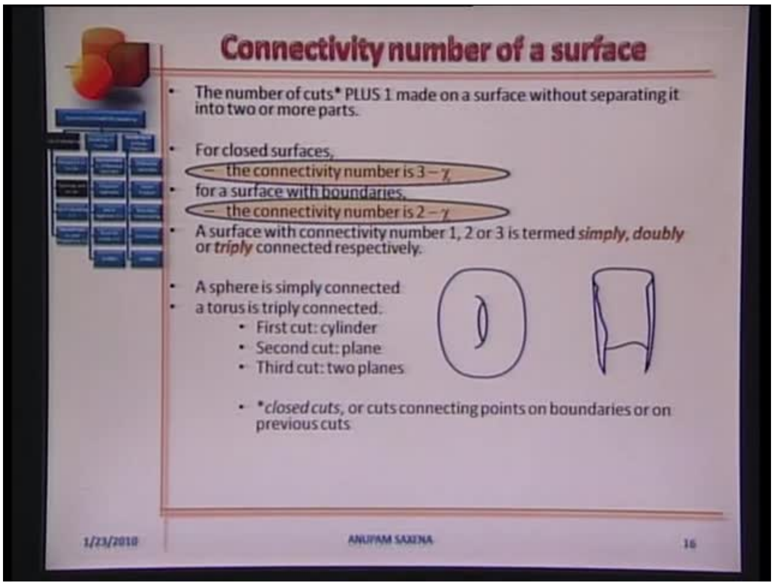

Finally, the connectivity number of a surface, the connectivity number is the number of cuts plus 1 made on a surface without separating it into two or more parts. These cuts here refer the close cuts or a cuts connecting the points on the boundaries or an previous cuts. Let me explain this on the board.

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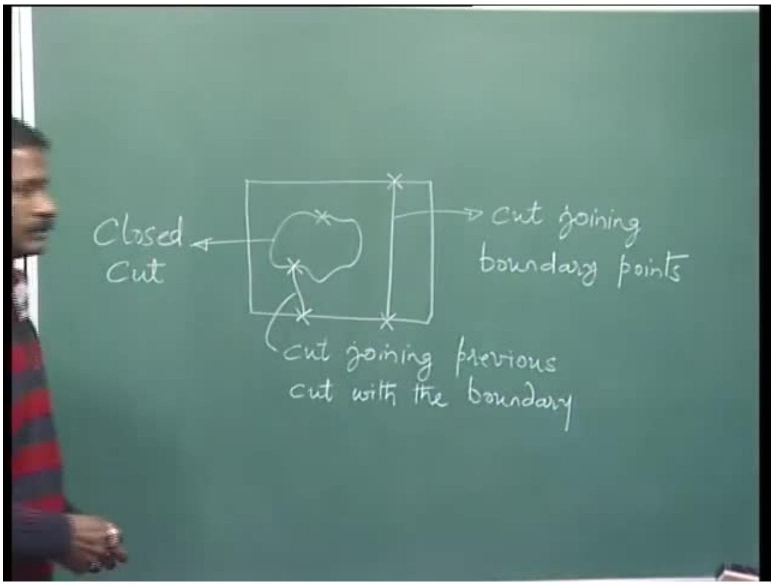

Imagine we have an rectangular piece of paper. This is a closed path, this is the cut that joins any two points in the boundary, and this is a cut that joins a point on the previous point with that on the boundary. For closed surface, the connectivity number is given by 3 minus its Euler characteristics. For a surface with boundaries, like a desk for example, the connectivity number is given by 2 minus the Euler characteristics. Let this be an exercise for you to find, as to how the connectivity number is related to the Euler characteristics? A surface with connectivity number 1 2 a 3 is called simple doubly or triply connected respectively.

A sphere is simply connected, back to the paper weighted example. I can take any point sketch a close path around it and cut the surface along that path. For a single closed path would make me separate this paper weight into two pieces. Hence, this paper weight are equivalently a sphere is simply connected, a torus is triply connected that you have. The first part would give us a cylinder. This is the first closed cut, that gives us an open cylinder.The second cut gives us the plane and the third cut would separate this plane into two parts.

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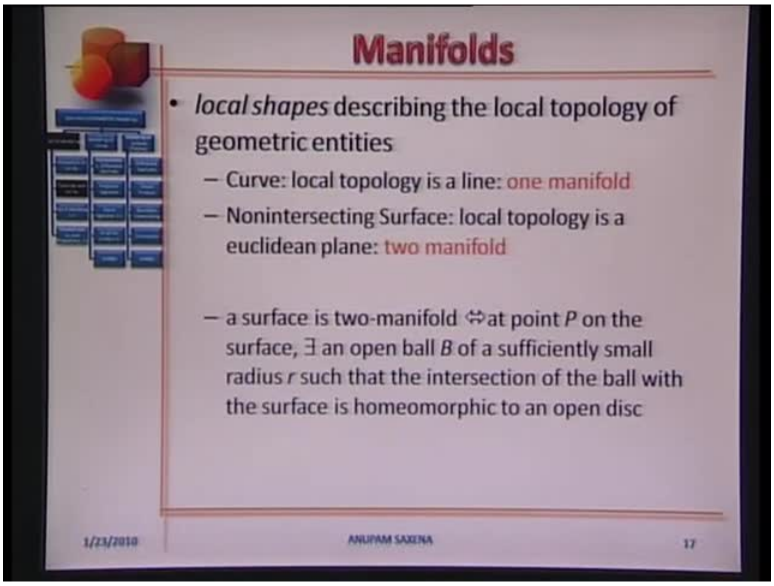

Manifolds, this is the final concept in the lecture. Manifolds refer to local shapes describing the local topology of geometric entities. For a curve a local topology is a line, that is the reason why a curve is called one manifold. For nonintersecting surface the local topology is an Euclidean plane, that is the reason why a surface a two manifolds a circle. A surface is two manifold if and only if at point P on the surface were exists an open ball B of a sufficiently small radius r, such that the intersection of the ball with the surface is homeomorphic to an open disc. Let us try to understand this statement via a few examples. If you remember what an open ball is, it is the collection of all the points within the ball and not on the surface of it.

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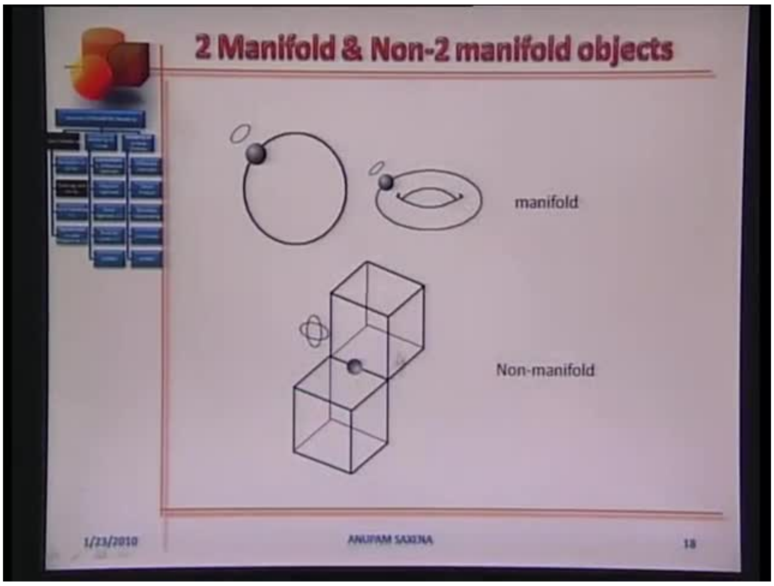

Consider a sphere for example, take an open ball, intersect it with the surface of the sphere and the result that intersection would be an open disc. Likewise, again take an open ball, intersect it with a toroidal surface, and that intersection result will be an open disc. They are therefore, two manifold or manifold solids ensure. Consider this example, this is like two cubes sharing a common edge, with each other. At any other point on this solid, if you take an open ball and intersect with any plane, you will get to see an open disc, but if you place that open ball on this edge, the result of the intersection will be two ended points disc.

So, this vertical disc here is the result of the intersection between the ball on this vertical plane and the horizontal disc here is the result of the intersection of the ball on this horizontal plane. So, the two disc are ended points like common hand looks. I cannot separate the two disc unless I take one of them half. So, here a solid is a non two manifold or ensure a non manifold solid. Of course, if you recall homogenous three dimensionality property of valid solids, this solids defy that property.

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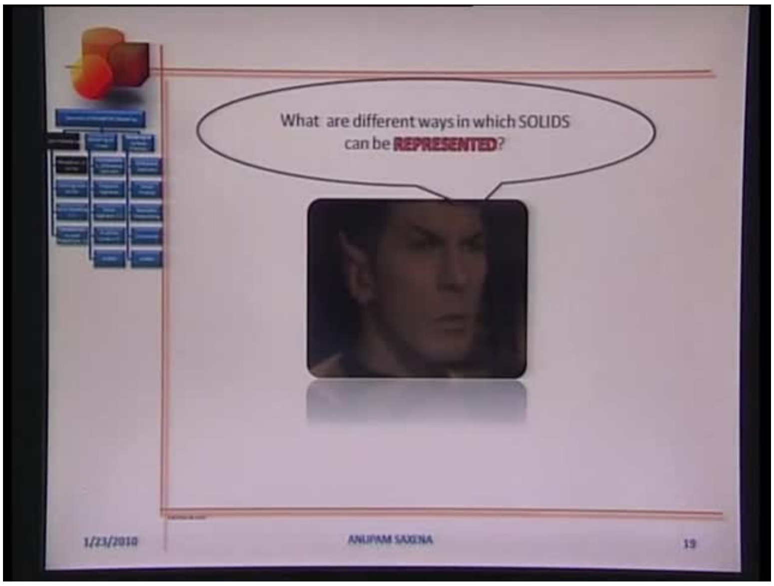

We conclude this lecture by having an implicit mind, as posed this question. What are different ways in which solids can be represented? We cover this in the following lecture.