Computer Aided Engineering Design
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Lecture - 36

Good morning, we will continue with our discussion on surface patches. As you would note, we are on the third column of our course layout now.

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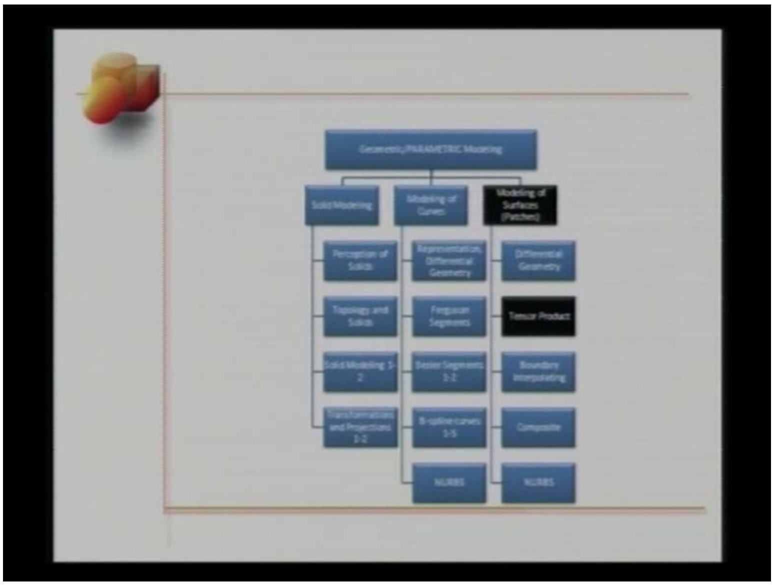

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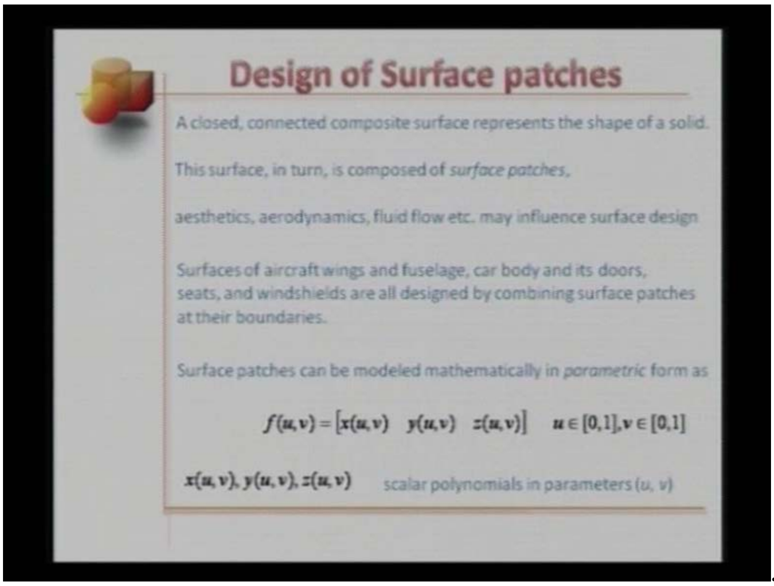

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Today, I will brief, you with different models of surface patches, and I will focus on only one, namely the Tensor Product Surfaces. Today is lecture number 36. Design of surface patches: This is something that I have been repeating over and over again, The Jordan curve theorem. A closed connected composite surface represents the shape of a solid. This surface in turn, is composed of surface patches, which is what we are going to design. Aesthetics, aerodynamics, fluid flow, etcetera are many physical factors that would influence surface design. Surfaces of aircraft wings and fuselage, car body and its doors, seats and windshields are all designed by combining surface patches at their boundaries.

Surface patches can be modeled mathematically in parametric form as a function of two parameters u and v, which is written in ordered form; a row matrix comprising x as a function of u and v, y as a function of u and v as well, and z also as a function of two parameters u and v. The bounds for two parameters are between 0 and 1, each. x, y and z are scalar polynomials or scalar fields in parameters u and v.

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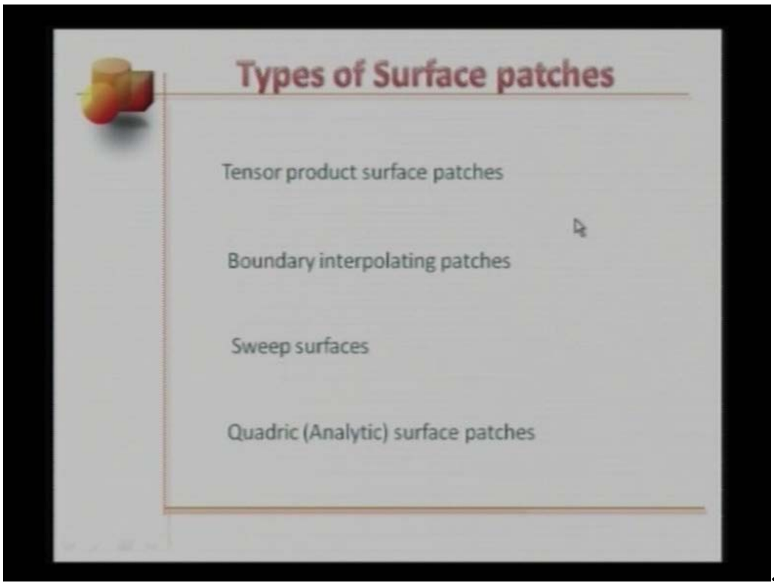

The different kinds of surface patches that we can design: The first one, which we are going to be focusing on in this lecture, is the tensor product surface patches. The second, which we will deal with later, is the boundary interpolating patches. The third one would be the sweep surfaces and the fourth would be a class of quadric or analytic surface patches. Today, we will be focusing mainly on tensor product surface patches.

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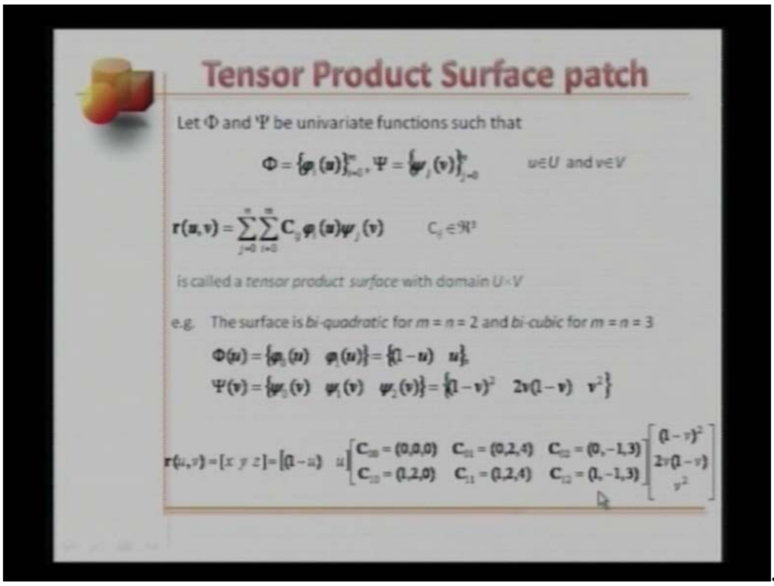

Now, what is a tensor product surface patch? Let us assume that phi and psi are two univariate functions, such that phi is given by a set of functions phi i all of them as functions of parameter u, the index i going from 0 to m and psi is again a set of functions psi j, all of them as functions of v; the index j going from j equals 0 to end. Here, the values of parameter u belongs to a set capital u which we have seen 0 and 1 and so is the case for values of the parameter v belonging to a set capital v which again is defined by the interval 0,1.

Using these two function sets, we can define a tensor product surface r as a function of u and v as summation index j going from 0 to n, again summation index i now going from 0 to m, c sub i j phi sub i as a function of u times psi sub j as function of v. This is the tensor product surface in the domain u times v or u cross v. Here, the coefficients c i j are all real variables and they belong to the Euclidian space. For example, the surface is biquadratic for m equals n equals 2 and bi-cubic for m equals n equals 3.

Let us take a look at a case, where pi i is given by a set of linear functions pi 0 and pi 1; these linear functions are for example, 1 minus u and u; 1 minus u is pi 0 and u is pi 1, and psi of v is a set of functions psi 0, psi 1, and psi 2, all of them as quadratic; for example, psi 0 is 1 minus v square; psi 1 is 21 v times 1 minus v psi 2 is v square. If you observe, these are linear Bernstein polynomials each of degree 1 and these are quadratic Bernstein polynomials of degree 2.

Using this model, here, we can write r of u v as the ordered set of x y z, which is equal to the row matrix 1 minus u, u times the geometric matrix here c 00, c 01, c 02, c 10, c 11, c 12 - these are the two rows in a matrix and we will have the column containing the functions psi j 1 minus v square 2 v times 1 minus v and v square. Let us say, for example, that c 00 is (0,0,0) c 01 is (0,2,4) c 02 is (0, minus 1,3) c 10 is (1,2,0) c 11 is (1,2,4) c 12 is (1, minus1 and 3). These are in a way the x, y, and z coordinates.

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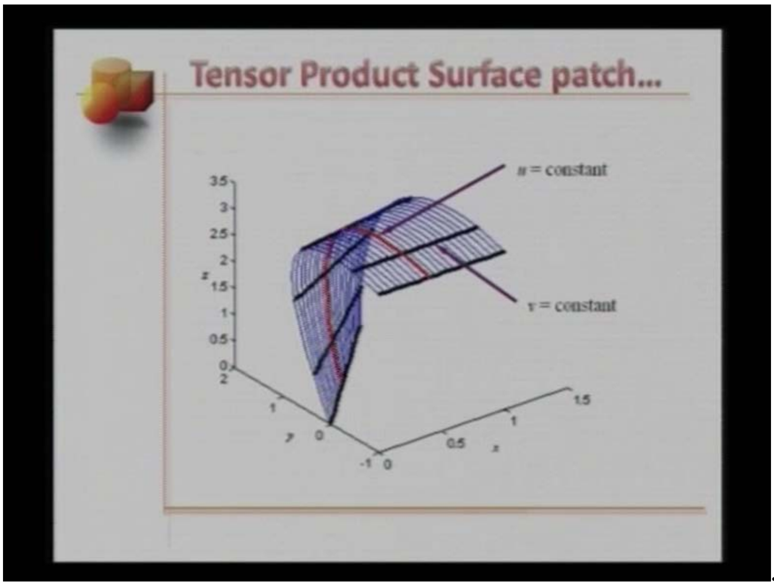

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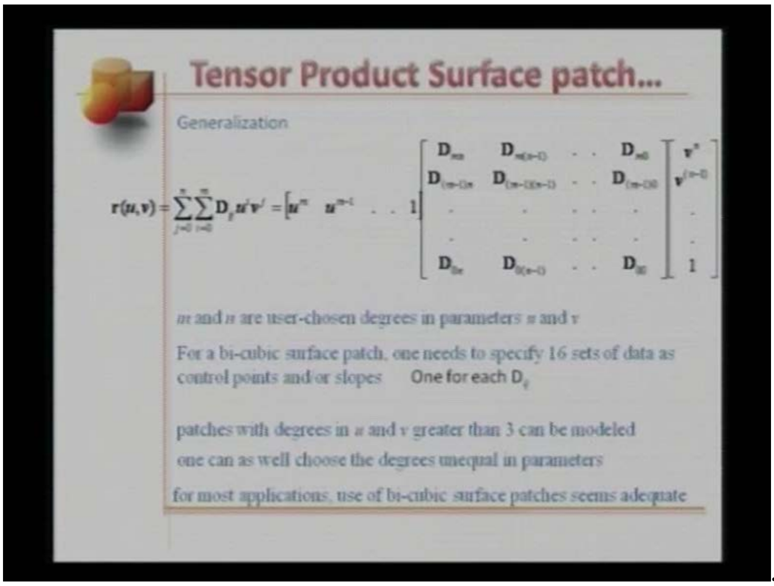

This is how the surface patch looks like. The curve in red corresponds to the values u as constant; the curve in black, which is actually a line corresponds to v equals constant. If you observe, this line in black is being swept along this curve to give us the tensor product surface. We can also think of this surface being a sweep surface, but we will reserve our discussion on sweep surfaces for later.

We can generalize and say that the tensor product surface r as the function of u and v, the two parameters can be given by summation j going from 0 to n summation i going from 0 to m of D sub i j u raised to i v raised to j; D sub i j would be some constant in case it would be or they would be unknown coefficients. This is equal to a row matrix containing all the u's, rather all the powers of u; for example, u raised to m, u raised to m minus 1 after 1 times the coefficient may fix given by D m n D m n minus 1 until D m0 in the first row; D m minus 1 n D m minus 1 n minus 1 after D m minus 10 for the second row; we can continue until we get the last row as D 0 n D 0 n minus 1 up to D 00, and this is followed by a column vector comprising all degrees in v; for instance v raised to n v raised to n minus 1 upto 1.

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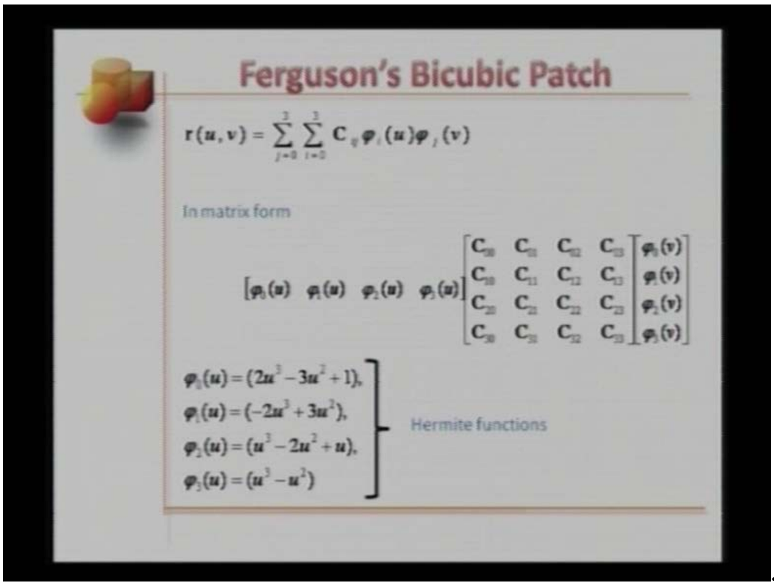

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Here, m and n can be user chosen degrees and parameters u and v; they may be the same; they can be different. For example, for a bi-cubic surface patch, one would need to specify 16 sets of data as control points or slopes. Why 16? For a bi-cubic we will have 4 rows and 4 columns. For each coordinate, they we will be having 16 coefficients of variables. As I said, one for each D sub i j patches with degrees u and v greater than 3 can be modeled. This generalization does not stop us from designing surface patches of any degree in parameters u and v. As I said before, one can as well choose degrees unequal in the two parameters, but for most of application use of bi-cubic surface patches seems adequate.

Let us consider the first tenser product model: Ferguson's bi-cubic patch. This is given by r of u and v equals summation j going from 0 to 3, i going from 0 to 3 c sub i j phi of phi as a function of u and phi of j as function v. Here, the two sets are the same; that we have seen. What do you expect these function will be? Take a guess. Try also to recollect our discussion on Ferguson's curves. Would you think these will corresponds to the Hermite cubic polynomials? Yes, they work.

Coming back to this tensor product model, we can write this in matrix form as phi 0 of u, phi 1 of u, phi 2 of u, phi 3 of u; this comprises the first row matrix times the 4 by 4 coefficient matrix; remember that we will be using different coefficient matrices for different coordinates x, y, and z. The entries are c 00 , c 01, c 02, c 03, c 10, c 12, c 13 and so on, uptil the last 4 by 4 term as c 33. And correspondingly, we will have a column vector comprising of the functions phi 0 phi 1 phi 2 and phi 3 in v.

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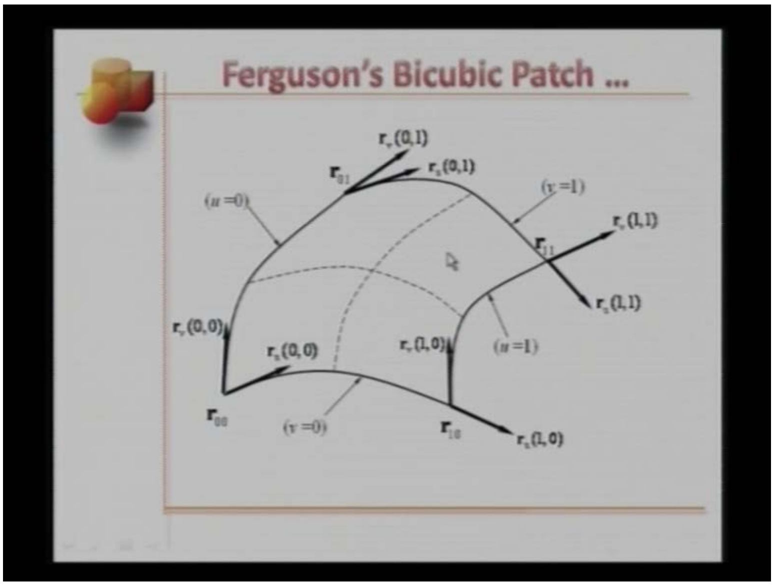

As I said before, c ij's will be ordered; phi 0 of u is given by 2 u cube minus 3 u square plus 1 by 1 of u is minus u cube plus 3 u square phi 2 of u is u cube minus 2 u square plus u phi 3 of u u cube minus u square. Likewise, phi 0, phi 1, phi 2, and phi 3 as functions of v can be constructed. As we discussed before, these are Hermite cubic functions.

Let us try to figure how we can determine the unknown coefficients c ij. See, we have a curve; we have another one, the third one, and fourth one. These are four bounding curves for this Ferguson's bi-cubic patch. This one here corresponds to v equals to 0. The u parametric direction is along this curve, let us say, this curve here corresponds to u equals to 0; this one here for v equals 1 and the last one for u equals 1.

Let us name different geometric attributes that are required to construct or to design the surface patch. The first point is r sub 00; the second one is r sub 0 sub 1; third one here is r 10 and the fourth one is r 11. The first index corresponds to parameter u and the second index corresponds to parameter v. This is the best way to remember. You start at a point that corresponds to u equals 0 and v equals 0. Correspondingly, you identify this point as r sub 0 sub 0. If u move along the u direction, since this curve corresponds v equals to 0, you hit at this point which corresponding u equals 1 and v equal 0. Therefore, you can think of naming this point as r 10.

Likewise, if u move along this curve here, this is for u equals 0; this point here would correspond values u as 0 and v as 1. We can name this point as r 0 1. Similarly, this would be r 11. Now, these are only four other sixteen conditions that we need completely determine this patch. How about the other twelve conditions? Remember that when discussing Ferguson's segments, we were dealing with slopes. Let us consider slopes here as well.

This is the slope along the u direction at this point (Refer Slide Time: 20:04) and this is the slope along v direction at r 00. The slope along u, the slope along v, and slope along u slope, along v; the same for this point as well. Let us try to name this slope as well, but before, let us identify the curves on this patch, which would correspond to constant values of u and v. This curve here would to correspond to a constant value of v; try to observe this and this here, and this curve right here would correspond to a constant value of u.

Now this slope is along the u direction. So, we have r sub u here evaluated at u equals 0 and v equals 0. What is r sub u? It is partial r over partial u. Likewise, the slope here corresponds to partial r over partial v evaluated at u equals 0 and v equals 0. For this point r 01, the slope corresponds to r u at 0 and 1, and this point here corresponds to r sub v. At 0 and 1 the slope right here corresponds to partial r over partial u evaluated at u equals 1, v equals 0. The slope here is r v at 1 and 0. Likewise, these two slopes here are r u and r v respectively evaluated at u equals to 1 and v equals 1. What is next?

Now, we have 12 conditions of 16 that we want. We will have to generate 4 more conditions to be able to completely determine this patch.

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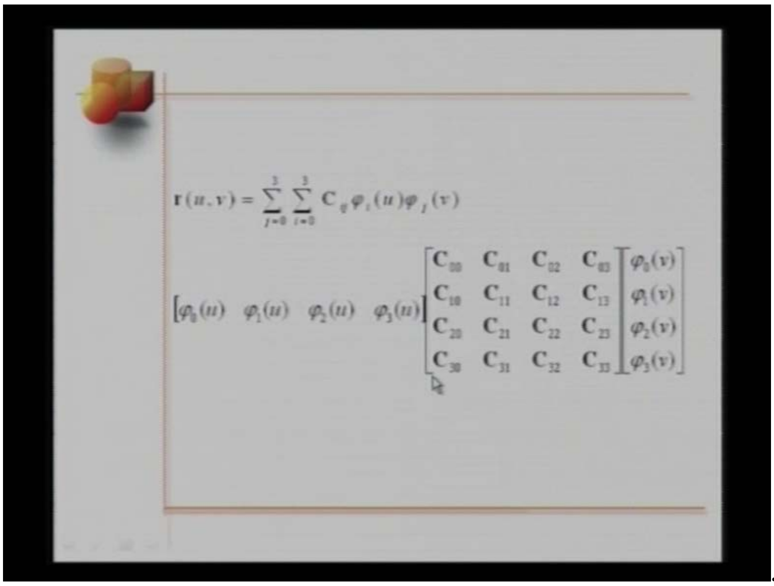

Let us recall this mathematical form for Ferguson’s bi-cubic patch; r of u and v equals summation j going from 0 to 3, i going from 0 to 3 and c i j phi i of u and phi j of v. This was written in this form. Here, we had a row vector comprising this Hermite functions, column vector again comprising the Hermite cubic functions in terms of v, and then these unknown coefficients which were 16 of them.

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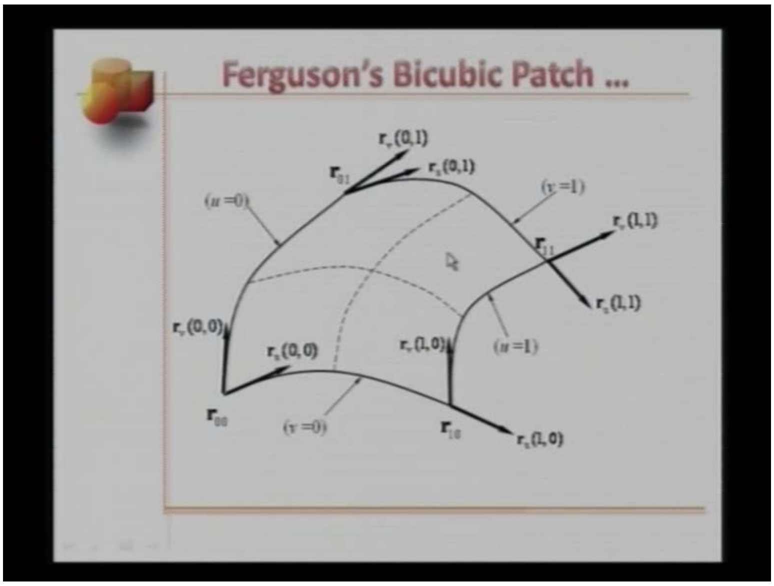

Let us try to set this information in the mathematical form and try to determine the unknown coefficients c ij.

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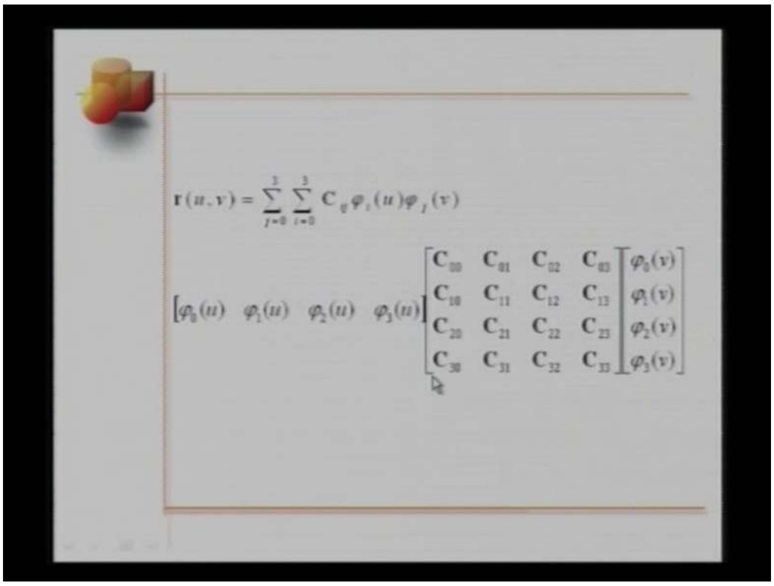

All we would need to do is for the points plug in the values of u as 0 and1, plug in the value of v as 0 and 1, and determine may be four of these coefficients. For the slopes along the u direction, we will have to differentiate at this expression at this u and plug in the corresponding information. For slopes along the v direction, we will have to differentiate the same expressions now with respect to v. We can determine four more of these coefficients. You can do the algebra all by yourself. Let me give you the results.

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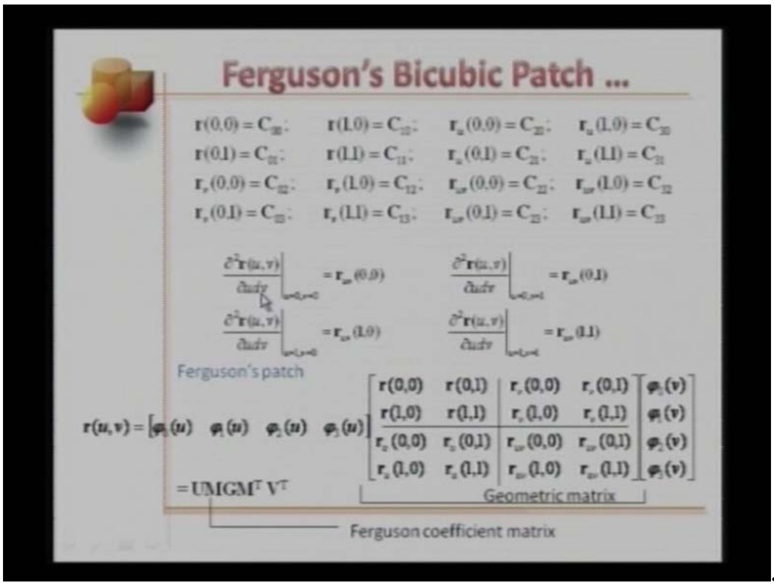

So, c 0,0 is the first point r 0,0; c 10 is r 1,0 point corresponding to u equals 1 and v equals 0; c 20 is the slope along the u direction at the first point; c 30 is a slope again along the u direction at a point corresponding to u equals 1 and v equals 0. Likewise, c 01 is r evaluated at 0,1; c 11 is r evaluated at 1,1 c 21 is the slope along the u direction evaluated at 0,1 if 0,1 c 31 is the u slope evaluated at 1,1; c 02 is r v 0,0; c 12 is r v 1,0; c 22 is r u v 0,0; I will talk about that in a moment; c 32 is r u v 1,0. c 03 is the slope along to the v direction at v equals 0 and v equals 1. c 13 is r v 1,1; c 23 is r u v is evaluated at u equals 0 and v equals 1, and c 33 is r u v at 1,1.

We had seen the information previously in terms of points, the slopes along the u direction and the slopes along the v direction; these were the twelve sets of information. Where did these come from? Well, since we are working with the slope information or the corner points, to generate four more conditions we will need mixed derivative information as well; this is what these four conditions pertain to; r u v is the mixed derivative of r with respect to u and v evaluated at u equals 0 and v equals 0.

Likewise r u v at 0,1 which is c 23 is partial to over partial u partial v at u equals 0 and v equals 1; r u v 1,0 is c 32; it is partial to r partial over u partial v and u equals 1 and v equals 0. And finally, r u v 1,1 which is c 33 which is equals to partial u r and over partial u partial v at u equals 1 and v equals 1. So, now we have 16 conditions corresponding to which we can determine 16 unknowns, and then we have the Ferguson's bi-cubic patch at our hand, completely determined.

Ferguson's patch is now given by this expression here: r of u v is equals to the row matrix comprising the Hermite cubic functions in u times all the c is now, all the c ij is now which are determined here, multiplied by a column vector corresponding to the Hermite cubic functions in v. I missed to mention before that it becomes a little difficult for the designer to work with the slope information. That was the drawback with Ferguson's segments and it is also a drawback with Ferguson's bi-cubic patch.

Let us consider some examples, but before that I can write this complex looking right hand side in a very compact form as a row vector u, recall what this was when we discussed Ferguson's segments, times a matrix m times g and times m transpose times v times transpose. Now, this corresponds to row vector that contains the elements 1, u, u square, and u cube; likewise, this column vector will have elements 1, v, v square, and v cube. You might contour figure the order in which they appear. M is the Ferguson's coefficient matrix and G is the geometric matrix; so, this is G and this Ferguson's co efficient matrix that we have seen in case of Ferguson's matrix segment. I can set you right this geometric matrix into 4 parts. This is the way I would remember the matrix G. This part here on top left correspond to the point data 0,0 0,1 1,0 1,1. So, these would correspond to the coordinates and the four corner points of a patch.

The sub matrix on the top right would correspond to the v slope information, partial r over partial v. They will be evaluated in the same order as here 0,0 0,1 1,0 and 1,1. This sub matrix here in the bottom left corresponds to the u slope information that is partial r over partial u. Again, evaluated in the same order, u equals 0 v equals 0 and u equals 0 v equals 1; u equals 1 and v equals 0; u equals 1 and v equals 1, and this one here in the bottom right corresponds to the mixed derivative information. In CAD literature, this information has a name: partial 2 r over partial u and partial v are called twist vectors.

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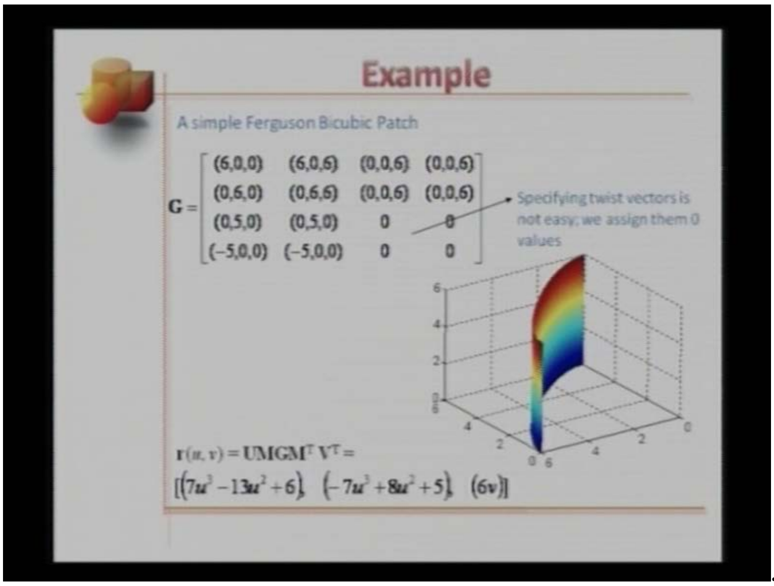

Now, for the examples; let us look at a simple Ferguson's bi-cubic patch. The geometric matrix is given by (6,0,0) (6,0,6) (0,0,6) (0,0,6) (0,6,0) (0,6,6) (0,0,6) (0,0,6). Looks like 6 happens to be a lucky number here. (0,5,0) (0,5,0) we have here (0,0,0). We represent this by a single number here. Likewise, (0, minus 5, 0) (0, minus 5, 0) (0,0 and 0). Try to identify in this geometric matrix, which ones of these entries would correspond to the point's information to the slopes along the u direction, to the slopes along the v direction, and to the twist vectors. As I said, we can divide this matrix into four sub systems. Draw a horizontal line here; draw a vertical line here These entries will correspond to, you guessed it, the points information.

How about this matrix? They are the slopes along the v direction at u and v as 0; u as 0 v as 1 u as 1 v as 0, and u and v both as 1. Try to remember this order. These entries are the slopes along the u direction again evaluated at (0,0) (0, 1) (1,0) (1,1), and of course these four entries are the twist vectors. Now, as a designer, you would be comfortable specifying these four points. You may be okay satisfying the slopes along with v direction and satisfying slope along with u direction, although I must mention that these numbers are not included for all of us to specify. How about the twist vectors? It will be a little difficult for us to predict a shape for any given set of twist vectors; a totally non-intuitive information. So, let us say we have assigned these twist vectors zero values and try to see how a Ferguson's bi-cubic patch looks like.

R of u v is given as capital u times capital m times capital g times capital m transpose times capital v transpose. You can do the algebra. I will give you the final result here. So, if I use only the first entries in these ordered sets, I will get the scalar field in x. The field is 7 u cube minus 13 u square plus 6. Even though the variable v does not appear here, this field is actually a function of u and v, both. Likewise, if I consider both second entries, the second entries in these ordered sets, I would be able to compute the scalar field y of u and v; for this case, it is given as minus of 7 u cube plus 8 u square plus 5. Finally, considering the third seven entries will give me the c u v field, which is 6v. This is I emphasize a very simple example. Now, this is how a Ferguson bi-cubic patch looks like for this geometric matrix.

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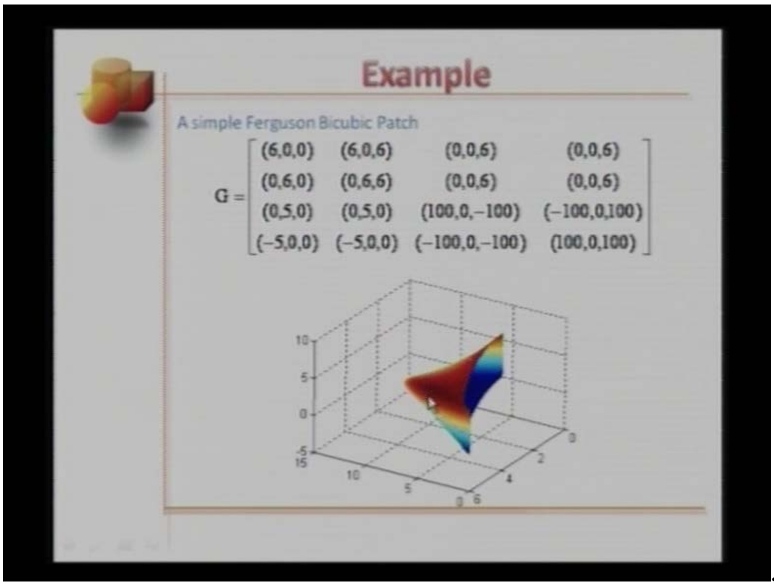

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Let us experiment with this for a while. Let me change a twist vector (( )). So, here I have entries like (100, 0, minus 100). Here (minus 100, 0, 100) (minus 100, 0, minus 100) (100, 0, 100), how would a patch look like? I could see a bulge somewhere at the center here.

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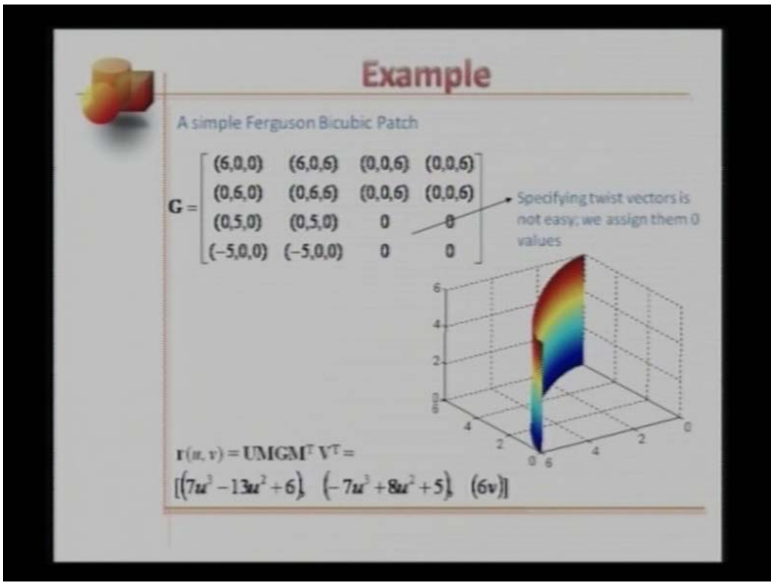

This bulge was not present here, but did I know that I would get such a change in the shape of the surface patch, if I introduced non-zero values for this for these twist vectors? May be not. You can try with different geometric matrices to convince yourselves that indeed predicting the shape of a bi-cubic Ferguson's patch by changing the v slopes, the u slopes, and the twist vectors may not be very easy. This gives us the motivation of studying surface patches that involves only specifying design points or control points. Let us say, as a designer we would not like to specify higher order information like the slopes, the twist vectors, second derivatives, third derivatives, and so on. With this set, we will try designing different tensor product surfaces that involves only design points and we will do that in subsequent lectures.