Computer Aided Engineering Design
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Lecture - 35

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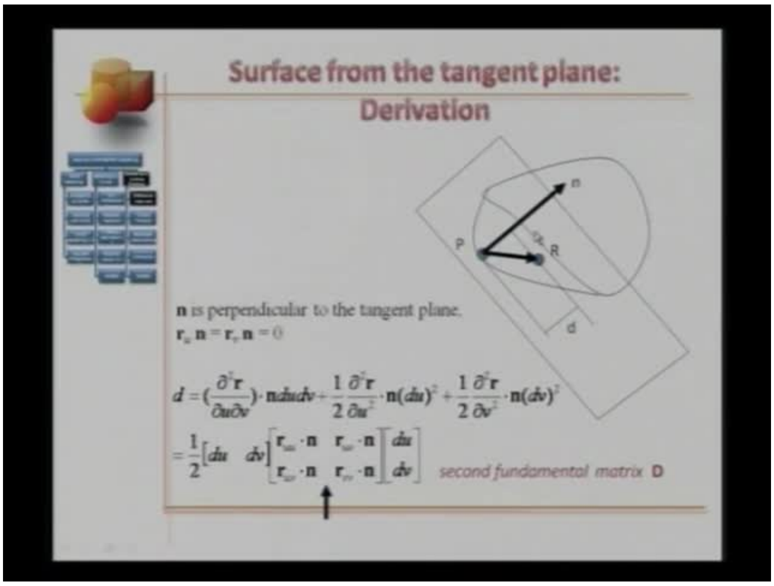

Hello and welcome, let us continue with differential geometry of surfaces. A little recap, the last time we had classified points on surfaces with respect to the tangent plane. Let us revisit that discussion. So, we have the surface patch, we had a point P, we had a tangent plane passing through that point, we had another point R on the surface patch. We had drawn vector P R and a normal to the tangent plane n at P.

We computed this distance d as the projection of P R on n. Let me quickly go through the math and highlight the salient points. So, this 2 by 2 matrix here was something we called the second fundamental matrix. It involved the second partial derivative of the parametric equations of the surface patch with respect to one of the parameter u mixed second partial derivatives of r with respective two parameters u and v and the second partial derivative of r with respective second parameter v and also this matrix involved the normal vector to the tangent plane.

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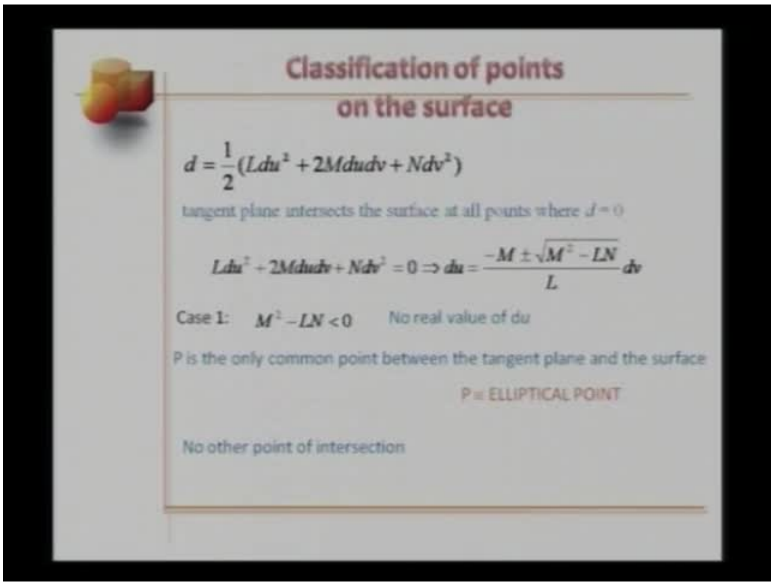

We had found the equation for d which is the projection of P R two points on a surface patch on the normal passing through the point P on the patch as half L d u square plus 2 M d u d v plus N d v square. The figure that the tangent plane intersects the surface at all points where d equals 0 and we had computed the corresponding condition for that. So, we had L d u square plus 2 M d u d v plus N d v square equals 0 and that gave us the expressions for d u in terms of d v and the three coefficients L M and N corresponding to the second fundamental matrix d.

So, we have d u equals minus M plus minus the square root of M square minus L N over L. We consider four cases. The first case corresponded to this discriminate M square minus L N smaller than 0. In that case, we had no solution. So, point P was the only point common between the tangent plane passing through it and the surface patch. We called that point the elliptic or elliptical point.

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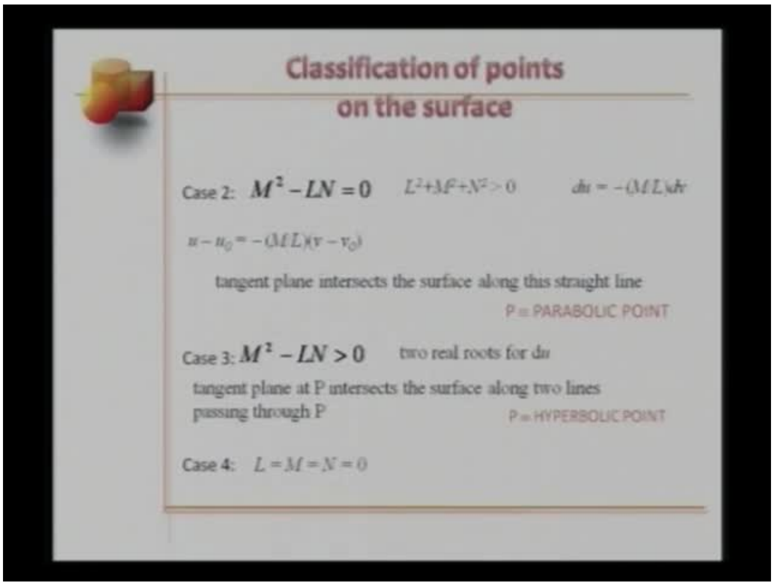

The second case where the discriminate M square minus L N equals 0 where L square plus M square plus N square is greater than 0. We have a condition, which corresponds to the equation of line u minus u 0 equals minus of M over L times v minus v 0. That implies basically that the tangent plane intersects the surface along a straight line. We call such a point P as a parabolic point.

Third case, when the discriminant is greater than 0. In this case, we have two real roots for d u to the tangent plane basically, at P intersects the surface along two lines passing through P. We name P in this case as a hyperbolic point and fourth case of course, is when the elements L M and N of the second fundamental matrix are all 0 in which case we call P as a flat point.

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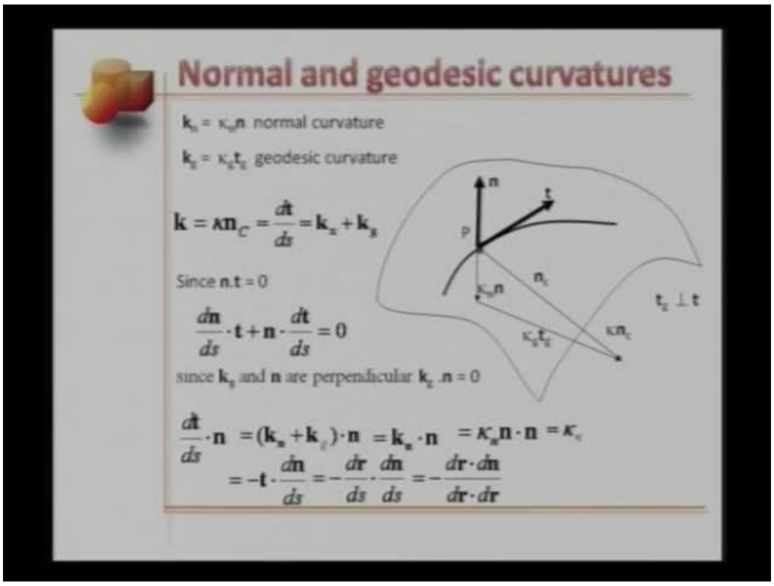

Today, we talk about Gaussian and mean curvature of surfaces in lecture number 35, but first let us talk about normal and geodesic curvatures. So, we have a surface patch and we have curve. Let us say playing on the patch, we will plot a point P on the curve and a tangent along this curve at point P. We also sketch a vector normal or perpendicular to this vector t and we call it n.

Now, here I would want you to follow the construction rather carefully. Let us sketch another vector and call it n sub c. So, n sub c happens to be the unit vector along this direction and the magnitude of this vector, let us say is given by kappa. So, this vector is kappa times n sub c. I will tell you what or how n c is related to these two vectors. Let us extend this normal vector n in the opposite direction and name this vector kappa sub n times n. Let us join the two heads of these two vectors and name the resulting vector as kappa sub g times t g. t sub g is also a unit vector along this direction.

The question you would want to ask yourself is how all these two vectors n sub c and t sub g related to t? n sub c is the first derivative of t with respect to the parametric out line. So, n sub c would be perpendicular to t and you will have to be a little observant here, if you realize t sub g is a vector that lies on the plane defined by this vector n and this vector n sub c. Consequently, t sub g will also be perpendicular to t. Let me repeat, the unique tangent here is perpendicular to the unit normal since n c is the first derivative of t with respect to the parameter s which is the arc length parameter, n c is perpendicular to t.

So, the plane containing n and n c is perpendicular to t or let me put it the other way round. Tangent t is perpendicular to the plane containing n and n c and by construction t g happens to lie on this plane. So, t g will be perpendicular to t. Once, you understand this geometric construction let us proceed with some algebra and normal curvatures of course. k sub n is equal to kappa sub n times n and we call k sub n as the normal curvature. k sub g is equal to kappa sub g times t sub g and we call this k sub g as geodesic curvature. k is equal to kappa times n c which is respect to here and that is given by the first derivative of the tangent with respect to the arc length parameter as I mentioned before and by vector algebra a vector law of addition k n, which is this vector plus k g which is this vector here equals kappa times n c which is k.

Now, we know that is normal and this tangent of perpendicular so n dotted with t is equals to 0, we differentiate this one time with respect to the arc length parameter s. So, we have total n over total s dotted with t plus n dotted with total t over total s equal 0. All we have done is we have differentiated this expression with respect to s. Now, since k sub g and n are perpendicular k sub g is this vector here and we have constructed this vector so that n and kappa g times t g make a 90 degree angle here.

So, we will have k sub g dotted with n as 0. Coming back to this result, d t over d s dotted with n is equal to k sub n plus k sub g this vector dotted with n. Now, if you realize this vector comes from this relation here. We expand this result here. So, we have k sub n dotted with n plus k sub g dotted with n and because of the construction because this angle is 90 degrees k g dotted with n is 0 so we are left with k sub n dotted with n.

We substitute for k sub n as kappa n times n dotted with n which is equal to kappa n the scale. Now, d t over d s dotted with n is equal to minus of d n over d s dotted with t. You bring this on the right hand side and this is equal to minus of d r over d s dotted with d n over d s. Simply, because the tangent t is defined as the first derivative of r with respect to s. What is r here? r is the parametric equation of this curve. Now, this is equal to minus of d r dotted with d n over d r with dotted with d r. How did this happen? One has to be careful; it gives you the impression that this dot product is getting transferred to the numerator and the denominator while this is not the case correctly. Let me explain this to you on the board. So, this how this happens.

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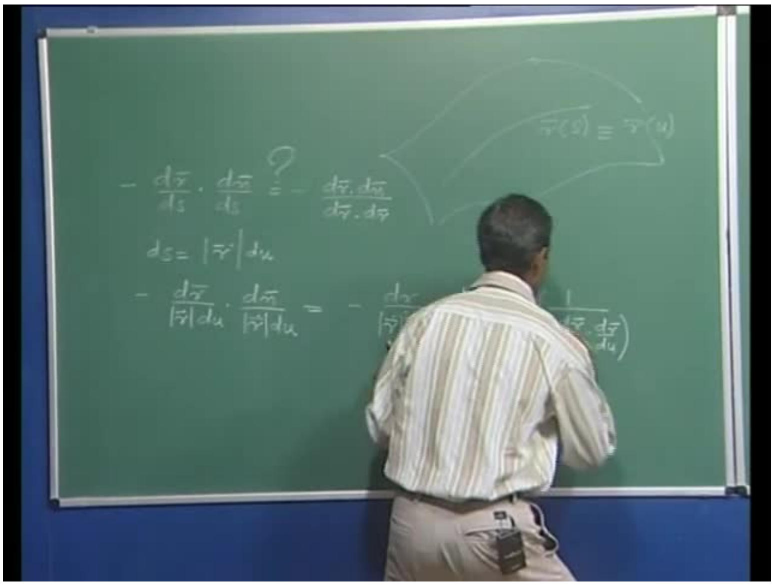

Let us say you have this curve r as the function of the outline parameter s, I can also write this thing as r as a function of another parameter u. This curve is lying over the surface patch that we are considering. Now, minus of d r over d s dotted with d n over d s. I said that this was equal to minus of d r dotted with d n over d r dotted with d r. How did this happen? Let us find out. So, I can write d s as modulus of r dot d u and I can substitute this result here. So, I get minus of d r over modulus of r dot d u dotted with d n over modulus of r dot d u. Since, these are scalars I can club them together.

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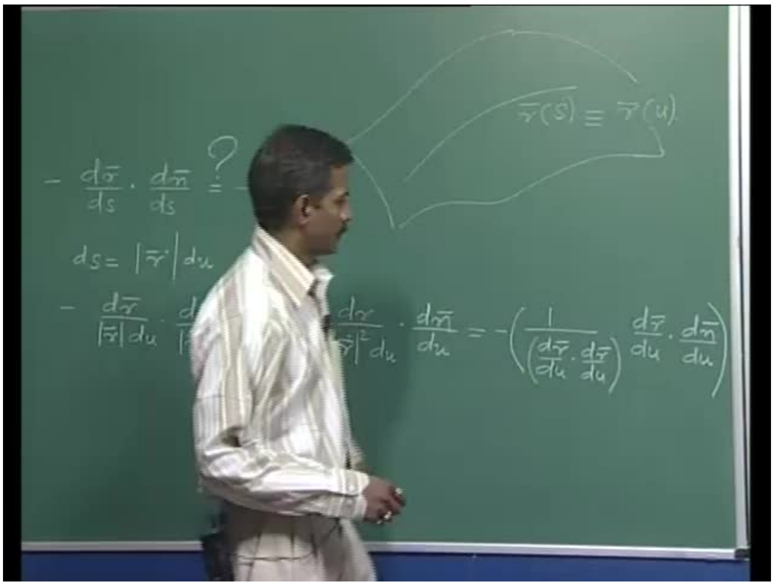

I can write this thing as minus of d r over modulus of r dot square d u dotted with d n over d u. This is equal to minus of 1 over what is r dot square; this is equal to d r over d u dotted with d r over d u times d r over d u dotted with d n over d u.

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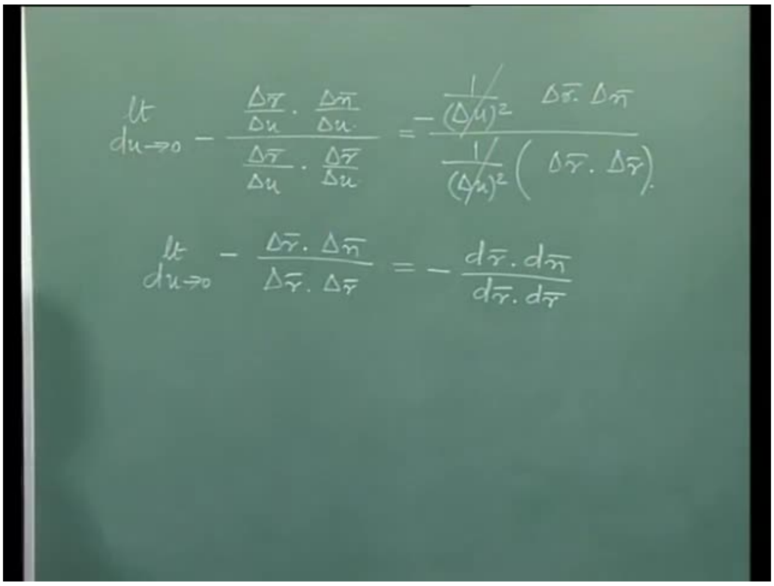

So, I will do a little hand waving here. If you notice this result is a limiting case of limit delta u tends to 0, delta r over delta u dotted with delta n over delta u over delta r, so these are vectors over delta u dotted with delta r over delta u. The numerators are vectors here while the denominators are scalars. Maybe I can take this scalars away and I can write thing as 1 over delta u square delta r dotted with delta n over 1 over delta u squared delta r dotted with delta r. I can cancel these. So, I am left with limit delta u tends to 0 delta r dotted with delta n over delta r dotted with delta r. And your conditions that delta r's and delta n are very small, you can change the notation and we can write this result as d r dotted with d n over d r dotted with d r. There was a negative sign here. This is how probably we get from here to here. What is next?

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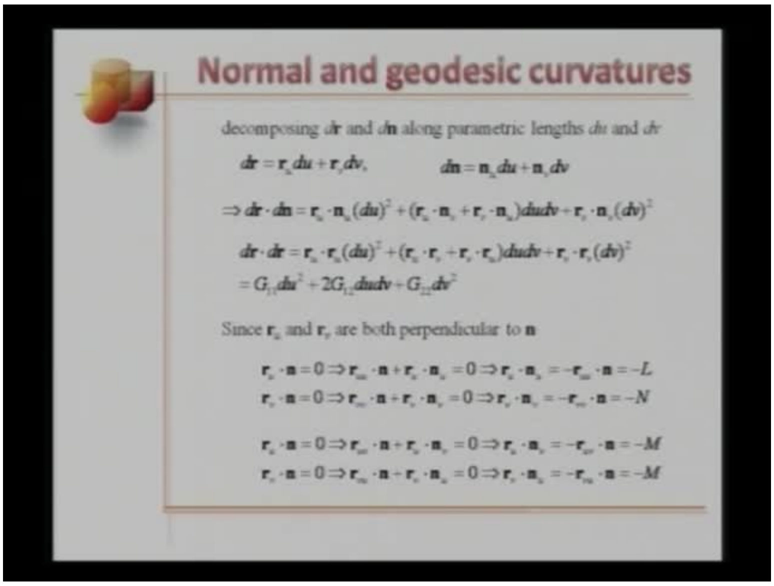

Now, what we can do is we can decompose d r and d n along parametric lengths d u and d v. So, here we are going to be using the information from the curve and try to convert that into the information pertaining to the surface patch on which the curve lies. So, d r equals r sub u times d u plus r sub v times d v. Likewise, d n equals n sub u times d u plus n sub v times d v. We can also think of these results being obtained from the first order Keller series expansion. Now, d r dotted with d n in numerator in the result that we have seen before is equal to r u dotted with n u times u square plus r u dotted with n v plus r v dotted with n u times d u d v.

Just in case, if you have forgotten while the subscripts being here, r sub u is the partial derivative of r with respect to u and r sub v is partial derivative of r with respect to v and the same goes for n sub u and n sub v. We have final term of d r dotted with d n as plus r v dotted with n v times d v square. Likewise, the denominator of the previous result d r dotted with d r is equal to r u dotted with r u times d u square plus r u dotted with r v plus r v dotted with r u times d u d v plus r v dotted with r v times d v square.

In terms of the elements of the first fundamental matrix G, r u dotted with r u is G 1 1 times d u square plus 2 times G 1 2 times d u d v, each of these is G 1 2 plus G 2 2 times d v square. This is what d r dotted with d r is. Now, we can use the fact that both the tangents r u and r v are the point on the surface are perpendicular to the normal to the surface at that point. So, we have r u dotted with n is equal to 0 and if we differentiate that with respect to u again, this implies the second partial derivative of r with respect to u dotted with n plus r u dotted with n u equals 0.

This implies r u dotted with n u equals negative of r sub u u dotted began. Now, if you look at this expression here, this corresponds to the first element L of the second fundamental matrix. Likewise, r v dotted with n equals 0 can be differentiated with respect to v to get r v v dotted with n plus r v dotted with n v equals 0. This implies that r v dotted with n v equals minus of r v v dotted with n and what is this? This is equal to minus n the fourth element of d, the second fundamental matrix.

Let us use this relation again and maybe differentiate this with respect to v. So, you will get the mixed derivative here r u v dotted with n plus r u dotted with n b equals 0 and this implies that r u dotted with n v equals minus of r u v dotted with n. And this result here is equal to minus of M and this is the second or the third element of d, the second fundamental matrix. We can likewise differentiate this expression with respect to u to finally get minus of r v u dotted with n equals minus M. Where are we heading? Let us find out.

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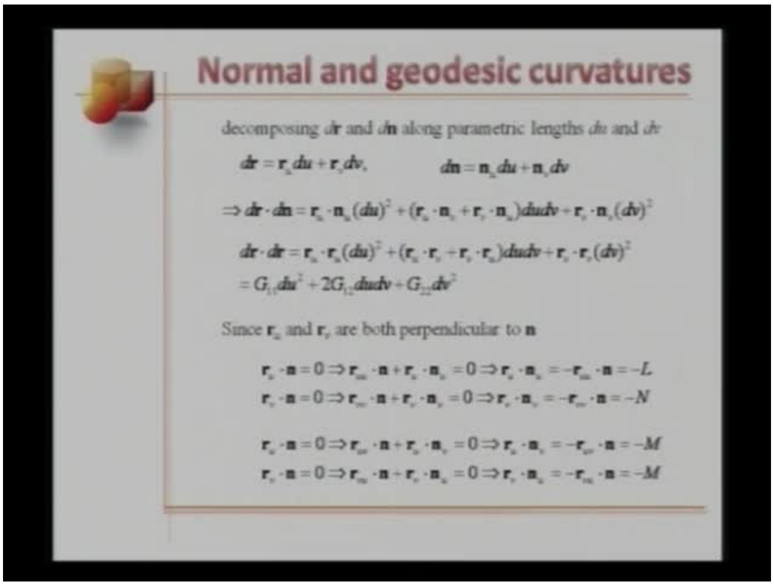

If you notice we were after computing the expression for the normal curvature kappa sub n and this would be equal to L d u square plus 2 M d u d v plus N d v square over G 1 1 d u square plus 2 G 1 2 d u d v plus G 2 2 d v square. This is equal to L plus 2 M times mu plus N times mu square over G 1 1 plus 2 G 1 2 mu plus G 2 2 mu square. Let us get back to the expression here. Let us go back and try to recap what we have done.

So, here we had the expression of normal curvature kappa sub n. It was equal to minus of d r dotted with d n over d r dotted with d r. We expanded d r in terms of the tangents r u and r v, also we did the same for this normal here d n in terms of n sub u and n sub v. This is a quick recap. We computed d r dotted with d n and d r dotted with d r in terms of the elements of the first fundamental matrix G and the second fundamental matrix d. All we have done is we have substituted for the numerator and denominator for kappa sub n, right here.

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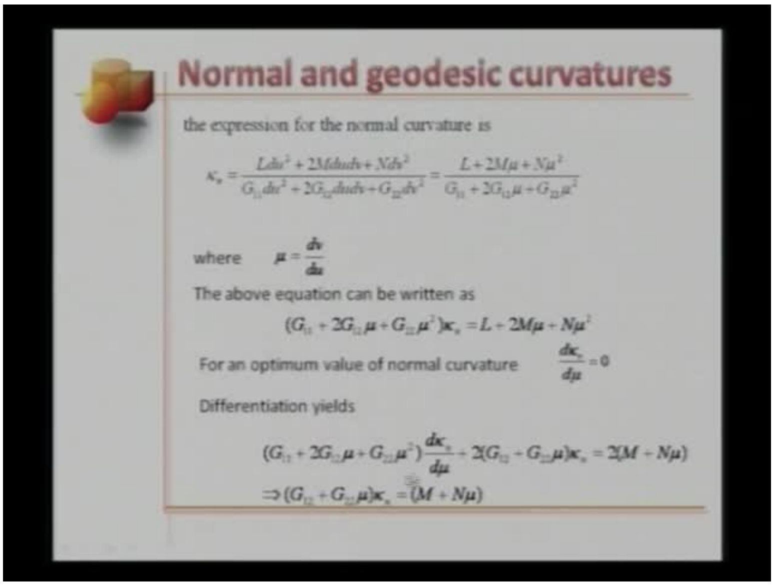

This is d r dotted with d n and this is d r dotted with d r. The minus sign has gotten absorbed in the algebra. What is mu here? mu is d v over d u. All we have done is we have divided the numerator and denominator by d u square respectively. The above equation can of course, be written as G 1 1 plus 2 G 1 2 times mu plus G 2 2 times mu square times kappa n, we are transferring this denominator here and that is equal to L plus 2 M mu plus N mu square.

What do we have here? We have a quadratic N mu. That would allow us may be to compute the optimum value of the normal curvature kappa n. If kappa n is the objective here and if we are trying to find an extremum or extrema for kappa n we will have to use the first order conditions. That correspond to the total of the kappa n with respect to mu and that should be equal to 0 to get a set of stationary points.

So, if we differentiate this expression with respect to mu, we have G 1 1 plus 2 G 1 2 mu plus G 2 2 mu square times total kappa sub n over total mu plus 2 times G 1 2 plus G 2 2 times mu times kappa n and that is equal to 2 times M plus N times mu. So, this here corresponds to the differentiation of the expression on the left hand side and this here corresponds to the same for the expression on the right hand side. All we need to do is we need to set d kappa sub n over d mu equals 0. That would give us G 1 2 plus G 2 2 times mu times kappa n equals M plus N times mu. This expression goes off to 0.

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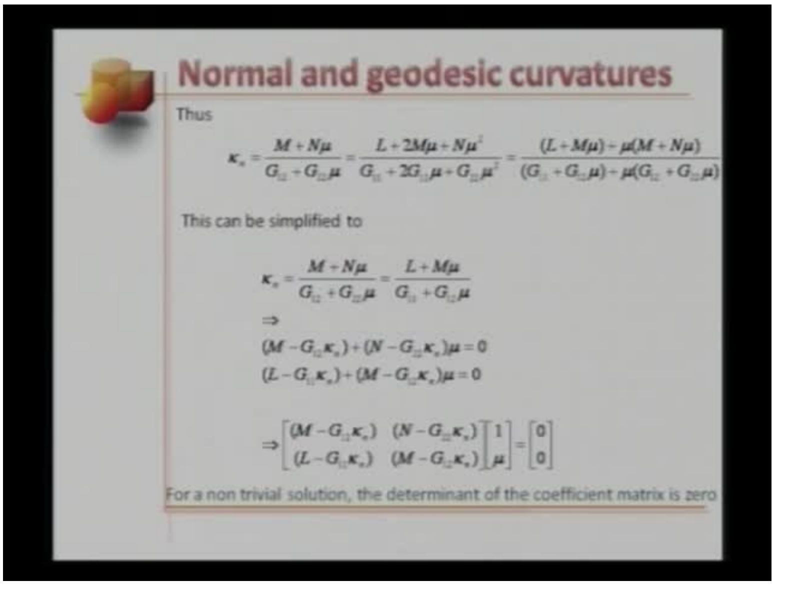

Therefore, kappa sub n is equal to M plus N mu over G 1 2 plus G 2 2 times mu and this is equal to L plus 2 M times mu plus N times mu square over G 1 1 plus 2 G 1 2 mu plus G 2 2 times mu square. Where did this expression come from? If you notice this is how kappa n was determined before. This here gave the original expression for kappa n. So, I can write this expression as L plus M mu plus mu times M plus N mu over G 1 1 plus G 1 2 times mu plus mu times G 1 2 plus G 2 2 times mu. All I have done is I have separated this term here and likewise I have separated this term here and I have rewritten this addition in the numerator here and correspondingly the addition and the denominator here. We can simplify this to kappa n equals M plus N mu over G 1 2 plus G 2 2 mu, which is equal to L plus M mu over G 1 1 plus G 1 2 mu which implies that M minus G 1 2 times kappa n plus n minus G 2 2 times kappa n times mu is equal to 0.

I would advise you to work and get this expression by hand. We also have L minus G 1 1 times kappa n plus M minus G 1 2 times kappa n times mu equals 0. So, this equation here which are two equations if you think about it can be rewritten in these forms. In matrix form I can write these two equations as M minus G 1 2 times kappa n N minus G 2 2 times kappa n L minus G 1 1 times kappa n M minus G 1 2 times kappa n. These four elements form this 2 by 2 matrix times the column vector containing 1 and mu and this is equal to 0 column vector. If you want to have a non trivial solution to these two equations this matrix has to be salient, in other words the determinant of this matrix as to be 0.

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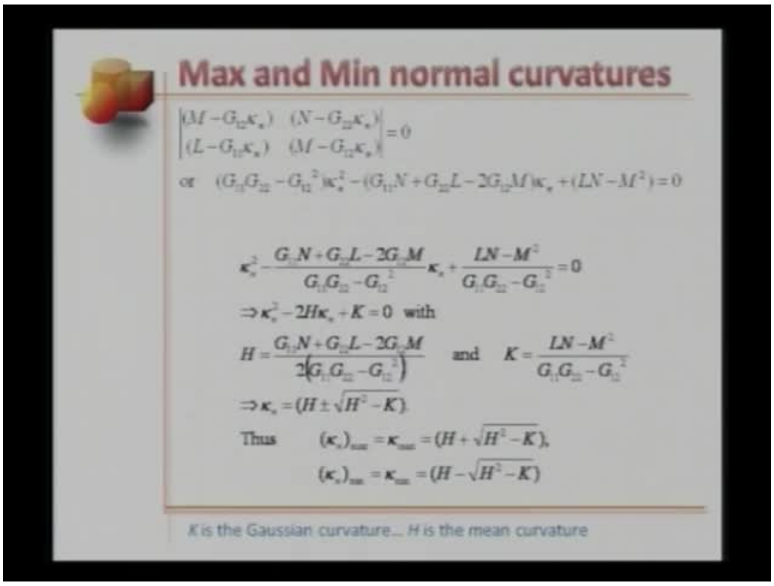

That would give us G 1 1 G 2 2 minus G 1 2 squared times kappa sub n squared minus G 1 1 times N plus G 2 2 times L minus 2 times G 1 2 times M times kappa sub n plus L N minus M squared is equal to 0. We have another quadratic in terms of kappa sub n. So, kappa sub n squared minus G 1 1 N plus G 2 2 L minus 2 G 1 2 M over G 1 1 G 2 2 minus G 1 2 squared times kappa n plus L N minus M squared over G 1 1 G 2 2 minus G 1 2 squared is equal to 0; this equation is the rewritten form of this one.

In short we have kappa sub n squared minus 2 times capital H times kappa sub n plus capital K equals 0, where H has this coefficient here G 1 1 N plus G 2 2 L minus 2 G 1 2 M over 2 times G 1 1 G 2 2 minus G 1 2 squared and K as L N minus M squared over G 1 1 G 2 2 minus G 1 2 squared, which is this constantly. So, if you notice the normal curvature kappa n is expressed in terms of the elements pertaining to the first and the second fundamental matrix. To be more specific we are also talking about the extremum points for kappa sub n.

Of course, we know by now how to solve a quadratic equation kappa sub n equals H plus minus square root of H squared minus K. So, we will have one extremum kappa n max or K max as H plus square root of H squared minus K and a minimum kappa sub n min equals kappa min which is equal to H minus square root of H squared minus K. This co efficient here H is called the mean curvature and K is called the Gaussian curvature. So, today we did quite a bit of math. Let me try to solve an example for you on board so that we can absorb the discussions in this lecture. Let us try to determine the Guassian and mean curvatures for a monkey saddle.

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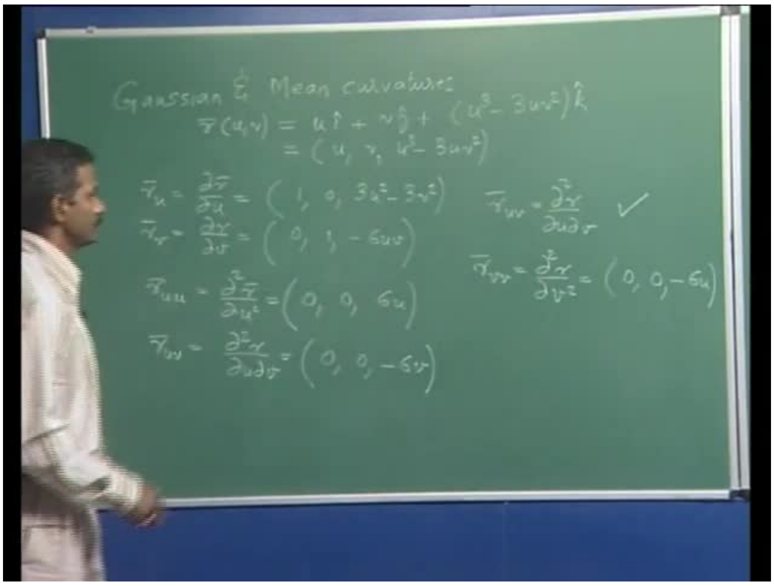

The monkey saddle is given by r a function of two parameters u and v as u times vector i plus v times vector j plus u cube minus 3 u v squared times vector k. I will show you later how this looks graphically, but for now let us try to compute the elements of the first and the second fundamental matrices. In ordered form I can write this thing as u comma v comma u cube minus 3 u v squared. So, we have r sub u which is partial r over partial u, this is equal to 1 0 3 u squared minus 3 v squared. We have r sub v equals partial r over partial v, this is equal to 0 1 this would be 0 will have minus 6 u v.

You would need this second partial derivatives as well. Let us try to compute them. So, r u u equals partial 2 r over partial u square will be equal to 0 0 derivative of this expression with respect to u this would give us 6 u, partial r u v equals partial 2 r over partial u partial v which is equal to I will have to differentiate this with respect to v, so I will have 0 0 and this is minus of 6 v. The mixed second derivative r u v is partial 2 r over partial u partial v, I have already computed that. The second derivative of r with respect to v r v v equals partial 2 r over partial v square, this is equal to, we will have to differentiate this expression here with respect to v will have 0 0 and minus 6 u.

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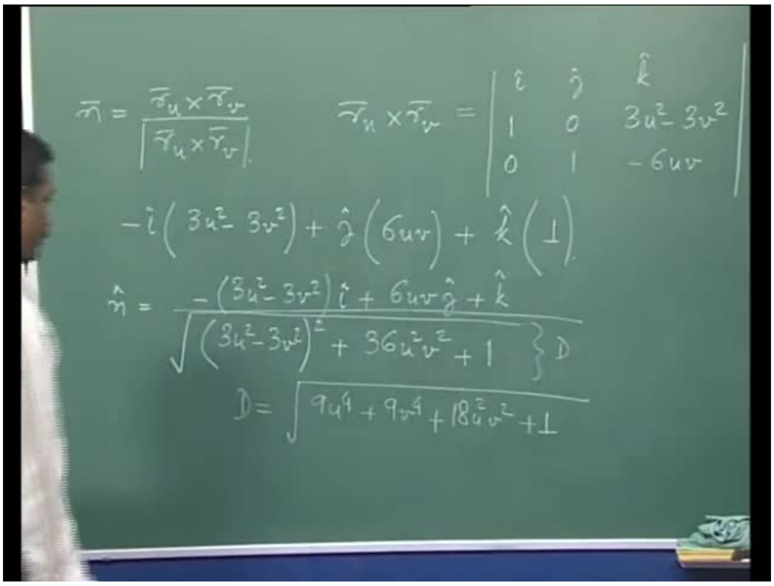

We would also need the expression for the normal as you would know this is equal to r u crossed with r v over the absolute value of this numerator r u cross with r v. Let us try to determine this cross product first, r u cross with r v is equal to the determinant of i j k. r u is this expression here 1 0 3 u squared minus 3 v squared, r v is this expression here 0 1 minus of 6 u v which expand this determinant. So, we have i times 0 minus this times this which is minus of 3 u squared minus 3 v squared plus j times this times this minus of this times this, this minus sign goes off we have 6 u v plus k times this times this minus this times this, we have 1.

So, the unit normal n is equal to minus 3 u squared minus 3 v squared times i plus 6 u v times j plus k over square root of 3 u squared minus 3 v squared, the whole squared plus 36 u squared v squared plus 1. Let me write this denominator as D and I can simplify D as under root of 9 u 4 plus 9 v 4 minus 18 u squared v squared, we will have 36 u squared v squared from here. So, this becomes plus 18 u squared v squared plus 1. We will be needing this side of the board. So, what I will do is I will transfer this result right over here.

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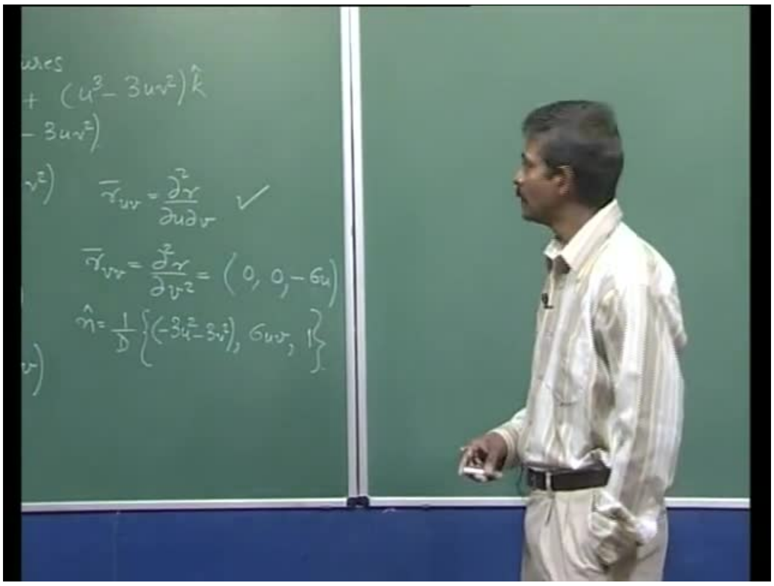

So, n cap equals 1 over D where you know what D is, times minus 3 u square minus 3 v squared comma 6 u v comma 1 in the order form. Let us start computing the elements of the first and the second fundamental matrices.

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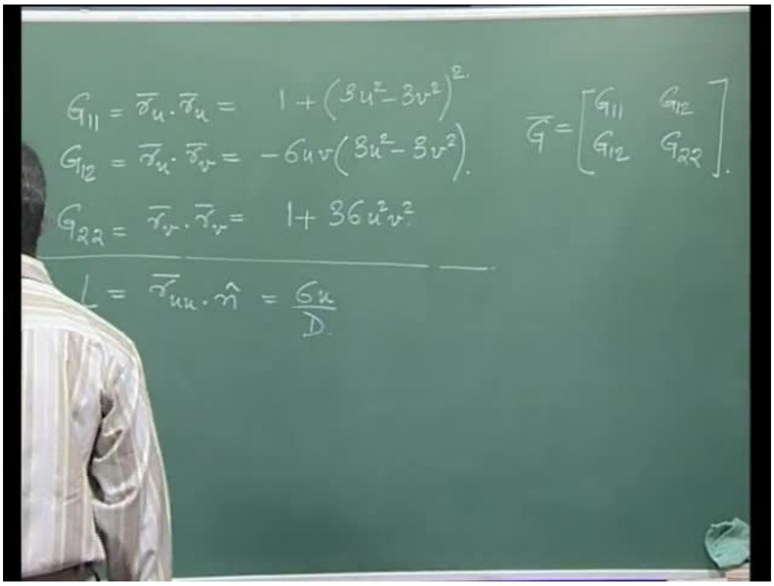

G 1 1 equals r u dotted with r u and that is given by 1 plus this squared 3 u squared minus 3 v squared the whole square. G 1 2 is given by r u dotted with r v which is this dotted with this. So, this multiplied by this is 0, this multiplied by this is 0. We are left with minus 6 u v times 3 u squared minus 3 v squared and G 2 2 equals r v dotted with r v and that is equal to 1 plus 36 u squared v squared.

You know what the first fundamental matrix is given by G 1 1 G 1 2 G 1 2 which is the same G 2 1 G 2 2, how about the second fundamental matrix? The elements are given by L equals r u u dotted with n. r u u is here, n is here, so what do we have? This dotted with 0 plus this dotted with 0 plus 6 u. So, we have 6 u over D here. Let us not confuse this simple D with the second fundamental matrix itself. Let me change that symbol.

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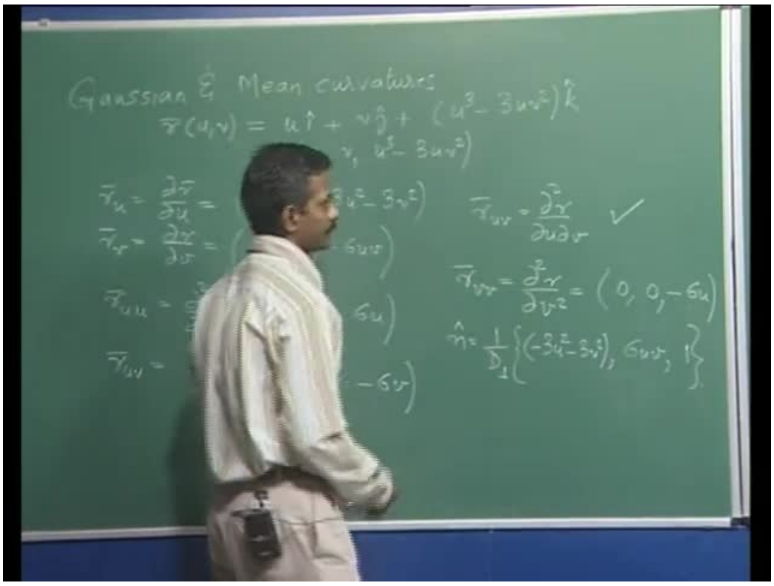

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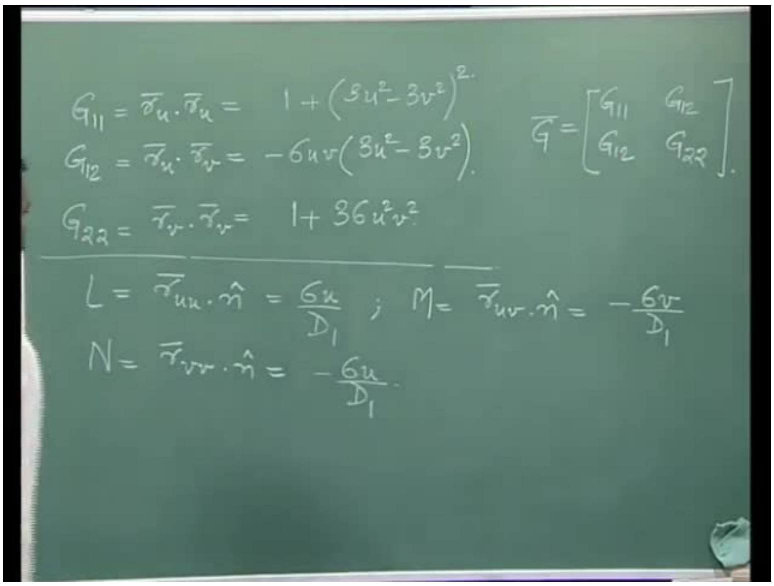

Let me call D as D sub 1 just to signify that D 1 is nothing but the absolute value of this vector. So, we have D 1 here. The second element M is given as r u v dotted with n, this is equal to minus of 6 v over D 1, this dotted with 0 plus this dotted with 0 plus this dotted with minus 6 v; and the third element N is r v v dotted with n, which is equal to minus 6 u over D 1.

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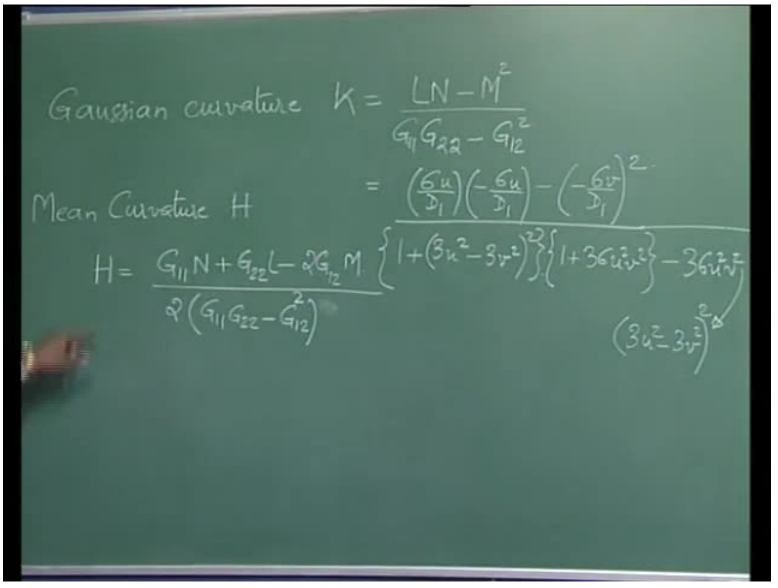

The Gaussian curvature K is given by L N minus M squared over G 1 1 G 2 2 minus G 1 2 squared. This is equal to 6 u over D 1 times minus 6 u over D 1 minus minus 6 v over D 1 squared over G 1 1 is 1 plus 3 u squared minus 3 v squared squared G 2 2 is 1 plus 36 u squared v squared minus G 1 2 squared, which is 36 u squared v squared times 3 u squared minus 3 v squared, the whole square.

The mean curvature H is given by H equals G 1 1 N plus G 2 2 L minus 2 G 1 2 M over 2 G 1 1 G 2 2 minus G 1 2 the squared term appears here. We have the expression for G 1 1 G 2 2 and G 1 2 from here. You also know what N L and M are from here. All you would need to do is substitute for these variables or elements to get a mean curvature. Once, you have the Gaussain and mean curvatures you can compute the minimum and maximum values of kappa n, the normal curvatures.

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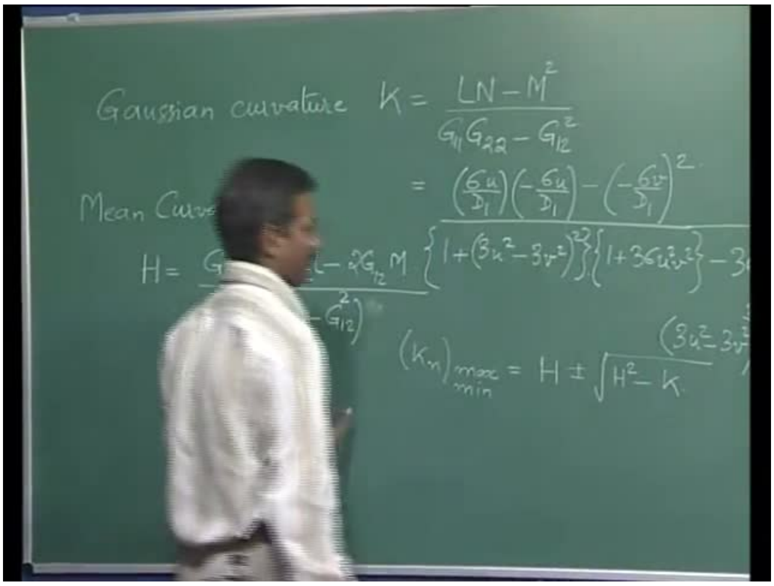

Kappa n max min as you know is given by H plus minus under root of H squared minus K.

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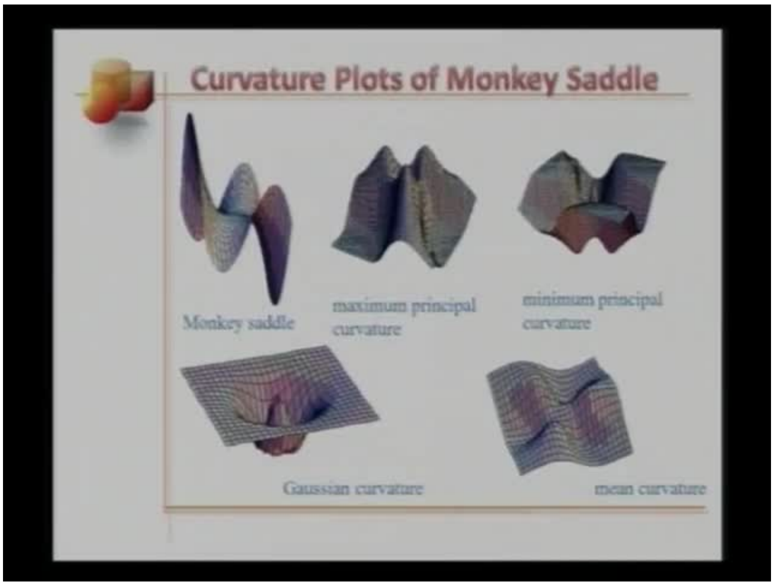

For the example that we have just worked out this figure here shows the monkey saddle. This shows the maximum principle curvature kappa sub n, shows the minimum principle of curvature kappa sub n, this is the plot for the Gaussain curvature and this is the plot for the mean curvature.

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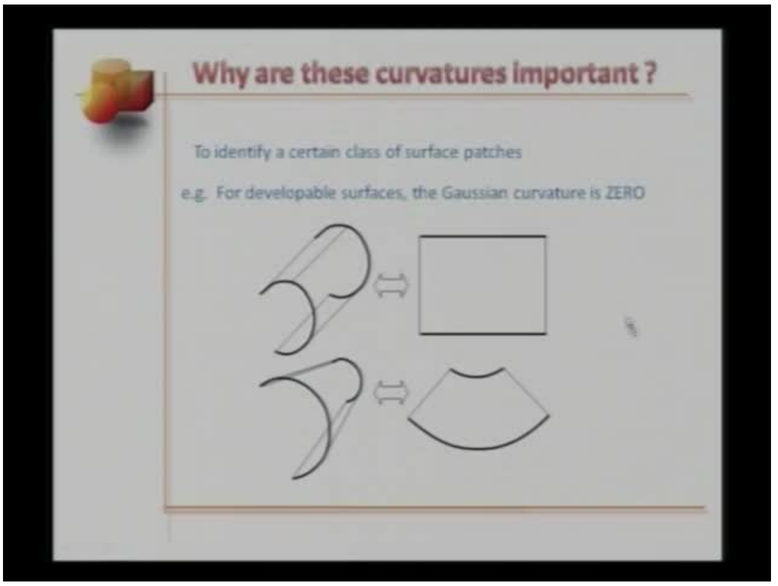

Why are these curvatures important after all? With the help of these curvatures, we can identify certain classes of surface patches for example, for developable surfaces the Gaussain curvature is 0 as you would know cylindrical and conical surfaces are developable surfaces for which the Gaussion curvature is 0.