Computer Aided Engineering Design
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Lecture - 38
Boundary Interpolating Patches

Good morning, today we move further. In lecture number 38, we discuss a new model of surface patches namely Boundary Interpolating Patches.

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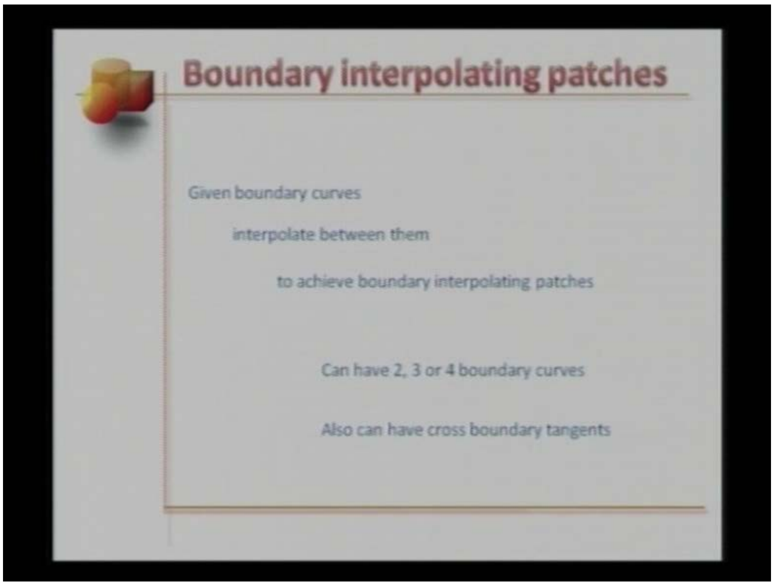

Here, you are going to be working with only boundary curves. So, given boundary curves, interpolate between them to achieve boundary interpolating patches this is what, the idea is. One can have 2, 3 or 4 boundary curves, as you would have observed in case of Bezier patches of these pan patches, the boundary curves in those cases or Bezier or Bezie plane curves themselves. So, in a way, the tenser product patches and the boundary interpolating patches are interrelated. One can also have slope information and the boundary curves or what we call, cross boundary tangent, I tell you in a little while, what I mean by them.

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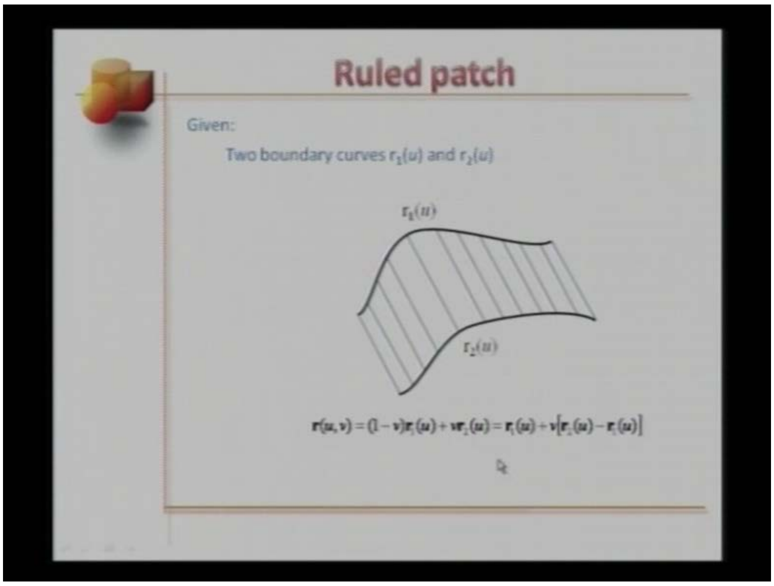

The first model ruled patch so, given two boundary curves r 1 of u and r 2 of u, this is the first curve, this is the second curve, r 1 of u and r 2 of u. One can perform linear interpolation between two corresponding points on the respective curves like so, it is like sketching or drawing different straight lines, joining two corresponding points of these two curves. Mathematically, a ruled patch can be expressed as, r as a function of u and v equals 1 minus v times r 1 of u plus v times r 2 of u.

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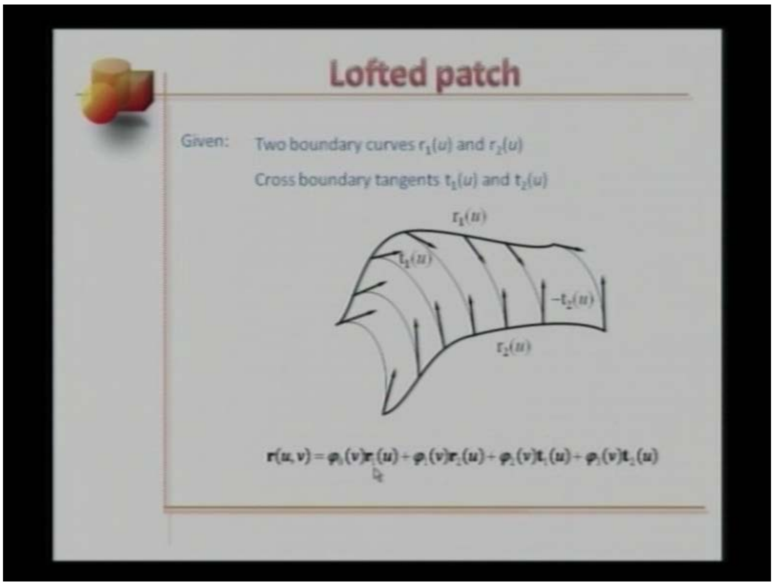

So, these are the u parametric representations, these are the v parametric representations so, you can notice, it is a linear interpolation model. We can rewrite these expressions as, r 1 of u plus v times r 2 of u minus r 1 of u, this vector here gives some sense of the tangent information between the two curve models r 2 of u minus r 1 u. The second model lofted patch, given, again two boundary curves r 1 of u and r 2 of u and also given additional information but any two cross boundary tangents t 1 of u and t 2 u.v. For example, these are the two curves r 1 of u and r 2 of u, and we have these as cross boundary tangents, denoted by t 1 u and t 2 of u, the directions shown here are opposite.

What we can do is, for any value of u, we can choose a corresponding point on this curve a corresponding point on this curve and loft a curve using these two corresponding tangents like so. And that is the basic difference between a ruled patch model and a lofted patch model, in a ruled patch model, we simply joint the corresponding points by a straight line but here, we use a higher order curve in this case, cubic to create the lofted effect.

Mathematically, a lofted patch can be written as, r of u v equals pi 0 of v r 1 u plus pi 1 of v r 2 u plus pi 2 of v t 1 u plus pi 3 of v t 2 u. r 1, r 2, t 1 and t 2 are given to us, any guess, as to what pi 0, pi 1, pi 2 and pi 3 all as functions of v can be. You are right, these are the Hermite polynomials, the Hermite cubic polynomial, that we had seen in case of (( )).

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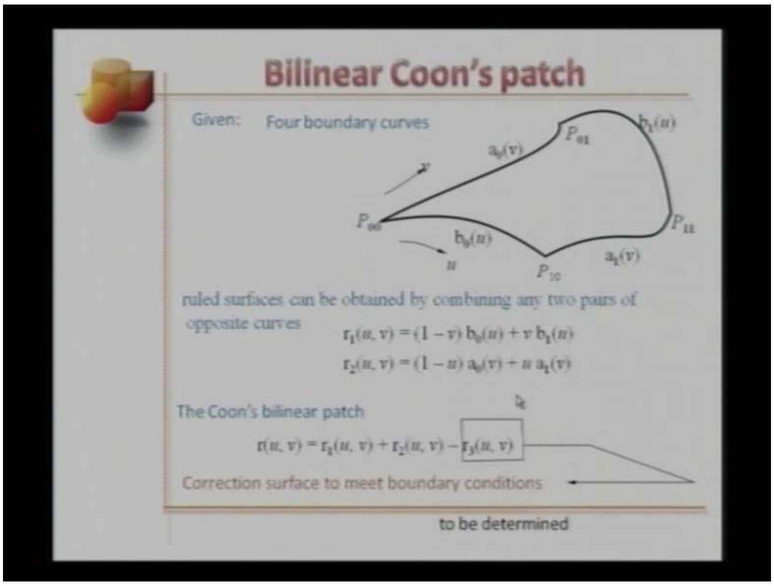

The next model bilinear coons patch, given, four boundary curves, in all the previous model, we had considered two boundary curves and in the lofted patch, we had also considered cross boundary tangents at those two curves. But here, we work with four boundary curves of those, this is the first one, this is the second one, this boundary curve here is denoted by b sub 0 function of u, b sub 1 function of u.

We have two more, these are denoted by a sub 0 as v and a sub 1 as a function of v, this is the u parametric direction and that is the v parametric direction. There will be four points of intersection, the first one is represented as P 0 0, this was first found for values of u and v, both are 0. This is P 1 0, for u equals 1 v equal 0, P 1 1, for u and v equals 1 and P 0 1, for u equals 0 and v equals 1. Here, we do not have any additional information so that, probably, using ruled surface models, we can obtain them by combining any two pairs of opposite curves.

For example, we can combine these two curves or combine; these two curve let us say, the first ruled patch model r 1 of u v is given by a linear combination of b 0 (u) and v 1 (u), this model and this curve model 1 minus v times v 0 u plus v times b 1 of u. Likewise, you can have another rule patch, r 2 u v equals 1 minus u times a 0 v plus u times a 1 v.

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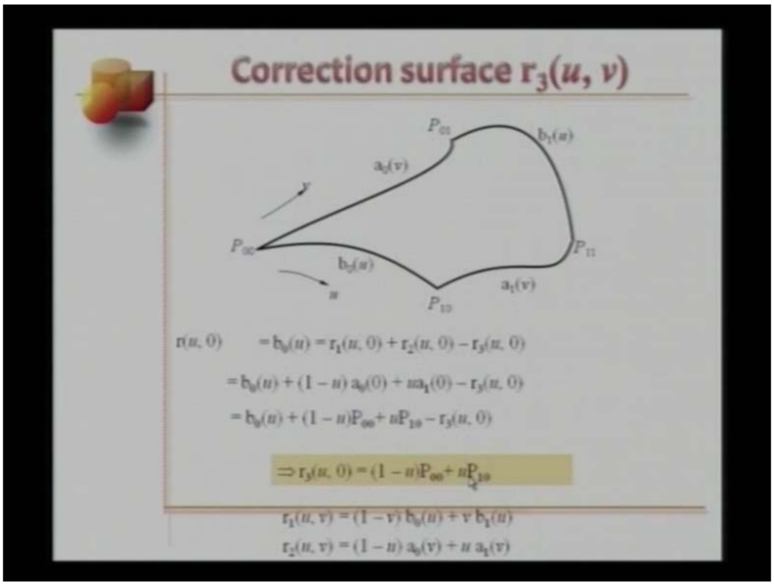

Once again, we have a rule patch by combining these two models and we have another rule patch r 2, by combining these two opposite curves. The coon’s bilinear patch can be obtained by adding r 1 and r 2, r u v equals r 1 u v plus r 2 u v or wait a minute, why are we subtracting r 3 u v. We do not know anything about it, we will probably have to make this adjustment to make sure, that all the boundary conditions are met. So, r 3 of u and v is like a correction surface, to ensure the boundary conditions are properly addressed, we need to determine r 3 and we will do this next.

So, the correction surface, how do we determine this so, this is the previous picture of bilinear coon's patch, I have copied for expressions for r 1 here and r 2 here. Now, let us take a look at some of the boundary conditions to determine r 3, remember that, r is r 1 plus r 2 minus r 3. Now, r u 0 is 1, corresponding to values of v equal 0, r u 0 is the curve v 0 u, which is equal to r 1 u 0 plus r 2 u 0 minus r 3 u 0, from the coon's patch model itself.

Now, look at, what r 1 u 0 can be, we should plug in v equals 0 here, this expression will be 0 and this will simply be b sub 0 of u. So, r 1 u 0 is b 0 of u plus 1 minus u, a 0 0 plus u a 1 0 these two terms come from here, if you plug in v equals 0, we have 1 minus u a 0 0 plus u a 1 0 minus r 3 of u 1 0. If u look at a 0 of 0, we are here on this curve and for a value v equal 0, we are here on this point P sub 0 0, which is what this is.

Likewise, if you look at a 1 of 0, we are on this curve and for v equals 0, we are on this point P 1 0, which is this here. So, we consider the right hand side and the left hand side, v 0 of u would cancel out and eventually, will have r 3 u 0 equals 1 minus u P 0 0 plus u times P 1 0. Let us say, this result, let us proceed further bilinear coon's patch, you are now looking at r u 1, r u 1 will be r 1 u 1 plus r 2 u 1 minus r 3 u 1.

For value of v equals 1, what is the surface r u 1, this is this curve here, this bounding curve b 1 of u now, plug in the value of v equals 1 here and v equals 1 here, what do you get. For v equals 1, we get b 1 of u from r 1 u v and for v equals 1, we get a 0 1 a 1 1 from r 2 u v so, the right hand side can be solved as b 1 of u plus 1 minus u times a 0 of 1 plus u a 1 of 1 minus r 3 u 1. The (( )) a 0 for v equals 1 is, we are on this curve for v equals 1, we are at point P 0 1, which is this. What is a 1 for v equals 1, we are here for v equals 1, we are at point P 1 1, with this right hand side, with this as left hand side, the one of u cancels out.

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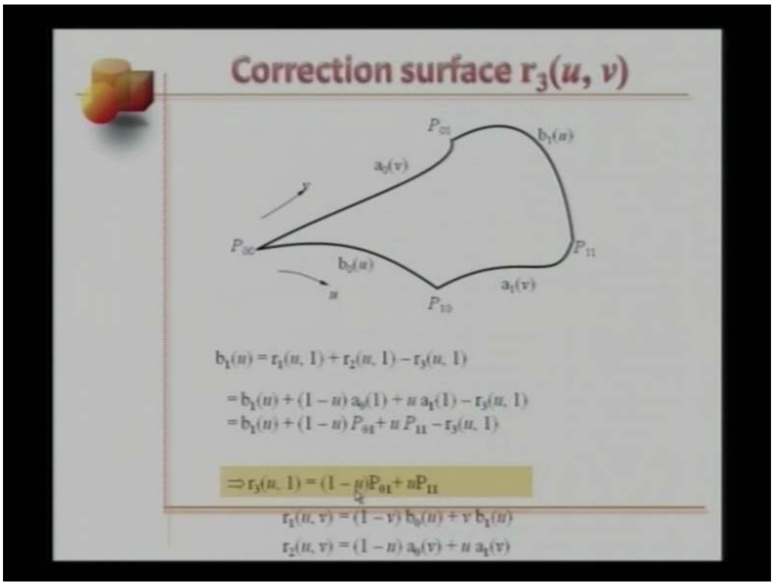

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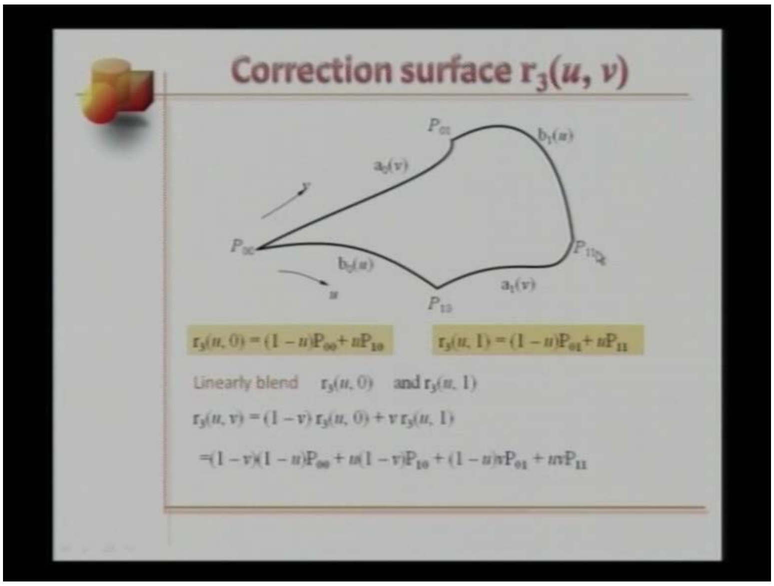

So, we have r 3 u 1 equals 1 minus u times P 0 1 plus u times P 1 1 so, where are we healing. Let us copy the two results from before, one corresponding to r 3 of u 1 1, which is equal to 1 minus u times v 0 1 plus u times P 1 1 and the second corresponding to r 3 u 0, which is equal to 1 minus u times P 0 0 plus u times P 1 0. What can we do with these two results, well we can linearly blend them in this manner so, we can linearly blend r 3 u 0 and r 3 u 1 to get r 3 u v that is, r 3 of u v equals 1 minus v times r 3 u 0 plus v times r 3 u 1 where, r 3 u 0 is this here and r 3 u 1 is this here.

That is equal to 1 minus v times 1 minus u times P 0 0 plus u times 1 minus v times P 1 0 plus 1 u v P 0 1 plus u v P 1 1. If you work the algebra properly, you will notice that, if you use r 3 u v in this form, r u v which was the original model for this bilinear coon's patch, which was the sum of r 1 u v linear blend between b 0 and b 1, r 2 u v, which was the linear blend between a 0 and a 1. And if you subtract r 3 u v from that sum, r 3 u v is given by this expression, you would realize that all your boundary conditions are met. What are your boundary conditions, these are your four corner points and these four are corresponding boundary curves.

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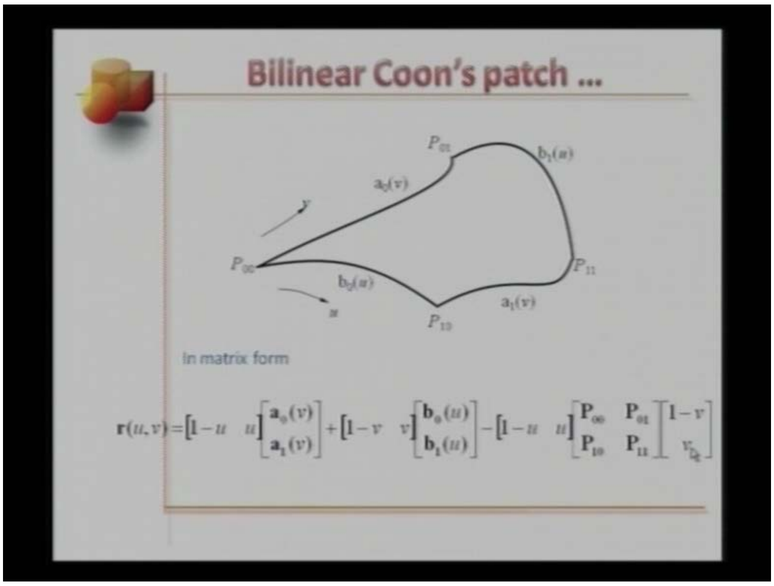

In matrix form or in short form, the bilinear coon's patch is given by the row vector 1 minus u, u times a 0 of v, a 1 of v arranged in column form plus again, a row vector involving parameter v, which is 1 minus v, v times the column b 0 of u, b 1 of u, which are of these two boundary curves minus your correction surface, which is 1 minus u, u times P 0 0, P 0 1, P 1 0 and P 1 1 times the column vector, 1 minus v v.

So, this is the mathematical expression for your bilinear coon's patch in a similar manner, can you think about creating a bicubic coon's patch. With these four bounding curves and to loft these curves, we would need cross boundary tangent information. Once again, with these four boundary curves and with the respective cross boundary tangent information, is it possible for us to create a bicubic coons patch. We will see this now, let me warn you before hand, that although the principle is relatively simple, the mathematical expressions involved are interrogate. I would like you to pay full attention to the following discussion.

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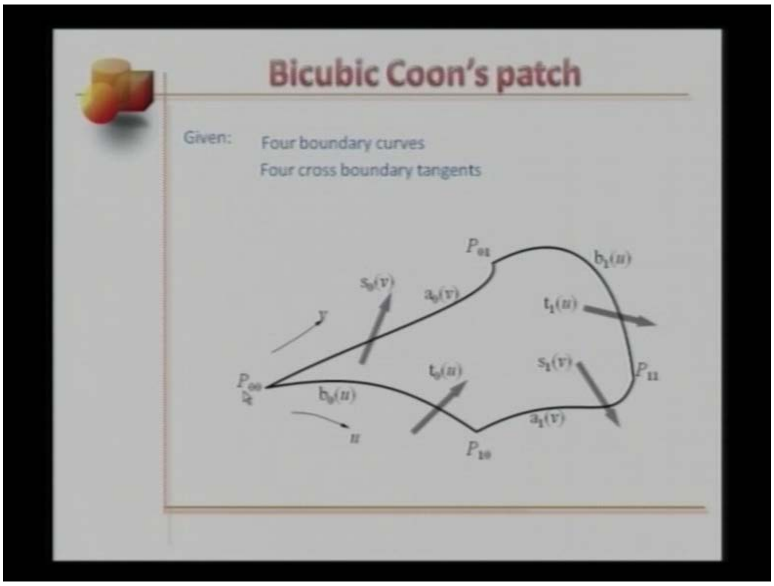

So given, four boundary curves and four cross boundary tangents so, these are two, of the four curves b 0 of u and b 1 of u, these are the other two, a 0 of v and a 1 of v. For b 0 of u, we have the tangents as t 0 of u and for b 1 of u, We have the tangents as t 1 of u. What could these cross boundary tangents mean, well physically, if you stand at any of these points, t 0 of u would be a set of directions, which would be tangents along the v parameter direction.

Likewise, t 1 of u will be tangents, if you stand on any of these points on v 1 of u, again these would be the directions along the parameter v direction. Likewise, the other two sets of corresponding tangents s 0 of v and s 1 of v will be pointing along the parameter u direction like so. Recall now, what we did, when we created a lofted patch, for a given u, we choose a point, we choose the corresponding tangent direction.

For the same value of u, we choose a point here on this curve v 1 of u correspondingly, we also have the tangent information and we lofted a curve along the v direction, we are going to be doing something very similar here. As we did in case of bilinear coon's patch, we will create two lofted surfaces r 1 of u v and r 2 of u v and we will subtract a correction surface from this sum. So, these are the two parameter directions u and v, as you would know, these are four corner points P 0 0, P 1 0, P 0 1 and P 1 1.

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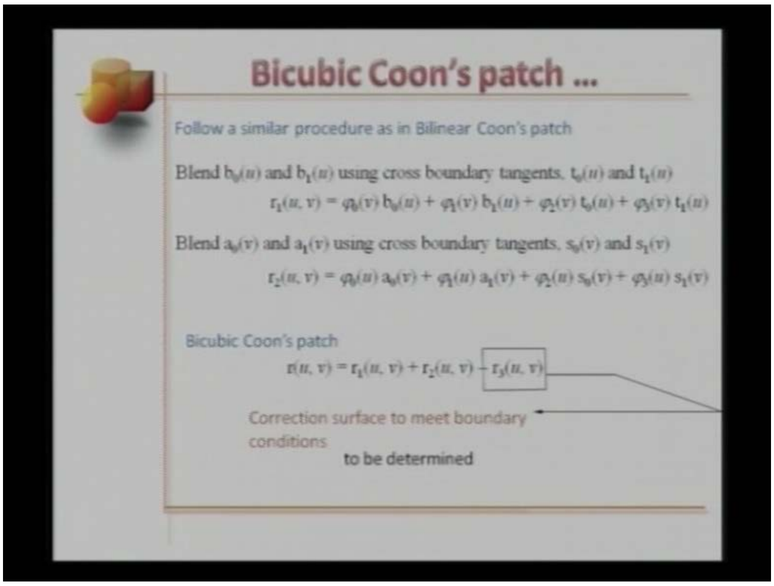

As I mentioned before, we will follow a very similar procedure as in the bilinear coon's patch, to blend two boundary curves b 0 of u and b 1 of u using cross boundary tangents t 0 of u and t 1 of u to get r 1 of u v equals pi 0 of v times b 0 of u plus pi 1 of v times b 1 of u plus pi 2 of v times t 0 of u plus pi 3 of v times t 1 of u. Pi 0, pi 1, pi 2 and pi 3, as you would realize, are might a bicubic polynomials in v.

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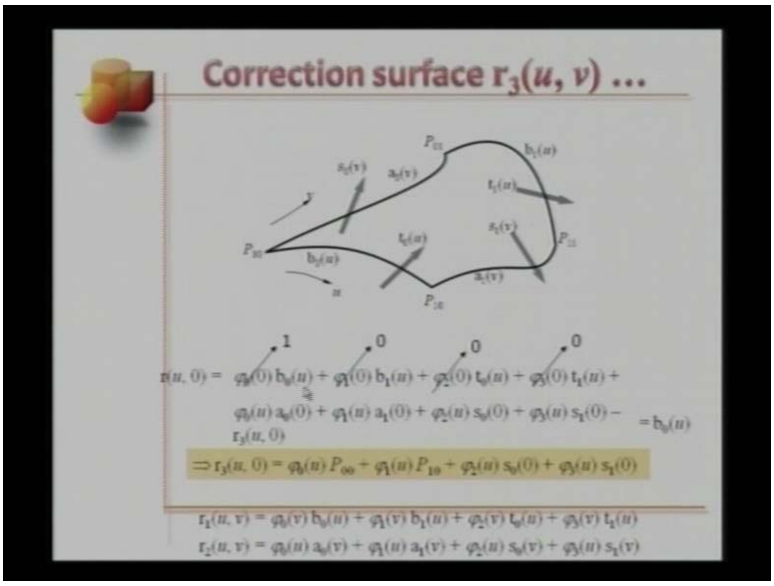

Likewise, we are going to be blending the two other boundary curves a 0 of v and a 1 of v and we will use cross boundary tangents s 0 of v and s 1 of v. And this way, we will create the other lofted patch mark r 2 of u and v is equal to pi 0 now, it is going to be a function of u times a 0 v plus pi 1 of u times a 1 v plus pi 2 of u times s 0 v plus pi 3 u times s 1 v. So, our bicubic coon's patch is given by r of u and v is equal to r 1 of u v plus r 2 of u v minus the corrections surface, as in the case of bilinear coon's patch, which is r 3 of u v. Once again, we will determine this correction surface to meet all the requisite boundary conditions.

Let us continue so, I have copied the expressions for r 1 u v here and r 2 u v here, we will need these expressions to compute the corrections of this r 3 of u and v. So, this is the information that we have, the four boundary curves and the four sets of cross boundary tangents. Now, let us compute, what are u 0 is, plug in the value of v equals 0 here and here. So, we have pi 0 of 0 times b 0 u plus pi 1 of 0 times b 1 of u plus pi 2 of 0 times t 0 of u plus pi 3 of 0 times t 1 of u plus pi 0 of u a 0 0 plus pi 1 of u a 1 0 plus pi 2 of u is 0 0 plus pi 3 u s 1 0.

And of course, we are subtracting the correction surface r 3 of u and now, v equal 0, as I mentioned, the expressions will be quite interrogate and what is r u, for v equals 0, this is this boundary curve b 0 of u. Let us analyze this a little further, if you remember your Hermite cubic functions, you would notice that pi 0 of 0 is 1 and all the other three will be 0. What is next, look at a 0 of 0 along this curve for v equals 0, we have point P 0 0, about a 1 0, we are on this curve here v equals 0, we are standing on P 1 0.

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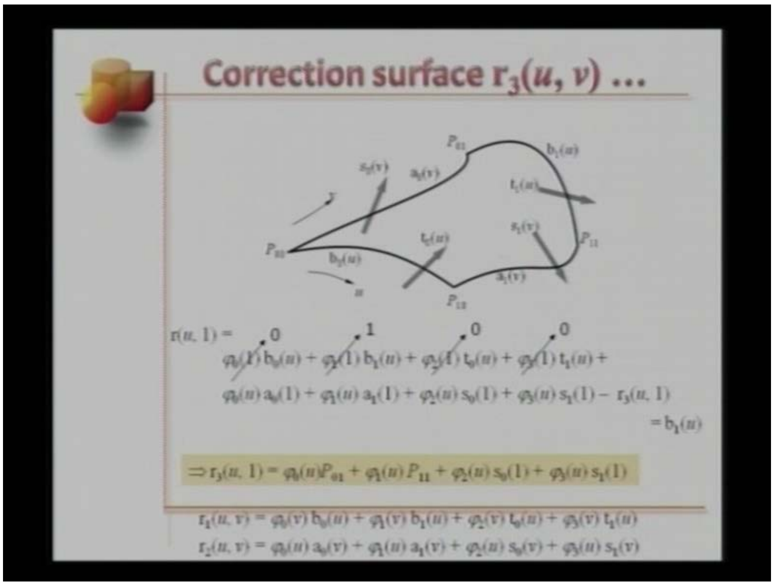

How about s 0 of 0 and s 1 of 0, we are looking at the cross boundary tangents s 0 of p, for v equals 0, the cross boundary tangents along the u direction will be defined at P 0 0, s 1 0. You are looking at these sets of cross boundary tangents and for v equals 0, s 1 of 0 will be defined at p 1 of 0. We will come to these later, for now, if we solve this equation we have r 3 u of 0 equals pi 0 of u times P 0 0 plus pi 1 of u times P 1 0 plus pi 2 of u times s 0 0 and pi 3 of u times s 1 0.

If you notice, this term here it cancels out with this term and here, we are taken this term on the right hand side here, let us proceed. So, we have this patch and now, we are interested in finding what are u 1 is, all we need to do is, to plug in value of v equals 1 here, here and we need to subtract r 3 u 1 on this sum. So, the sum is pi 0 of 1 times b 0 of u plus pi 1 of 1, b 1 of u plus pi 2 of 1, t 0 u plus pi 3 of 1 t 1 u.

Plug in v equals 1 in this expression to get this expression, again plug in v equals 1 here to get this expression, which is pi 0 of u a 0 1 plus pi 1 of u a 1 1 plus pi 2 of u s 0 1 plus pi 3 of u s 1 1. And of course, we have minus r 3 u 1 and after all, what is r u 1 1, this is for v equals 1 will have this boundary curve here, b 1 of u. Once again, if you remember the expressions for Hermite cubic functions, this term goes to 0, this term goes to 1, this goes to 0 and this goes to 0.

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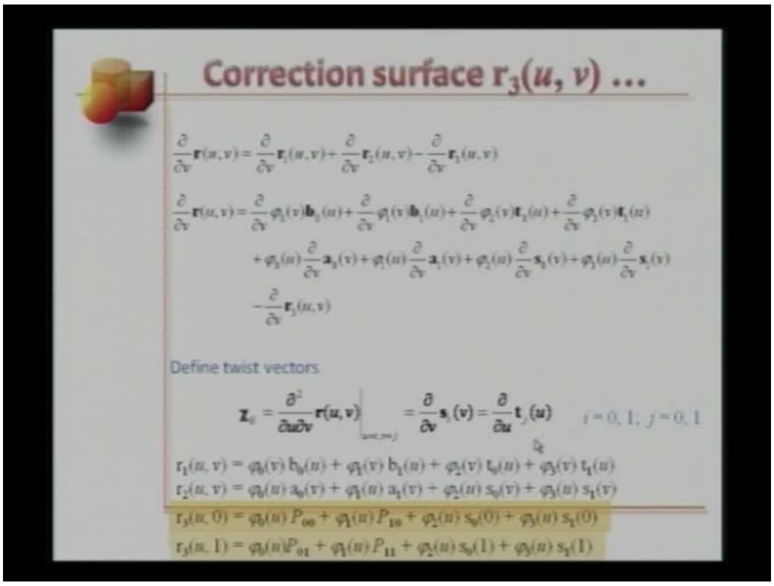

We can simply this complex looking equation as. r 3 u 1 equals pi 0 of u times p 0 1 plus pi 1 of u times P 1 1 plus pi 2 of u s 0 1 plus pi 3 of u s 1 1. Look at this expression here, where did this come from, a 0 1, v on this curve would be equals 1 to get P 0 1. Once again, a 1 1, v on this curve would be equals 1, we get P 1 1, these are the cross boundary tangents at v equals 1, here and here for this one. Do you have an tinkling, as to what we are trying to do here, in the previous slide, we computed r 3 u 0 here ,we compute r 3 u 1. And if you notice, r 1 of u v and r

2 of u v were two lofted patches, that were created using the boundary curves and the cross boundary tangents. We are trying to do something very similar here to find r 3, let us see, what is in (( )) next.

So here, I have copied the four important results, that we would need r 1 of u v, r 2 of u v, r 3 of u 1 0, r 3 of u 1 1, you must have noted down these expressions in your notes. All we would want you do now is, to think about creating a lofted patch to get r 3 and for that, we would need information of the first partial derivative r with respect to v. So, partial r over partial v is partial r 1 over partial v thus, partial r 2 over partial v minus partial r 3 over partial v.

So, thus unfolds the suspense, if we plug in value for v as 0 and 1 respectively, we will have first derivative information of r 3 with respect to v. So, with these two information and with these two, we can think of creating a lofted patch for r 3. So, partial r over partial v will be equal to partial of this expression, with respect to v plus partial of r 2 with respect to v minus partial of r 3 with respect to v.

So, the first expression is, partial of pi 0 with respect to v times b 0 u plus partial of pi 1 with b times v 1 u plus partial of pi 2 with v times t 0 u plus partial of pi 3 with v times t 1 of u plus pi 0 u times partial of a 0 v over partial v plus pi 1 u times partial of a 1 v over partial v plus pi 2 u times partial of s 0 v over partial v plus pi 3 u times partial of s 1 v over partial v, minus partial of r 3 over partial v, which is this term here.

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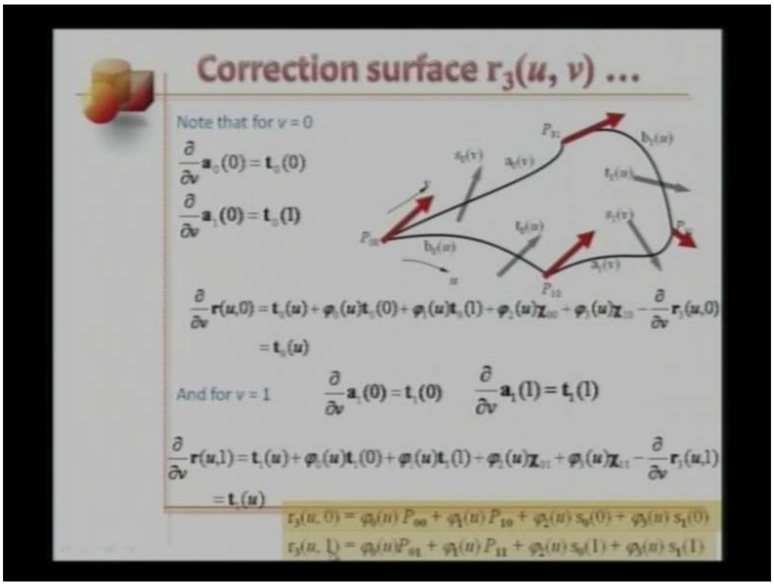

Once again, you can compute the first derivatives with respect to v to these expressions here, let us now, define twist vectors denoted by xi sub i j, which is equal to the next derivative of r u v with respect to u and v. Evaluated at u equals i and v equals j, i and j would go from 0 to 1, these makes derivatives are equal to partial s i v over partial v, which are also equal to partial t j u over partial u, as I said before, I can assume any value 0 or 1 and so is the case with j.

We will need these expressions r 32 of u 0 and r 3 of u 1, so I have copied these equations right here. Now, for a value v equals 0, partial a 0 of 0 with respect to v so, in a sense that, in this we are computing partial of a 0 v with respect to v and evaluating that, at v equals 0 and that, will be equal to t sub 0 evaluated at 0. Likewise, partial of a 1 v over partial v where, v equals 0 is equal to t sub 0 evaluated at 1. This is our coon's patch, this vector here will correspond to t sub 0 evaluated at u equals 0 likewise, this vector here will be equal to t 0 for u equals 1.

Partial of r over partial v, for v equals 0 is equal to t 0 u plus pi 0 u t 0 at 0 plus pi one u times t 0 at 1 plus pi two u times xi 0 0 plus pi three u times xi 1 0 minus partial r 3 with respect to v evaluated at u equals 0 and v equals 0. And that, is equal to t 0 of u, for v equals 1, partial of a 1 with respect to v, for v equals 0 is equal to t 1 of 0 and partial of a 1 with respect to v, for v equals 1 is t 1, evaluated at 1.

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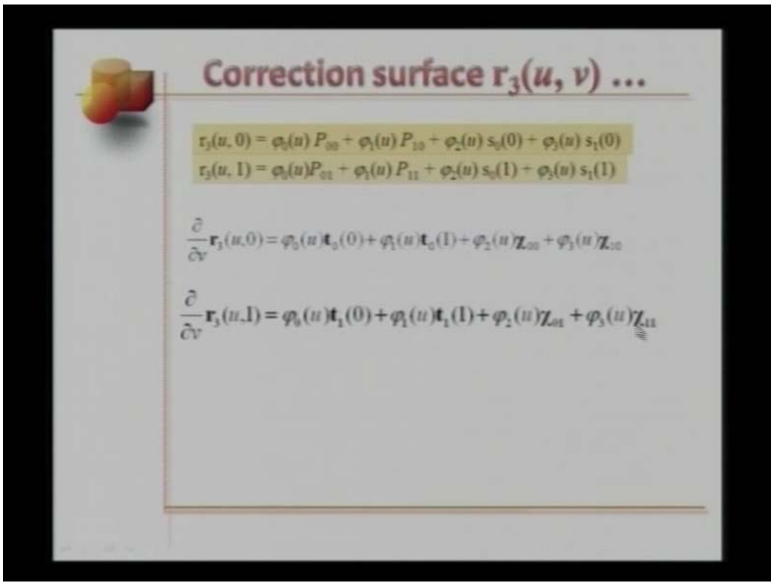

These two vectors are given by these red arrows, t 1 at u equals 0 and t 1 at u equals 1. So, we will have partial of r with v, for u and v equals 1 is equal to t 1 of u plus pi 0 of u t 1 at 0 plus pi 1 u t 1 at 1 plus pi 2 u times xi 0 1 plus pi 3 u times xi 1 1 minus partial of r 3 with v, for u equals u and v equals 1 and that, will be equal to t 1 at u. So, from this equation, we can compute partial r 3 with respect to v at u and 0, and from this equation, we can compute partial of r 3 with v at u and 1. And we already have information pertaining to r 3, for u and 0 and r 3, for u and 1, all we need to do is, use the set of four data to create a lofted surface r 3.

To summarize, r 3 of u 1 0 is pi 0 u, P 0 0 plus pi 1 u p 1 0 plus pi 2 u s 0 0 plus pi 3 u s 1 0, r 3 at u and v equals 1 is equal to pi 0 u v 0 1 plus pi 1 u p 1 1 plus pi 2 u s 0 1 plus pi 3 u s 1 1. And just about now, we had computed what partial r 3, for partial v or for v equals 0 and v equals 1. Here, respectively, pi 0 u t 0 0 plus pi 1 u t 0 1 plus pi 2 u xi 0 0 plus pi 3 u xi 1 0 and pi 0 u t 1 0 plus pi 1 of u t 1, evaluated at 1 plus pi 2 u xi 0 1 plus pi 3 u xi 1 1.

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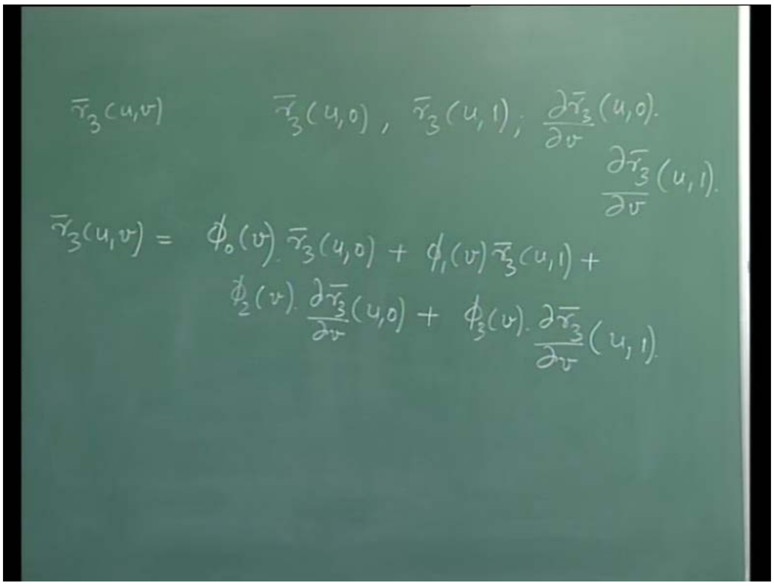

For the correction surface, r 3 u v, this is the information we have, r 3 u 0, r 3 u o1, partial r 3 over partial v at u and 0, and partial r 3 over partial v at u and 1. With this information, it is very natural for us to create a lofted surface r 3 u v, as pi 0, now a function of v, times r 3 u 0 plus pi 1, function of v, times r 3 u 1 plus pi 2 function of v, times partial r 3 over partial v u 0 plus pi 3 of v times partial r 3 over partial v u 1. These are the expressions, we will have as functions of u and pi 0, pi 1, pi 2 and pi 3 will be Hermite polynomials in v. We can do the math and verify that, r 3 u v constructed in this manner would satisfy all the boundary conditions, pertaining to the corner points, pertaining to the bounding curves and also relating to the cross boundary tangents. By this time, you would already have the expression for r 3 u 0, r 3 u 1, the slope with respect to v evaluated at u 0 of r 3 and the slope with respect to v of r 3 evaluated at u 1.