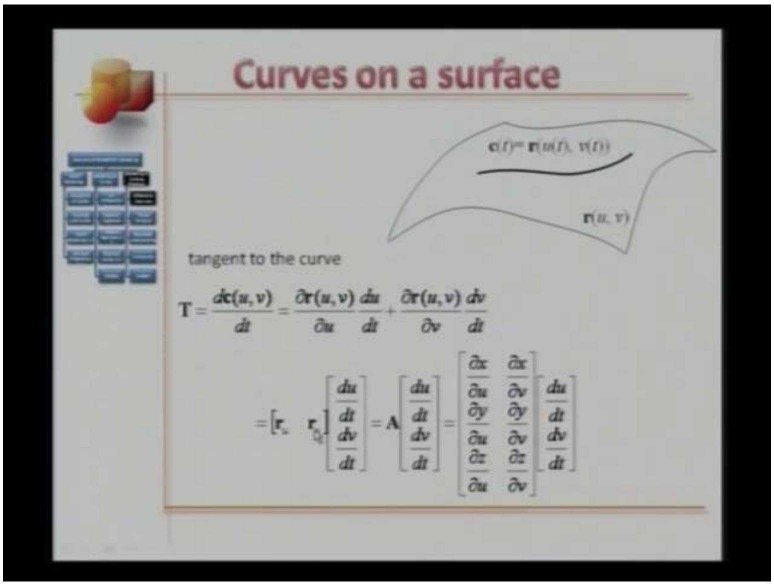
Computer Aided Engineering Design
Prof. Anupam Saxena
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 34

Good morning and welcome again, we continue with our discussion on differential geometry of surface patches. This is lecture number 34.

(Refer Slide Time: 00:31)

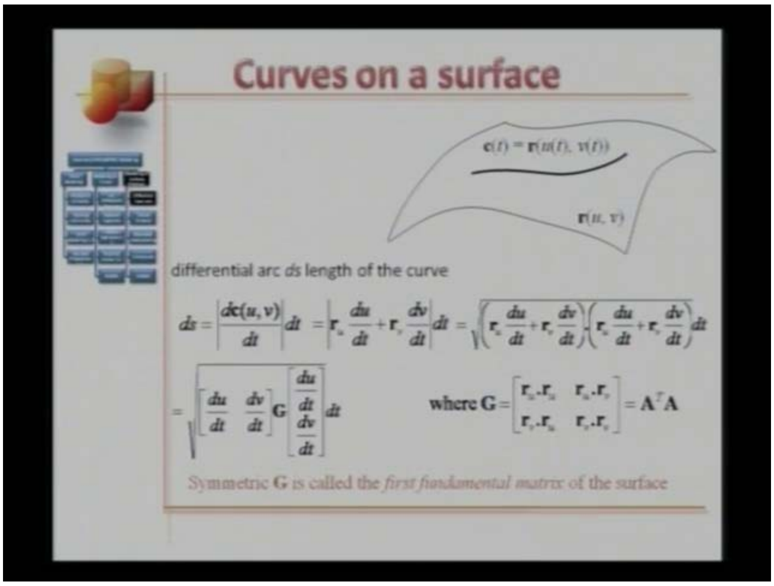


Let us talk about curves on a surface patch, this is a typical surface patch given by a vector function r of u and v the two parameters. Let us say we have a curve and this curve is represented by c of t, less compute the tangent to this curve at any point. The tangent is given by total c over total t, c is a function of u and v here. Using the change row we have partial of r of u and v over partial of u times d u over d t plus partial r over partial v times d v over d t.

We can do the math and I can express this right hand side as a multiplication of two matrixes are row matrix and column matrix. This row matrix comprises the tangent along the u direction, the tangent along the v direction. r sub u is given by partial r over partial u, r sub v is given by partial r over partial v and then in the column you will have d u over d t and d v over d t. You can think of writing this row vector as bold a and then we have this column vector write here.

In terms of scalar functions x, y, and z where each function x, y, and z is dependent on parameters u and v. We can write a as partial x for partial u partial x over partial v that is in the first row, in the second row we have partial y over partial u partial y over partial v and in the last row partial z over partial u and partial z over partial v. This is not surprising, because both these tangents r sub u and r sub v are vectors and these are the components of these vectors. So, this column here are components of r u and this column right here are the components of r v.

(Refer Slide Time: 03:50)



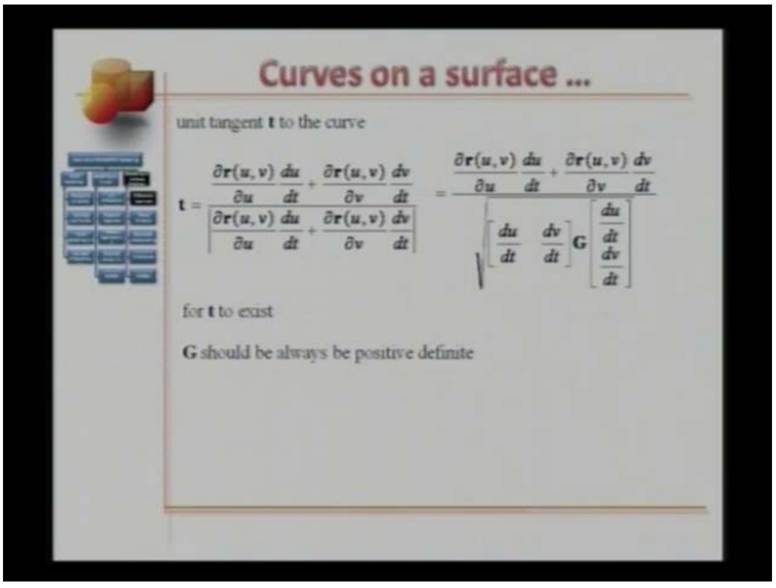
Let us now compute the differential arc length d s over a portion of this curve. Now, d s is given by the absolute value of total c over total t times d t. Now, d c over d t can be written as r sub u times d u over d t plus r sub v times d v over d t and then we have this d t here. Instead of using the modulus sign I can replace the expression by this one, the radical sign and remember this is a vector. So, what we have doing is we are dotting this vector by itself. So, within the radical sign we have r sub u d u over d t plus r sub v d v over d t dotted with itself.

We can write this result in a matrix point hear, if you look at this expression this can be written as the rho matrix d u over d t, d v over d t times the column matrix r u r v dotted with r u r v times d u over d t, d v over d t times d t. I can compute the dot product now. So, we have d s equals within the radical sign the row matrix d u over d t d v over d t times this 2 by 2 matrix here now, whose elements are r u dotted with r u, r u dotted with r v, r v dotted with r u and r v dotted with r v. And then we have this column matrix d u over d t d v over d t.

This 2 by 2 matrix has a special name an accordingly a special significance, will come to that. What I would request you is to remember this expression of this matrix. We write that 2 by 2 matrix in short as capital G in bold you rest of the expressions remain that. As a set G is this 2 by 2 matrix with r u dotted r u, r u dotted r v, r v dotted with r u, r v dotted with r v as four elements.

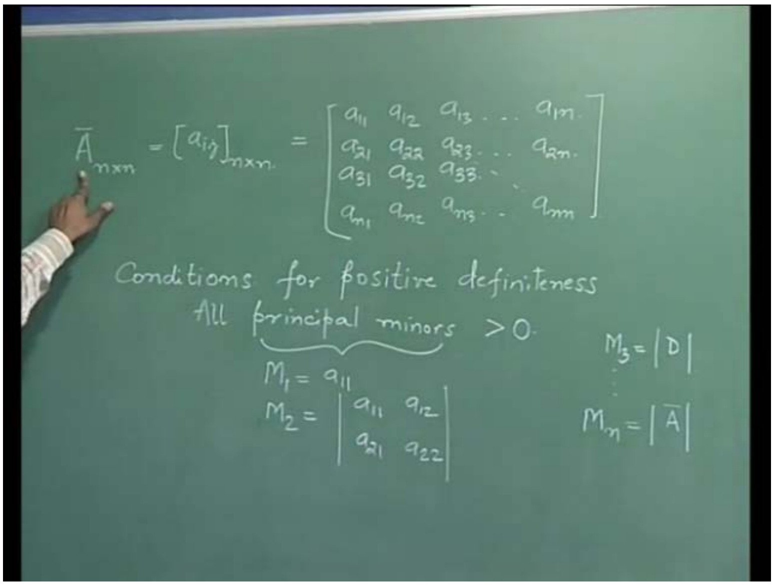
Recall from our previous discussion what A was, in terms of A G can be written as A transpose A. Let me take you to where we describe A, so this matrix A is the components of r u arranged in the first column and the components of r sub v arranged in the second column. If you realize G is the symmetric 2 by 2 matrix and it is called the first fundamental matrix of the surface. The definition of G is related in the slide to be infinites mile arc length t s, along curve c t. We will now see certain properties of G, one of the properties that I had mentioned is that G is symmetric.

(Refer Slide Time: 09:09)



Well us talk about the unit tangent bold t to the curve, the unit tangent t is given as partial r over partial u which, is r sub u in short times d u over d t plus partial r over partial v times d v over d t, over the absolute value other modulus of this vector. From before we had seen that the modulus of r sub u d u over d t plus r sub v d v over d t is given as square root of d u over d t d v over d t, this row matrix times G, that we have define in the previous slide times d u over d t d v over d t. And we retain this numerator right here to compute the unit tangent. We have to insure that the denominator in particular this expression here should be non zero and it should be positive. That is for the unit tangent t to exist at any point on the curve c of t on a sub patch bold G should always b positive definite.

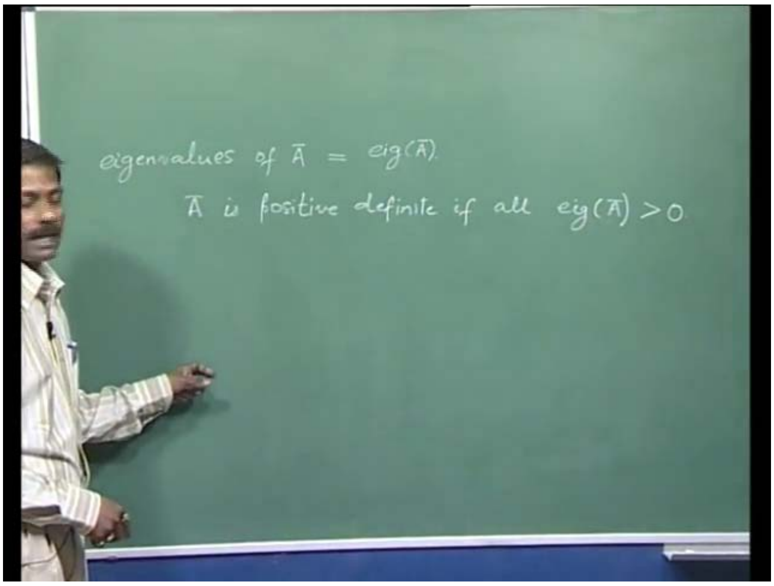
(Refer Slide Time: 11:27)



Well what do I mean by a positive definite matrix, say we have a square matrix of size n by n and this matrix has elements a i j, in the expanded form I can write this matrix as a 1 1, a 1 2, a 1 3, up to a 1 n, a 2 1, a 2 2, a 2 3, up to a 2 n and the end through as a n 1, a n 2, a n 3 a n n. For this matrix to be positive definite, we has certain conditions. Conditions for positive definiteness, all principal minors have to be greater than 0.

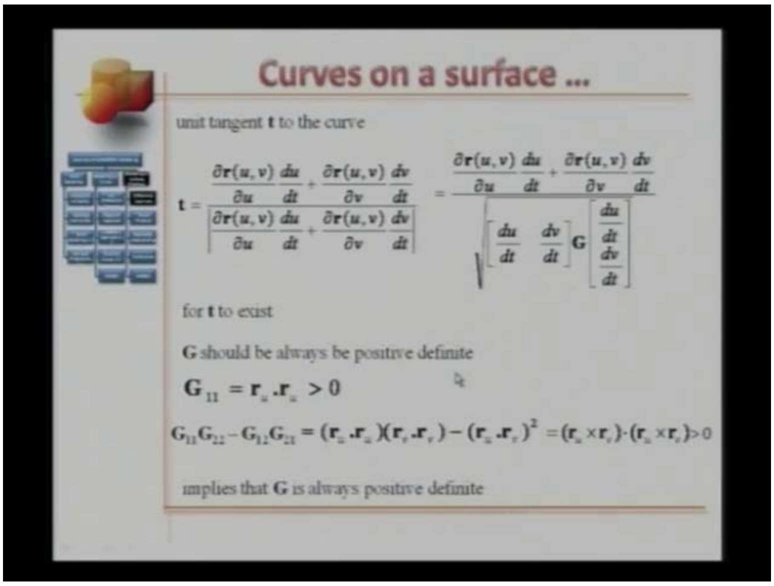
Now, what are principle minors, they are the set of determines that you construct on this (( )). For example; the first principle minor will simply be the first element a 1 1, the second principle minor will be the determinant are this 2 by 2 matrix. If you right the third row here, the third principle minor M 3 will be the determinant of this 3 by 3 matrix, a 1 1, a 1 2, a 1 3, a 2 1, a 2 2, a 2 3, a 3 1, a 3 2, a 3 3 in short is written as D here. We can continue and till the last principal minor M n which is the determinant of the matrix itself. Alternatively we can also describe positive definiteness of the n by n square matrix in terms of the second values.

(Refer Slide Time: 15:00)



If I write the Eigen values this matrix, as I said e i g of A, then a is positive definite if all eigen values of A are strictly positive. If some of them are equal to 0 then we call a to be positive some i definite. If some of the Eigen values are negative, then a is not positive definite. Let us come back to where we were we had said that for a unit tangent t to exist on a curve the matrix G the first fundamental matrix should always be positive definite.

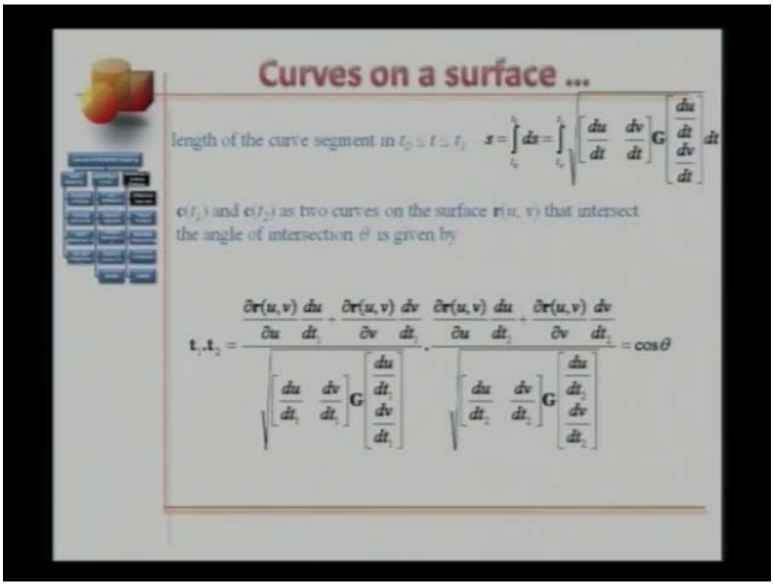
(Refer Slide Time: 16:09)



The first principle minor G 1 1 as you know is given by r sub u dotted with r sub u and for r sub u not equal to 0, this dot product is always greater than 0. So, the first principle minor of G is positive, the second principle minor is given by the determinant of G itself which is G 1 1 G 2 2 minus G 1 2 times G 2 1. The first element, the forth element, the second element in the first row and the first element in the second row. In terms of its constituent's determinant of G is r u dotted with r u times r v dotted with r v minus r u dotted with r v square.

But in do a better algebra to figure that this expression is nothing but r u cross with r v dotted with itself and you are seen that r u crossed with r v is the normal to a curve, in therefore, to a surface fact at that point the dot product of. The normal with itself has to be greater than 0 and sense the first two and the only two principle minors of G are greater than 0, G is impact positive definite.

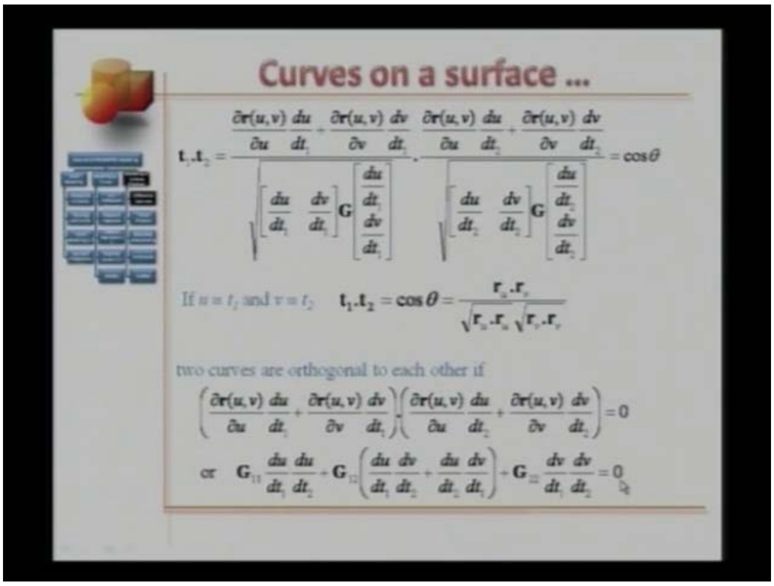
(Refer Slide Time: 18:48)



So, the length of the curve segment for parameter value t in between t sub 0 and t sub 1 is given as, s equals integration from t sub 0 to t sub 1 of d s which, is equal to integration from these two elements t 0 to t 1 of the square root of d u d t d v d t row matrix times the first fundamental matrix G times, the column vector d u over d t d v over d t times d t. So, this d t appears outside this for would sign. Let say we have two curves c as a function of some parameter t sub 1 and another one, c as a function of parameter t 2 which is deferent from t 1 and these two curves lie on the surface patch r of u and v and let these 2 curves intersect at angle theta.

This angle is given by the dot product between the two local tangents t 1 dotted with t 2 which, is equal to partial r or partial u times d u over d t 1 plus partial r over partial v times d v over d t 1 over the square root of d u over d t 1 d v over d t 1 times G times d u over d t 1 d v over d t 1 arranged in a column. So, here parameters u and v are assume to be functions of parameter t 1 perceptively. Likewise the second tangent t sub u is given by r sub u times d u over d t 2 plus r sub v times d v over d t 2 over the square root of d u over d t 2 d v over d t 2 times G and this column we have d u over d t 2 d v over d t 2. And sense t sub 1 and t sub 2 are unit tangents, this dot product is simply the angle rather the cosign on the angle between that.

(Refer Slide Time: 22:10)

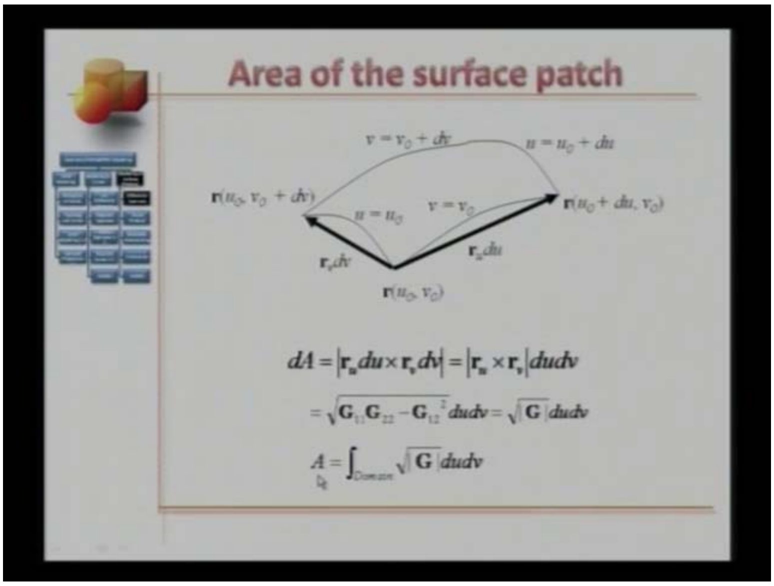


Let me copy the previous result here. Now, let us say if u is parameter t 1 itself and if v is parameter t 2 itself, then t 1 dotted with t 2 is given by r sub u dotted with r sub v over the square root of r u dotted with r u, times the square root of r v dotted with r v. You can verify this if u equals t 1 this expression is 1 if v equal to t 2 t 1 and t 2 are independent parameters. So, this result is going to be 0. Likewise this result is going to be 0 because u is t 1 and d t 2 over d t 2 where v is t 2 will be 1.

So, in the numerator will have r sub u, this expression goes to 0, in the denominator here will have 1 and 0 and this is 1 and 0. Correspondingly this expression can be simplified to r sub u dotted with r sub u and something similar can be computed for this term here. The numerator will be r sub v times 1, this expression will go to be 0 the denominator will have 0 and 1 and this row matrix 0 and 1 in this column vector. The multiplication here will result in the dot product r sub v dotted with r sub v within this square root sign. And of course, two curves are orthogonal of perpendicular to each other, if the cosign of the angle between them as the end section point is 0, that is if this dot product is 0.

Let me spell this out for you partial r over partial u times d u over d t 1 plus partial r over partial v times d v over d t 1 dotted with partial r over partial u times d u over d t 2 plus partial r over partial v times d v over d t 2. You can do your algebra simplify your result to get G 1 1 d u over d t 1 times d u over d t 2 plus G 1 2, note that G 1 2 and G 2 1 are the same and G 1 2 and G 2 1 are actually r u dotted with r v. Here we have d u over d t 1 times d v over d t 2 plus d u over d t 2 times d v over d t 1 plus G 2 2 which is r v dotted with r v times d v over d t 1 times d v over d t 2, this is a general expression. Here we not assuming that u is t 1 and v is t 2, for the two curves to be orthogonal this dot product between the two tangles has to be equals to be 0.

(Refer Slide Time: 26:43)

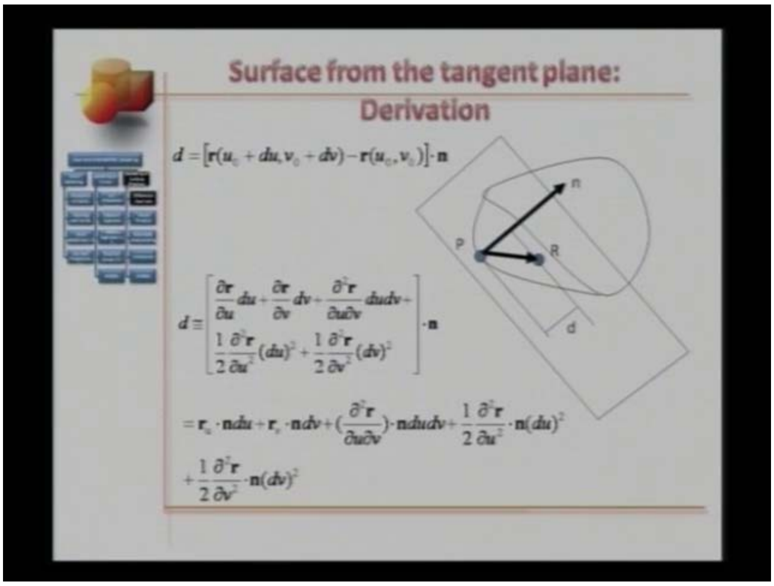


And let us try to compute the area of a surface patch. So, let us draw the bounding curves, this is u equals u sub 0 and u equals u sub 0 plus d u. This bounding curve corresponds to v equals v sub 0 and this point here is v equals v sub 0 plus d v. Let us say this point is the position vector r at u equals u sub 0 and v equals v sub 0, the values of the 2 parameters. This point here will correspond to r of u sub 0 plus d u and v sub 0, and this point here will be r of u sub 0 v sub 0 plus d v. This vector here is given by r sub u times d r sub u is partial r over partial v. This vector here is given by r sub v times d v, notice that this patch is very small size its infinite small size.

So, Taylor series expansion is allowed, the area of this patch is given by d A which, is equal to the absolute value of the cross product between these two vectors, r sub u times d u crossed with r sub v times d v, these are positive scalars. So, we can extract them out of these absolute values. We have the absolute value of r sub u crossed with r sub v times d u d v. If you would have realize from a previous discussion the absolute value of the cross product between r u and r v is the determinant of the first fundamental matrix G and of course, this square root sign. So, I can right d A as square root of G 1 1 times G 2 2 minus G 1 2 square and outside this sign, we have t u times t v as a set this expression here is the determinant of the first fundamental matrix write here.

So, if I want compute the area of the entire surface patch, I have to integrate over the entire domain that is over the values conversable values of u and v. So, A will be integration over the domain of the A which, is within the square root sign determinant of G times outside this for sign d u d v. You would also realize that it is the positive definiteness on this case on the particular. The second principal miner of G which is greater than 0, that allows computation of the area of the surface patch. Now, let us discuss relation between the surface and the tangent plane.

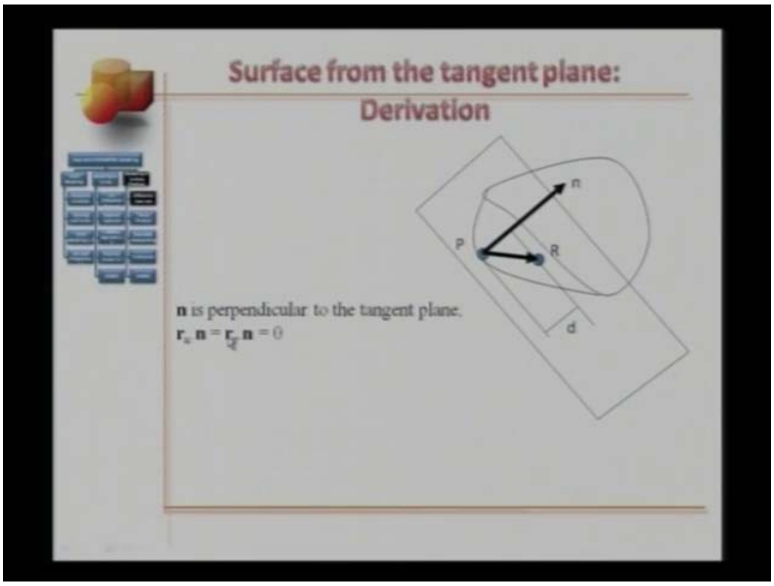
(Refer Slide Time: 31:41)



Let us see have the surface, we have a point on this patch this called P and we have a tangent plane passing through P. This tangent plane may or may not intersect with this surface patch. Let us try to figure the conditions for which, this plane will simply the plane or tangent plane to the surface or it may or may not intersect in the surface patch. Let us plot another point on surface and call it R. So, this is vector P R, this vector here is normal to the tangent plane this call it n. This distance d here can be seen as the projection of P R on this normal n.

So, f at P the position vector is r u 0 v 0, which is this point here and f at r the position vector is r u 0 plus d u v 0 plus d v. Then P R is given by the position vector of r minus the position vector of P, d is the projection of P R on this normal. So, d is dot product between P R and n, sense d u and d v are small, we can use Taylor series expansion. So, this projection d can be approximated as partial r over partial u times d u plus partial r over partial v times d v plus partial 2 r over partial u partial v times d u d v. These correspond to the next derivative plus half of partial 2 r over partial u square times t u square plus half of partial 2 r over partial v square t v square.

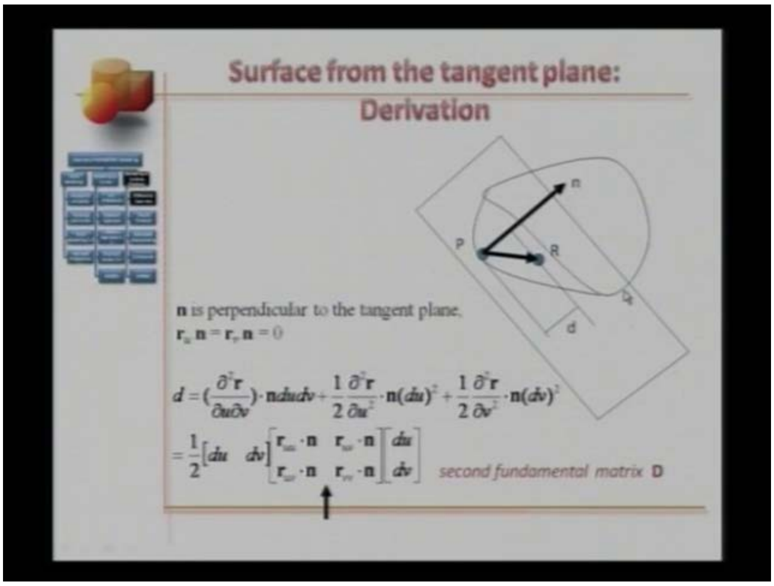
(Refer Slide Time: 36:30)



If you wondering this expression is the expansion, the Taylor series expansion up to second order terms. And of course, we have dotted with n here, let us try to simplify. So, this expression on the right hand side is equal to r sub u dotted with n. So, this expression r u times d u plus r sub v is here dotted with n times d v plus partial 2 r over partial u partial v dotted with n d u d v plus half of partial 2 r over partial u square dotted with n d u square plus half partial 2 r over partial v square dotted with n times d v square. Let us try to simplify this expression little further.

Just in case if you have missed the previous construction, we have surface patch we have a point on it P, to this point passes a tangent plane. We have another point are on this surface patch, we construct a vector P R, we let normal n pass through point P and this normal n is perpendicular to the tangent and we trying to compute the projection of P R on this normal n here. Now, you would note that n is perpendicular to the tangent plane, because of which r u dotted with n 0 and also r sub v dotted with n is 0 recall from before that vectors sub u and r sub v, in fact define the tangent plane. So, what will happen to this expression, sense this dot product 0 and this dot product is also 0.

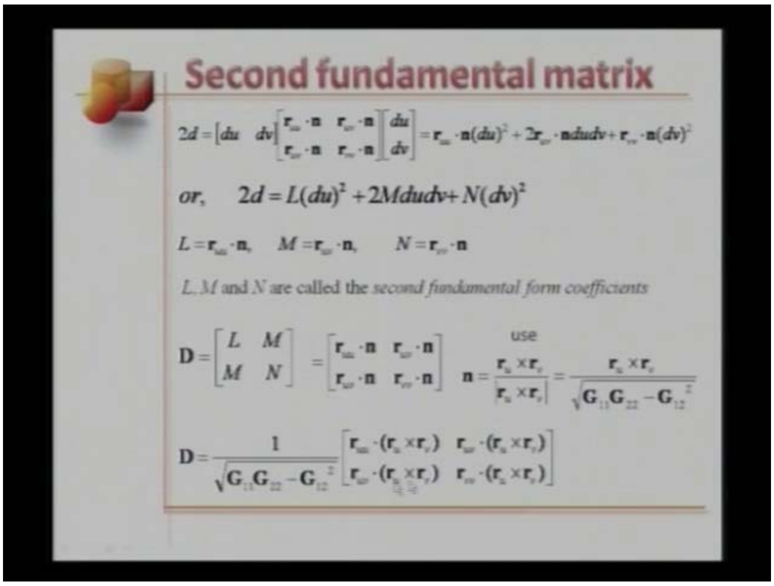
(Refer Slide Time: 38:14)



We have d equals partial 2 r over partial u partial v dotted with n d u d v plus half of partial 2 r over partial u square dotted with n t u square plus half a partial 2 r over partial v square dotted with n d v square. And we can express this right hand side in the matrix form as half of this row vector d u d v times, some matrix will come to that in a while, times the column vector d u d v.

Coming back to this 2 by 2 matrix the constituents are r u u dotted with n, r u v dotted with n, r u v dotted with n, r v v dotted with n. This is the second derivative of r with respect to u. Next, second derivative of r with respect to u and v, the same here and the second derivative of r with respect to parameter v, this matrix here as an a, is called the second fundamental matrix and is represented by bold D. This matrix allows us to classify points on the surface patch let us work towards sign.

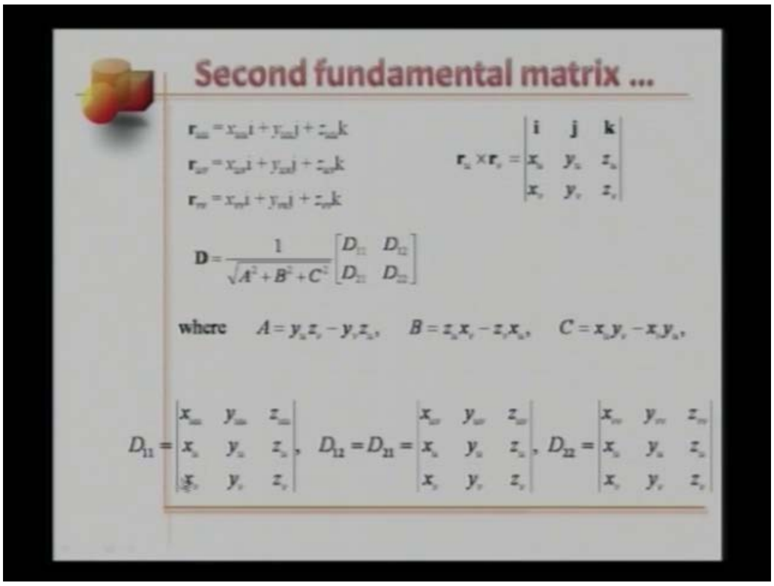
(Refer Slide Time: 40:25)



So, we have 2 d equals row matrix d u d v times, this 2 by 2 matrix the second fundamental matrix, d times column vector d u d v in algebraic form. This is r u u dotted with n d u square plus 2 r u v dotted with n d u d v plus r v v dotted with n d v square. Let us give some names to this dot product. So, r u u dotted with n lets its L r u v dotted with n is M and r v v dotted with n is N. I just said that L M and N can be called as the second fundamental form coefficients. So, this second fundamental matrix D is written as L M M and N, which is r u u dotted with n, r u v dotted with n, r u v dotted with n and r v v dotted with n.

Now, what is normal in terms of the partial derivatives of r with respect to u and respect to v. We know this from before the unit normal is the cross product between r sub u and r sub v, over the absolute value of this numerator here which, is equal to r u crossed with r v over 3 square root of the determinant of the first fundamental matrix, G 1 1 G 2 2 minus G 1 2 square. So, you use this result here, we get the second fundamental matrix D as 1 over the square root of the determinant of G times r u u dotted with r u cross r v, r u v dotted with r u crossed r v, the same element here and r v v dotted with r u crossed with r v, so these are scalar products. So, given a surface patch as a function of 2 parameters u and v you can compute both the first fundamental matrix G and the second fundamental matrix G.

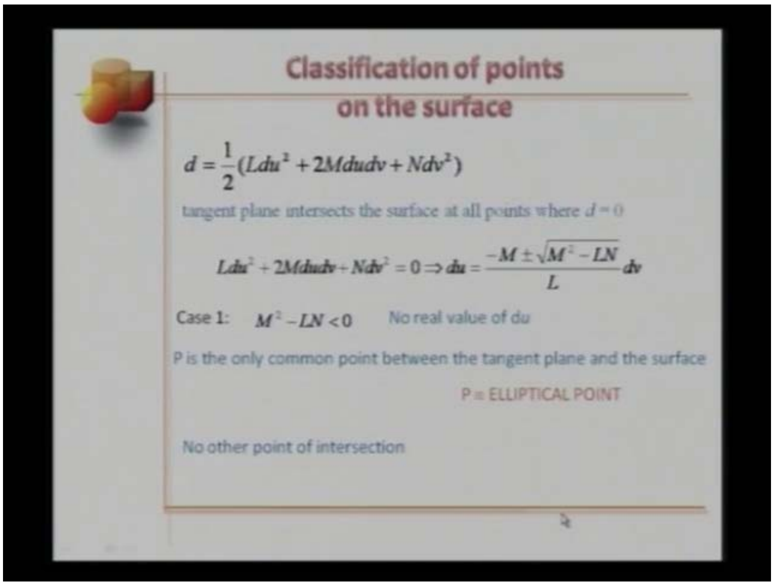
(Refer Slide Time: 44:08)



This is straight forward, the second partial derivative of r with respect to u is x u u times i plus y u u times j plus z u u times k, i j k are unit vectors along the x y and z, that is importance. Likewise you can compute the next second derivative and the second derivative with respect to parameter v, x y and z are scalar views along the x y and z coordinates. You can do this r u cross with r v is i j k, x u y u z u, x v y v z v the determinant of this 3 by 3 matrix.

So, the second fundamental form is equal to 1 over the square root of A square plus B square plus C square and then we have 2 by 2 matrix here with elements D 1 1 D 1 2 D 2 and D 2 2. A is y u times z v minus y v times z u, B is z sub u x sub v minus z sub v x sub u and C is x sub u y sub v minus x v y u. D 1 1 is the determinant of 3 by 3 matrix x u u y u u z u u, x u y u z u, x v y v z v. D 1 2 is the same as D 2 1 which is the determinant of the second 3 by 3 matrix x u v y u v z u v, x u y u z u, x v y v z v and of course, the forth element D 2 2 here is the determinant of x v v y v v z v v, x u y u z u, x v y v z v. So, in a sense if a surface patch is given as a function of u and v, as a set for the second fundamental form can be computed using this algebra.

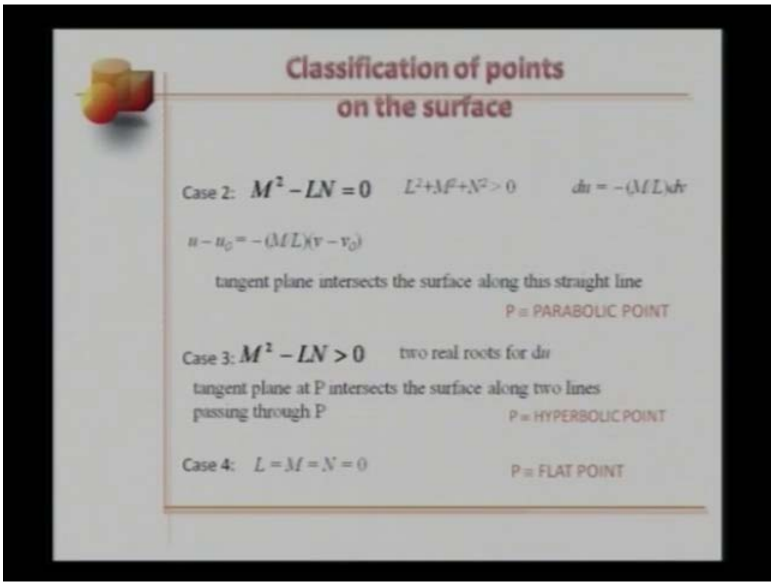
(Refer Slide Time: 47:05)



Let us classify some points on the surface know base on this, base on this second fundamental form. Let us start with the projection of a position rather a free vector P R where P n R are 2 points on a surface patch and 2 P a tangent plane passes, So, d is equals to half of L d u square plus 2 M d u d v plus M d v square. So, if tangent plane would intersect a surface, it would do so at all points where d is 0. That would mean that L d square plus 2 M d u d v plus N d v square is equal to 0. So, this is a quadratic equation in both d u and d v.

So, I can solve d u in terms of d v and the result is minus M plus minus square root sign M square minus L N over L. Of course, depending on how this expression behaves M square minus L N will have no 1 or multiple solutions. Case one; we do not have any real value of d u which, implies that we do not have any point on the surface, through which in this section plane on the tangent plane passes. P is the only common point between the tangent plane and the surface and there is no other point of intersection. So, that point is called the elliptical point.

(Refer Slide Time: 49:31)



Case two; if M square minus L N is equal to 0, we have L square plus M square plus N square is greater than 0. So, d u is given by minus of M over L times d v, I can write d u as u minus u 0 and I can write d v as v minus v 0. Well if you recall point P was the position vector corresponding two parameter values u sub 0 and v sub 0. So, this expression here is u minus u 0 equals minus of M over L times v minus v 0. Basically the tangent plane would intersect the surface along this straight line this one, these points are termed as parabolic points. Case three; we have M square minus L N is strictly greater than 0, will have two real roots for d u and so the tangent plane at point P will intersect the surface patch along two lines passing through P. So, we are talking about hyperbolic points. Finally case four where L is equal to M is equal to N which is equal to 0, the point P is called of flat point.