Computer Aided Engineering Design
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Lecture - 39

Good morning, this is our penultimate lecture on composite surfaces, lecture number thirty nine. Here, I am going to be talking about how different patches can be combined together along common curved boundaries. To exemplify this I will consider only two cases, those of ferguson patches and bezier patches. Using the concepts discussed in this lecture, you can later on extend the techniques for these types of surface patches or for other surface patch models.

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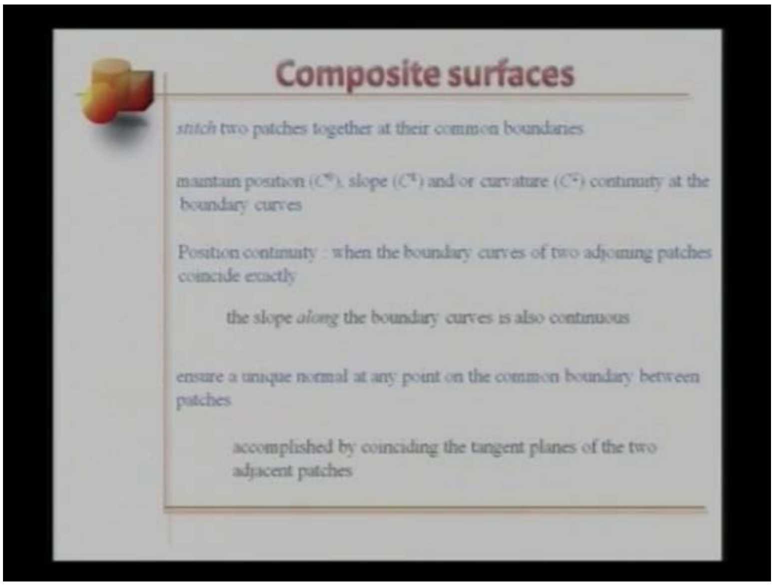

Composite surfaces; this is what the idea is, we need to stitch two patches together at their common boundaries. What does that mean means that we need to maintain position which is c 0, slope which is c 1 and or curvature which is c 2 continuity at the boundary curves. Position continuity implies when the boundary curves of two adjoining patches coincide exactly. This would mean physically, the slope along the boundary curves is also continuous. We also need to ensure a unique normal at any point on the common boundary between two patches. We can accomplish this by coinciding the tangent place of the two adjacent patches.

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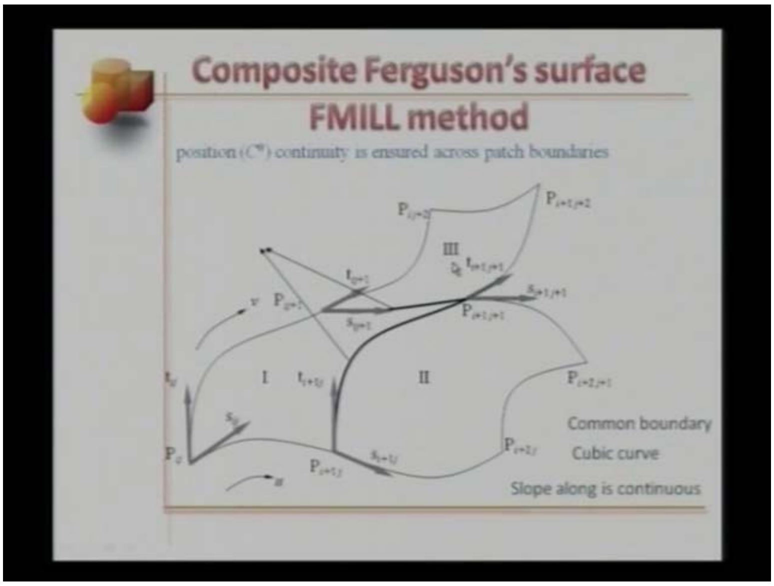

As I mentioned the first model is for the ferguson's surface patch and this is called the FMILL method, what we are going to be discussing now. Well position or c 0 continuity is ensured across patch boundaries. So, we have the two parametric directions here u and v, this is the first patch. These are the tangents along u and v, the tangents along u and v at this corner point likewise tangents along u and v at this point and the same for this point, lets name these tangents; this tangent here is S sub i j this here is t sub i j, this here is S i j plus 1, this direction here is t i j plus 1, this point here is S i plus 1 j plus 1, this is t i plus 1 j plus 1, this one here is S i plus 1 j and this direction is t i plus 1 j.

As you would notice the tangents along the u direction are represented by S and those along the v direction are represented by t. Index i is getting incremented along the u direction and index j gets incremented along the v direction let us have another patch, adjacent to this patch here along the u direction, likewise let us have a patch adjacent to this one here along the v direction. Let us say this is patch one this is patch two and this is patch three, this point here is P i j, this here is P i plus 1 j, this here is P i plus 2 j, this corner point is P i j plus 1, this one is P i plus 1 j plus 1, P i plus 2 j plus 1, P i j plus 2 and this point here is P i plus 1 j plus 2.

What is this curve here and what is this curve here, these are common boundaries. This one is common between patch one and two and this curve here is common between patch one and three. Since, all three patches are ferguson's bi cubic patches, each of these boundary curves are cubic and so the common boundaries are also cubic curves. And so happens that ferguson's models can neatly give us slope continuity along the curves. As you move along this boundary curve, the slope is continuous between one and two and as move along this common curve here, the slope is unique along the u direction between patch one and three.

What do we need to then worry about; we need to worry about the continuity in the cross boundary tangents. In other words if I stand at any point here, I would be able to complete a unique normal which, would represent the normal for patch one and two. Likewise the same here if I stand here I should be able to compute a unique normal which, is representative of patch one and three.

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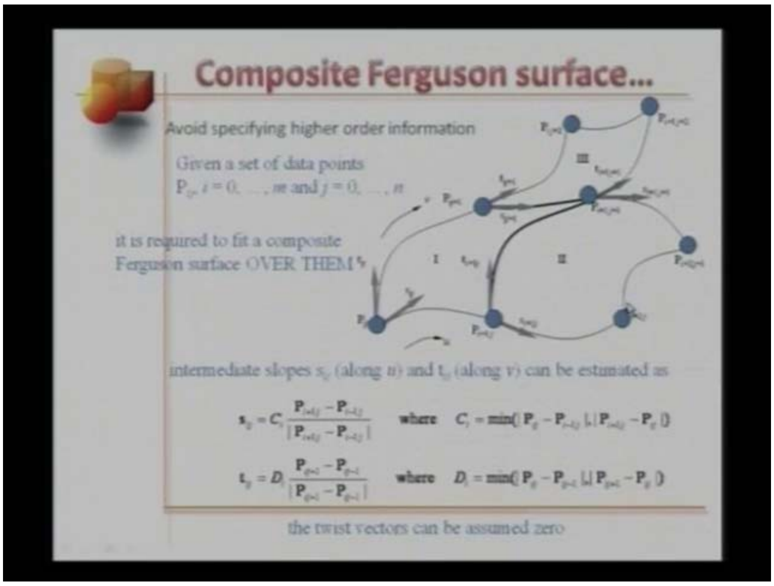

The same diagram here, let us say you want to avoid specifying higher order information. Let us work with the given set of data points P i j where, index i goes from 0 to m and index j goes from 0 to n. So, in a sense all we know are these points, let us see where this takes us. So, these are the points that we know and let us assume that we do not know anything else. The problem is that we require to fit a composite ferguson surface over these points. Since, we do not specify the slopes, as a design choices we need to estimate them.

So, the intermediate slopes s i j along u and t i j along the parameter direction v can be estimated as s i j equals C sub i times P i plus 1 j minus P i minus 1 j over the absolute value of P i plus 1 j minus P i minus 1 j. Note that these points are position vectors, they are ordered sets. They contain information pertaining to all the three Cartesian coordinates, C i is equal to minimum of the absolute values between P i j and P i minus 1 j and Pi plus 1 j minus P i j. So, physically what is happening, let us see P i plus 1 j is this point P i minus 1 j would be a point here somewhere.

So, P i plus 1 j minus P i minus 1 j over its absolute value will be some unit vector and C i would be the minimum of two absolute values P i j minus P i minus 1 j which is this minus some fictitious point here, P i minus 1 j and P i plus 1 j which is this point, minus P i j which is this point. So, in a sense we are considering three consecutive points along parameter direction u, to compute intermediate slope s i j. Let me get back not to compute, but to estimate, likewise the slope t i j can be given as D sub i times P i j plus 1 minus P i j minus 1 over the absolute value of the numerator.

So, this here is a unit vector, D i is minimum of these two distances, P i j minus P i j minus 1, this minus some point here P i j minus 1 P i j plus 1 which is this point, minus P i j which is this point, to compute or to estimate P i j, we are considering three consecutive data points along the parameter v direction like these. It would be easier for you to understand if we look at the estimation of t i j plus 1, there will be clear that we are considering P i j P i j plus 1 and P i j plus 2.

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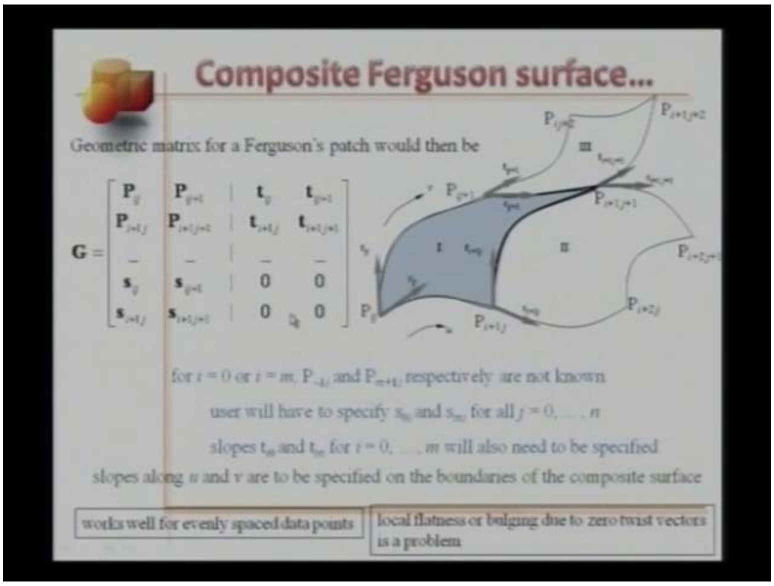

So, much for the slope information if patches one two and three happen to be ferguson patches recall from our previous lectures that we would need, the slopes along the u direction those along the v direction and also the crest vectors or the mixed derivatives of r at these points. For now we can assume that these crest vectors are 0.

So, a geometric matrix for a ferguson's patch, will then look like G equals the corner points in the top left 2 by 2 region P i j P i j plus 1 P i plus 1 j P i plus 1 j plus 1. Here we will have slopes pertaining to the v direction t i j t i j plus 1 t i plus 1 j and t i plus 1 j plus 1. In the lower left region we will have slopes along the u direction in the same order s i j s i j plus 1 s i plus 1 j and this i plus 1 j plus 1. This region pertains to the information on twist vectors for now we have assumed them to be 0. So this geometric matrix G would correspond to the first patch. Now, we have a slight problem for i equals 0 or i equals m, the points P minus 1 j and P n plus 1 j are not known.

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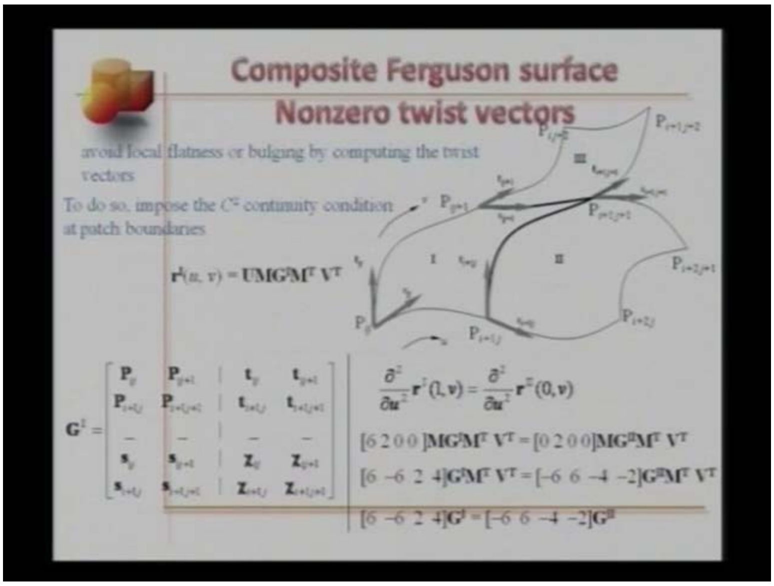

To counter this, the user now will have to specify the slopes s 0 j and s m j for all values of j from 0 to n. That is if you look at this figure if I move along j which, is along the v direction here, for j equals 0 which would be somewhere here let us say. We will have to specify s 0 j and for j equals m somewhere here, we will have to specify s m j, likewise slopes t sub i 0 and t sub i n for i going from 0 to m, we will also need to be specify. In summary slopes along the u and v parametric directions are to be specified on the boundaries of the composite surface. Now, this model works well for evenly spaced data points, but local flatness or bulging due to 0 twist vectors. So, these here happen to be a problem.

Let us now look at composite ferguson surface with non 0 twist vectors. So, we are working with the same figure here, we can avoid local flatness or bulging by computing the twist vectors as opposed to be assuming them to be 0. Well to do so we can impose the C 2 continuity condition at patch boundaries. Now, you have seen this compact form of ferguson's cubic patch before r super 1 which, is the equation for this patch in terms of u and v is equal to the row matrix U times the ferguson's coefficient matrix, the geometric matrix the transpose of M the transpose of V, u and v contain 1 linear quadratic and cubic terms in parameters u and v respectively.

The geometric matrix G super 1 for this patch as you know is P i j P i j plus 1 P i plus 1 j P i plus 1 j plus 1, the slopes along the v direction t i j t i j plus 1 t i plus 1 j t i plus 1 j plus 1, the slopes along the u direction in the same order and now the non zero crest vectors represented by kai. So, we have kai i j kai i j plus 1 which is here, kai i plus 1 j which is at his point here and the crest vector kai i plus 1 j plus 1 at this point.

To impose C 2 continuity condition we are saying that the second derivative of this patch, with respect to u evaluated at u equals 1 for all v which is here, if you recall the parameter u was from 0 to 1 here and v goes from 0 to 1 here. So, for this common boundary u equals 1 and v is any value between 0 and 1. This should be equal to the second derivative of this patch r super 2 with respect to u and for this patch for this boundary curve u equals 0. Where, we have expression right here for patch one we use the geometric matrix G 1, for patch two we use G super 2. Let me scale all the map and give you a set of final results.

Second derivative of r with respect to u, this would involve differentiating this row vector twice. We will get 6200 times M G super 1 M transpose V transpose and of course, this valuated at u equals 1. The second derivative of row vector u with respect to u evaluated at u equals 0 will give 0200 times M G super 2 the geometric matrix for this patch times M transpose times V transpose. We can post multiply M with this to get 6 minus 6 2 4 times G super 1 M transpose V transpose is equal to minus 6 6 minus 4 minus 2 times G super 2 M transpose V transpose. To get 6 minus 6 2 4 times G super 1 for this matrix equals minus 6 6 minus 4 minus 2 times G super 2 the geometric matrix for this patch.

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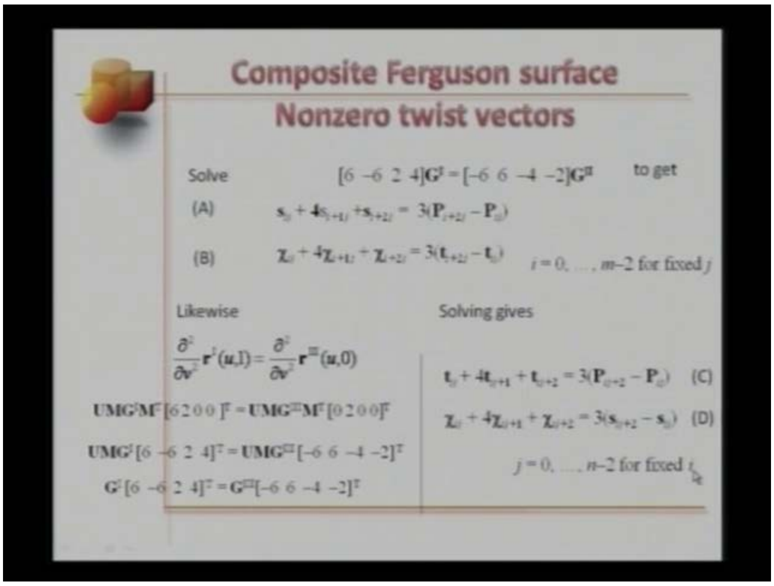

Now, solve this previous expression that may repeat 6 minus 6 2 4 times G super 1 equals minus 6 6 minus 4 minus 2 times G super 2. To get this equation in terms of the slopes along the u direction s i j plus 4 times s i plus 1 j plus s i plus 2 j equals 3 times P i plus 2 j minus P i j. Now, if you notice a very similar equation was encountered when discussing composite ferguson curves with C 2 continuity. We will have another equation coming out from this relation and that would be in terms of the crest vectors kai i j plus 4 kai i plus 1 j plus kai i plus 2 j equals 3 t i plus 2 j minus t i j, the right hand side corresponds to the slopes along the v direction.

Let us name these equations A and B and these equations are for index i going from 0 to m minus 2 for a fixed index j. Likewise if we compute the second derivative for patch 1 with respect to v and evaluate it for any u and v equals 1, using C 2 continuity condition, you have to equate it with the second derivative of the third patch which is raised over r 1 as you know from the figure. Second derivative of r 3 with respect to v for any value of u and v equals 0. You can do the math.

We have U M G 1 M transpose 6200 transpose, how do we get this, we need to differentiate now with column vector v or capital V with respect to this parameter v and evaluate that at v equals 1. Likewise you do the same for v equals 0 to get 0200 here. So, this expression here will be U M G 3 M transpose 0200 transpose. Let us skip the algebra and get to the final result. The geometric matrix for the first patch times 6 minus 6 2 4 transpose is equal to the geometric matrix for the third patch times minus 6 6 minus 4 minus 2 transpose. If we simplify we get another set of equations.

Now, in terms of the tangents along the v direction, t i j plus 4 t i j plus 1 plus t i j plus 2 equals 3 times P i j plus 2 minus P i j and this one here in terms of the twisted vectors kai i j plus 4 kai i j plus 1 plus kai i j plus 2 equals 3 times s i j plus 2 minus s i j. Now, these equations are for index j going from 0 to n minus 2 for a fixed index I, let us name them equations C and D. In summary we had equation A equation B for varying index i for fixed j and then we have equation C and equation D for varying j and fixed i.

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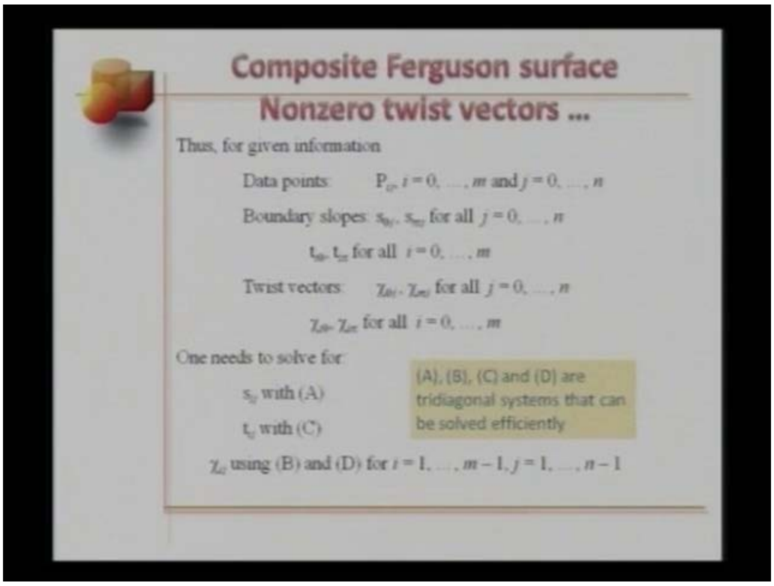

So, for given information which, is in terms of data points P i j i going from 0 to m and j going from 0 to n, with boundary slopes s 0 j s n j for all j going from 0 to n and t i 0 t i n for all i from 0 to m and twist vectors kai 0 j kai m j for all j from 0 to n and kai i 0 up to kai i n for all i from 0 to m, this getting a little complicated. One needs to solve for s i j with equation A from the previous slide t i j with equation C from that slide and kai i j using equations B and D for i equals 1 up till m minus 1 and j equals 1 up till n minus 1.

So, for the composite surfaces patch with non 0 twist vectors we will summarize again, we will need data points P i j, we will need boundary slopes s 0 j and s m j for all j's. We will also need the slopes along the v direction t i 0 and t i m for all i's from 0 to m and also we will need the twist vectors at the boundaries kai 0 j kai m j for all j's and kai i 0 kai i n for all i's. Once we have all this input we would be able to solve for all the intermediate slopes along the u direction along the v direction and all the intermediate twist vectors. You will need to spend some time all by yourself to understand this. A little note equations A B C and D are all tri-diagonal systems, that can be solved easily and efficiently using thermoses.

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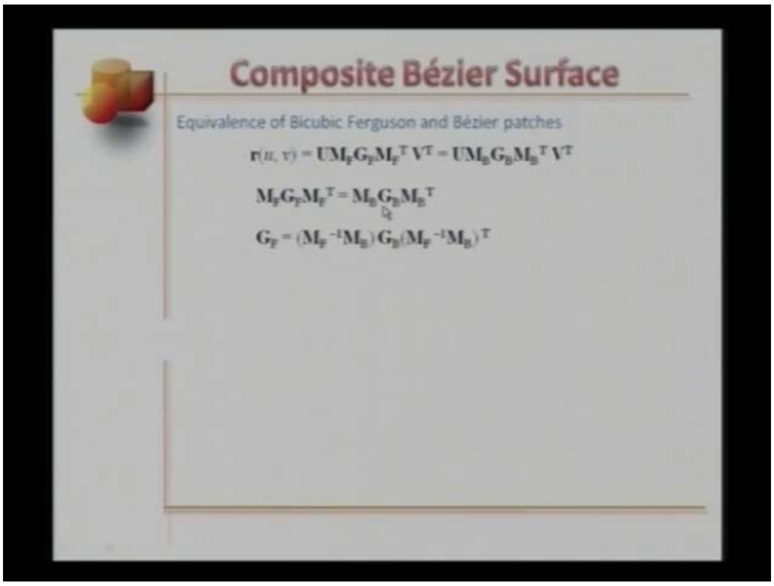

Let us now come to the second model, composite bezier surface, but before that let us establish equivalence between bicubic ferguson and bezier patches. So, this is the ferguson's patch here, r as function of u and v equals U M sub f the ferguson's coefficient matrix G sub F the ferguson's geometric matrix M F transpose V transpose is equal to U M B this is the bezier coefficient matrix G B the bezier geometric matrix M B transpose V transpose. So, if we let go of row vector here U and if we also let go of V transpose here, we will see that M F G F M G transpose is equal to M B G B M B transpose. So, this part here is equal to this part here.

Now, given information pertaining to bezier matrix we can get the corresponding geometric matrix for ferguson patch as M F inverse M B G B M F inverse M B transpose. Post multiply this relation by M F transpose inverse and pre multiply the same by M F inverse. Likewise, if you are given ferguson's patch you can determine the corresponding geometric matrix of bezier patch.

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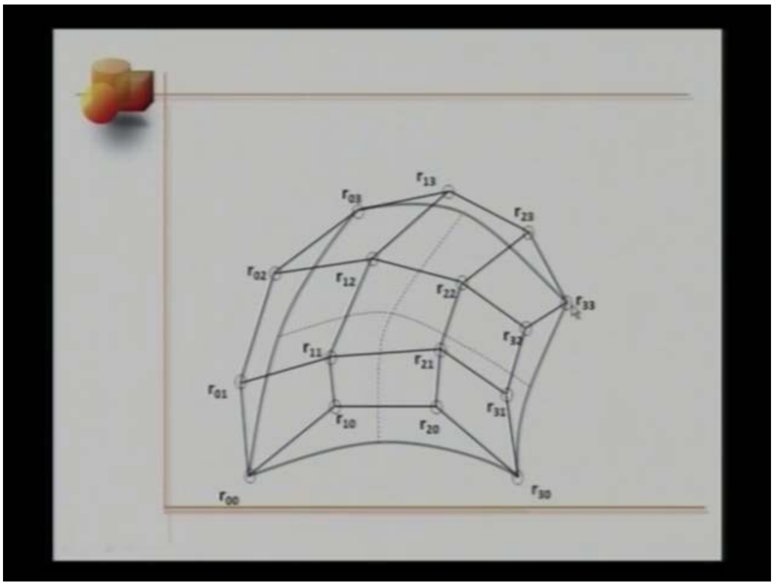

Let us revisit bicubic bezier patch with control points horizon points, r 0 0, r 1 0, r 2 0, r 3 0, r 0 1, r 1 1, r 2 1, r 3 1, r 0 2, r 1 2, r 2 2, r 3 2, and r 0 3, r 1 3, r 2 3, and r 3 3.

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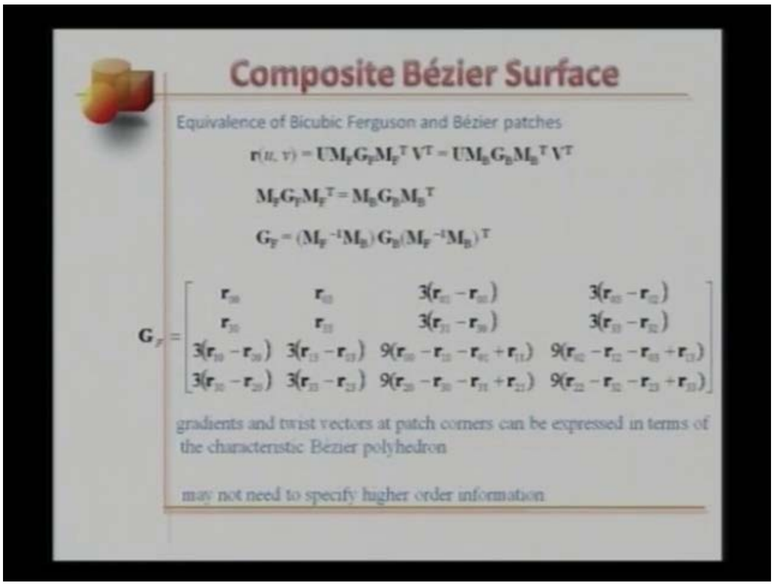

So, in terms of the design points that we have seen pertaining to G sub B, we can get the geometric matrix for the corresponding bicubic ferguson's patch. First row, is r 0 0 r 0 3, 3 times r 0 1 minus r 0 0, 3 times r 0 3 minus r 0 2. Second row, r 3 0 r 3 3, 3 times r 3 1 minus r 3 0, 3 times r 3 3 minus r 3 2, 3 times r 1 0 minus r 0 0, 3 times r 1 3 minus r 0 3, 9 times r 0 0 minus r 1 0 minus r 0 1 plus r 1 1, 9 times r 0 2 minus r 1 2 minus r 0 3 plus r 1 3. And the final row is 3 r 3 0 minus r 2 0 3 r 3 3 minus r 2 3 9 r 2 0 minus r 3 0 minus r 3 1 plus r 2 1 9 r 2 2 minus r 3 2 minus r 2 3 plus r 3 3, so in a sense gradients and twist vectors.

This region here as you know pertains to the slope along the v direction the slopes along the u direction and of course, the twist vectors here. So, the gradients and twist vectors at patch corners four of them can all be expressed in terms of the characteristic bezier polyhedron. That gives us one important hint as a design; we may after all not need to specify higher order information. In fact we do not need to go through unnecessary and intricate complications that we had seen in case of composite ferguson patches.

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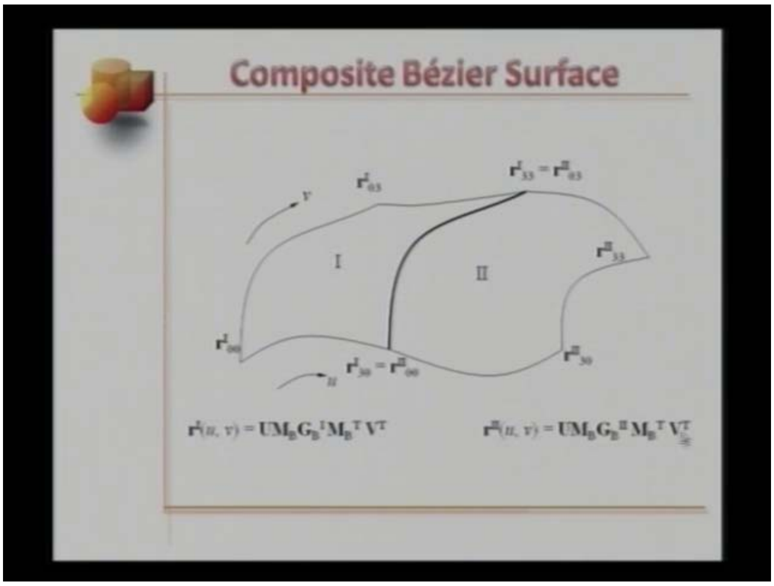

Let us continue with composite bezier surfaces. So, this here is the first patch the corner point is r sub 0 0 of patch 1, r sub 0 3 of the first patch, r sub 3 0 of patch 1, and we need to maintain position continuity here, point here will also be the 0 0 eth point of the second patch. Likewise the point here pertains to r 3 3 of the first patch this 1, which is the same as r 0 3 of the second patch, which is this one. As I said before these two points are the same to maintain position continuity at these two junction points, we will talk about this common boundary in a while. For the second patch this corner point here is r 3

0 and this point here is r sub 3 3.

Patch 1 patch 2 this corresponds to the u direction, this point here corresponds to the v direction. From what we know from before on bicubic bezier surface patches first patch r super 1 in u and v can be written as U M B G B super 1 M B transpose V transpose. Likewise the second patch are 2 u v and given by U M B G B super 2 M B transpose V transpose. Now let us look at positional continental.

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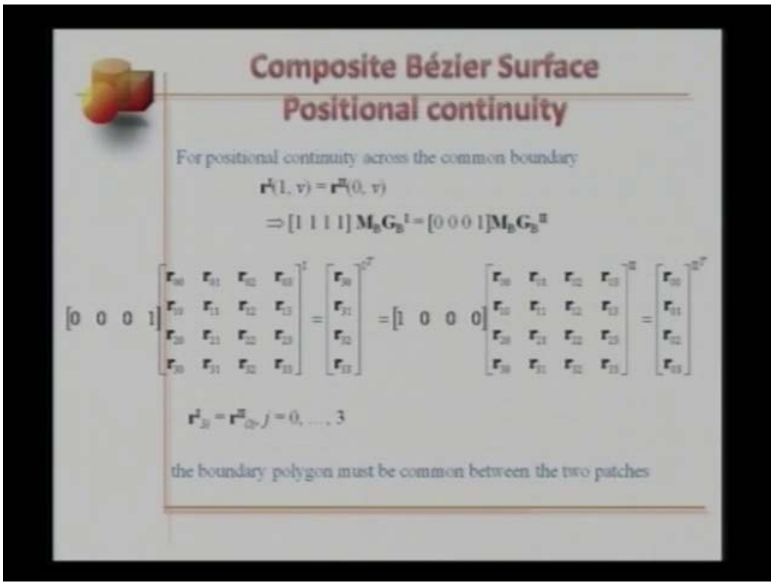

For that to the common boundary r 1 for u equals 1 and l v should be equal to r to u equals to 0 and any v this would mean that row vector 1 1 1 1 times M sub B for B is efficiency matrix times the geometric matrix for the first patch should be equal to 0 0 0 1 times M sub B times the geometric matrix for the second patch. Let me patch this realation for u here G B super 1 is this matric here and G B super 2 is this matrix here. Now, this row vector times M sub B is equal to 0 0 0 1 times G B over 1 which is this and this is equal to r 3 0 r 3 1 r 3 2 r 3 3 for the first patch a transpose of patch.

And this row vector times M sub B equals 1 0 0 0 times the geometric matrix for the second patch to invite matrix here which, is equal to r 0 0 r 0 1 r 0 2 and r 0 3 of the second patch the transpose. We can summarizer these results and say that r 3 j for the first patch equals r 0 j for the second patch for the values are going on 0 to 3. Physically that would mean that the boundary polygon must be come between two patches.

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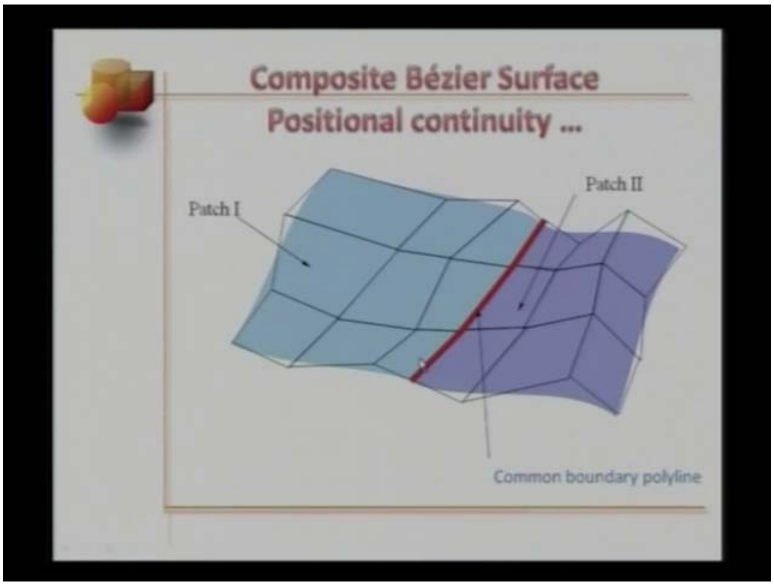

Here is the figure this is the control net for the first i cubic dispatch and this here is the control net for the second beziers patch; patch 1, patch 2. All the condition that we have seen in this slide says that the boundary polygon between these two nets should be the same, this here is a poly line.

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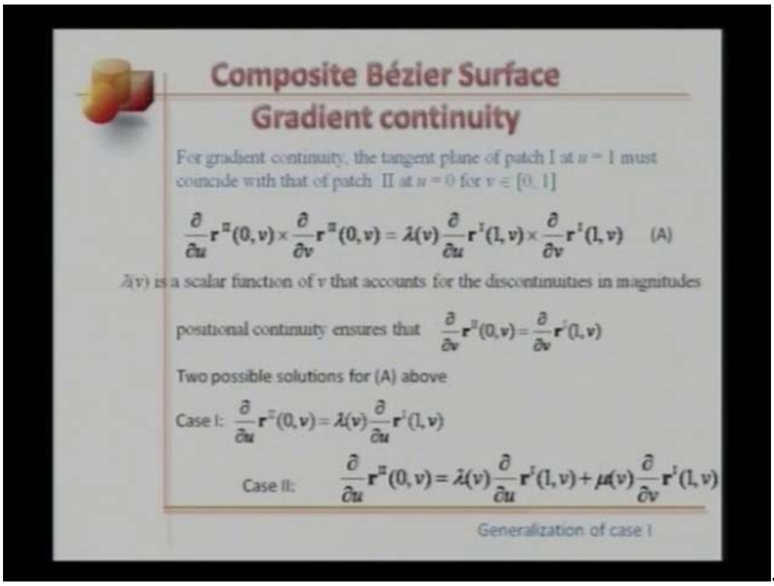

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Let us move forward an gradient continuity across the patch boundary. For gradient continuity the tangent plane of patch 1 at u equals 1 must coincide with the tangent plane of the second patch at u equals 0, for all values of v into 0 and 1. So, let us compute the tangent along the u direction for the second patch at u equals 0 and the tangents of the second patch along the v direction again at u equals 0 and let us compute cross product between these two tangents. So, the left hand side here will represent the normal of the tangent plane, likewise this expression here corresponds the tangent along the u direction for the first patch at u equals 1, this term here is the tangent for r super 1 along the v direction for u equals 1.

So, this cross product here is again a normal to this tangent plane. For the two tangent planes to coincide these normals must be scalar multiples of each other, that is for different values of v in between 0 and 1 and because of that, we introduce a scalar lambda which is a function of v. So, what we are saying is the directions of the 2 normals should be the same, but the magnitude may different. Positional continuity further ensures that the slope along the v direction of the second patch, at u equals 0 is the same as the slope along the v direction for the first patch at u equals 1.

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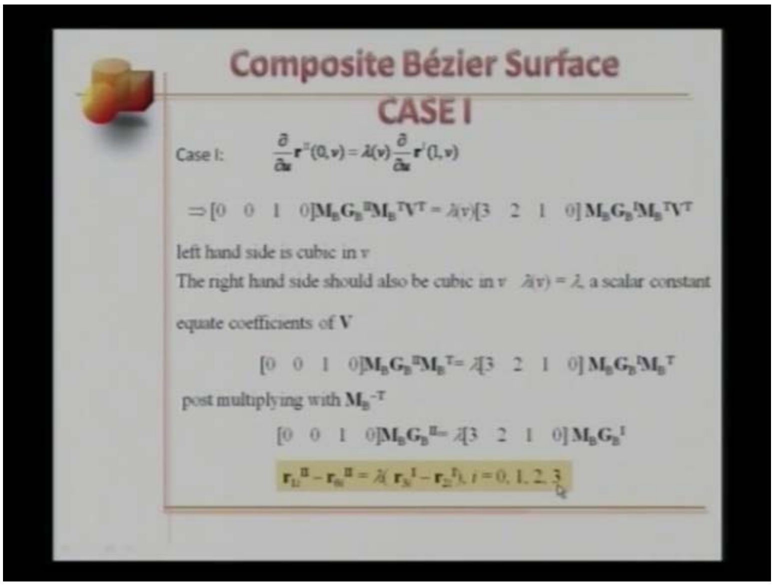

So, we have two possible solutions, this equation is equation A case one. The first derivative for the second patch with respect to u at u equals 0 is equal to lambda of v times. The first derivative of patch 1 with respect to u for u equals 1. And case two the first derivative of the second patch with respect to u at u equals 0 is a linear combination of the first derivative of patch 1 with respect to u and the first derivative of patch 1 with respect to v. The two scalars we use are lambda of v and mu of v. If you plug in this equation here you will see that you get the right hand side of course, these 2 is the generalization of case one.

Let us look at case one in detail, partial r 2 over partial u at u equals 0 and v equals lambda of v times partial r 1 over partial u at u equals 1 and v. This implies 0 0 1 0 M sub B G sub B of the second patch, geometric matrix times the bezier, coefficient matrix transpose V transpose equals lambda v times this row vector, here with elements 3 2 1 0 M sub B G B 1 M B transpose V transpose. And if you notice the left hand side is cubic in v, v transpose here comprises of terms 1 v v squared and v cube. For this equation to hold good the right hand side should also be cubic in parameter v.

We already have cubic term in V transpose here, for that to happen lambda of v should not be a varying function, but should be a scalar constant lambda. We can equate coefficients of V to get 0 0 1 0 M sub B G B of the second patch, M B transpose equals lambda 3 2 1 0 M sub B G B of the first patch M B transpose and if we further post multiply this equation by M B minus transpose. We get 0 0 1 0 M B G B 2 equals lambda times 3 2 1 0 times M B G B 1. Which, converts to r 1 i of the second patch minus r 0 i of the second patch equals lambda times r 3 i of the first patch minus r 2 i of the first patch, index i goes from 0 to 3.

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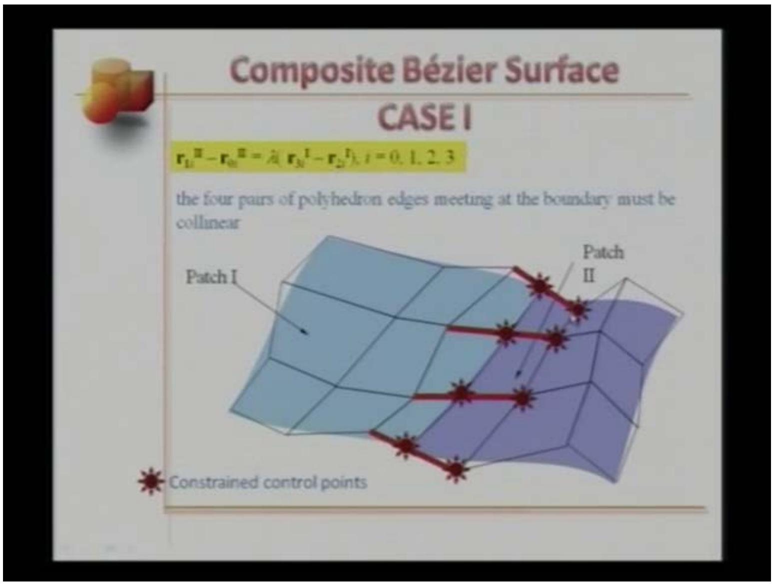

This is the condition from the previous slide geometrically, what this condition means is that the four pairs of polyhedron edges meeting at the boundary must be collinear. So, this is the first bicubic patch, this is the second bezier bicubic patch, these four lines must be collinear. From position continuity we had already seen that this was the common poly line between these two patches. So, if you notice on the design view point these eight points are constraint, these four points get constraint because of position continuity and these four points get constraint to lie along these respective lines, to maintain a unique tangent plane and this common boundary. In other words these are not so to speak free choices anymore words.

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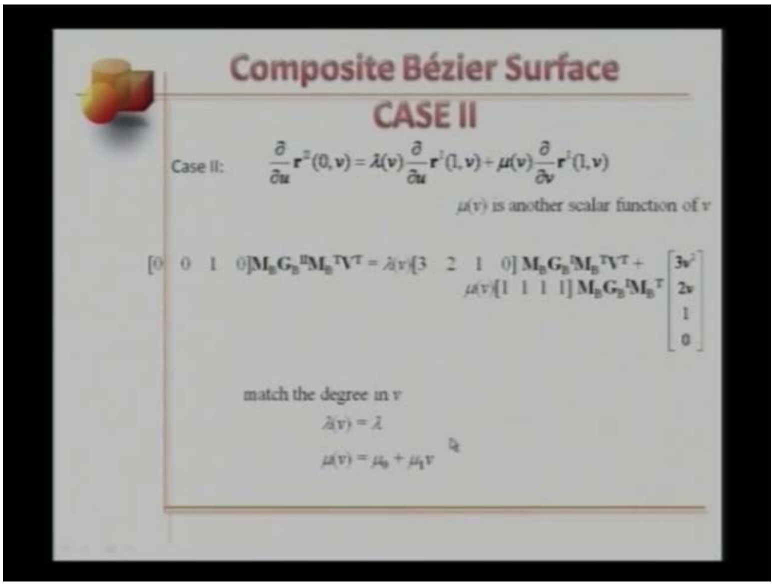

Let us look at case two now, partial of r of the second patch with respect to u at u equals 0 and v equals lambda of v times partial of the first patch, with respect to u at u equals 1 for any v plus any scalar as a function of v and partial of r 1, with respect to v at u equals 1 and any v, mu of v is another scalar function of v. This condition gives us 0 0 1 0 M B G B of the second patch, M B transpose V transpose equals lambda V 3 2 1 0 M B the geometric bezier matrix of first patch, M B transpose V transpose plus mu of V 1 1 1 1 M B G B of the first patch, M B transpose times, this column vector here 3 v square D 2 v 1 0, pertaining to partial over partial v of V transpose.

In this expression we must match the degree in v which, will mean that the scalar lambda is no longer the function of parameter v it is a constant, but we can have mu of v as a linear function in parameter v mu 0 and mu 1 are scalars, constant scalars.

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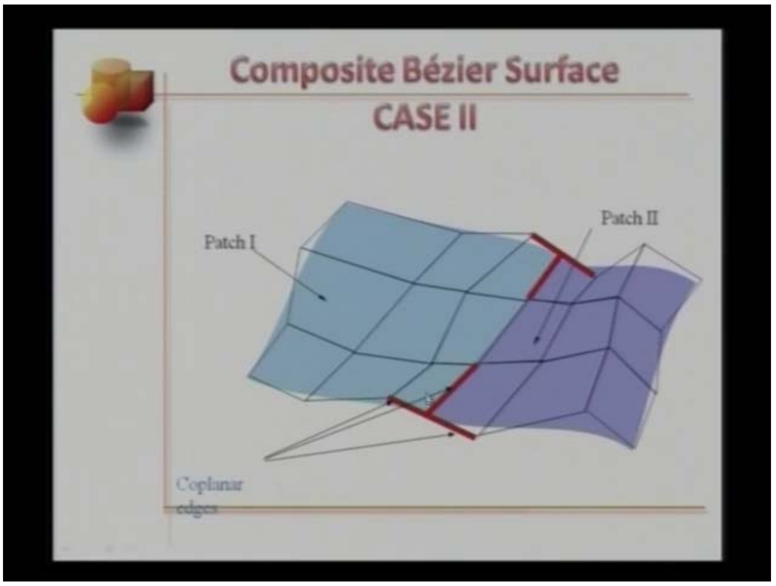

If we work out this condition and we try to figure what that condition means geometrically. This is the first patch, this is the second patch, the condition gives a more relaxed situation in terms of how a user can specify data points freely. Second case states that these three edges must be coplanar; likewise these three edges also need to be coplanar. So, what do we have, we have a common boundary line here that gives us position continuity between these two patches and then we have this point to be specified such that these four points lie on the same plane and likewise for this patch we have to specify this design point such that one two three and fourth point lie again on the same plane, these two planes can be different.

In this lecture I have covered quite a bit of material on composite ferguson and bezier surface patches. I would suggest that when you are watching this video, you go through each and every step slowly and try to work out the equations all by yourself on a piece of paper. Although I have considered only ferguson and bezier models, similar concepts can be used, can be extended to piece plane surface patches and composite piece plane surfaces. This is only to give you an idea as to how composite surface patches, composite surfaces get modeled in CAD.