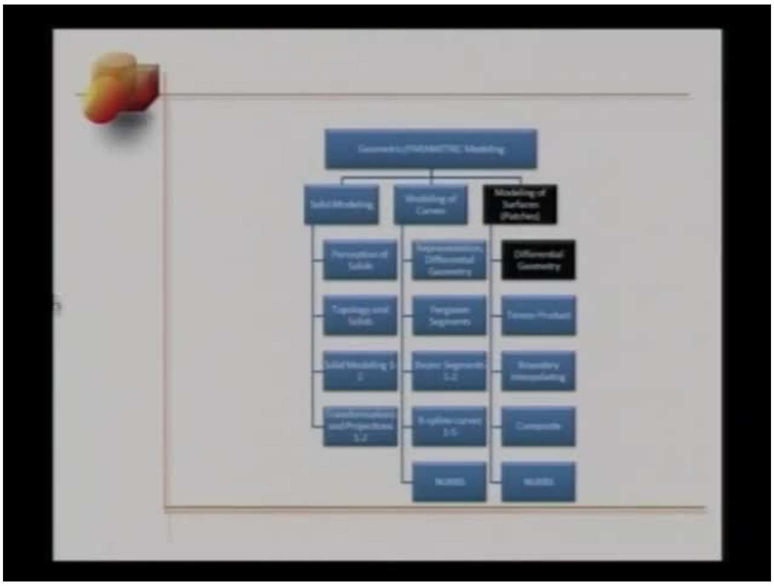
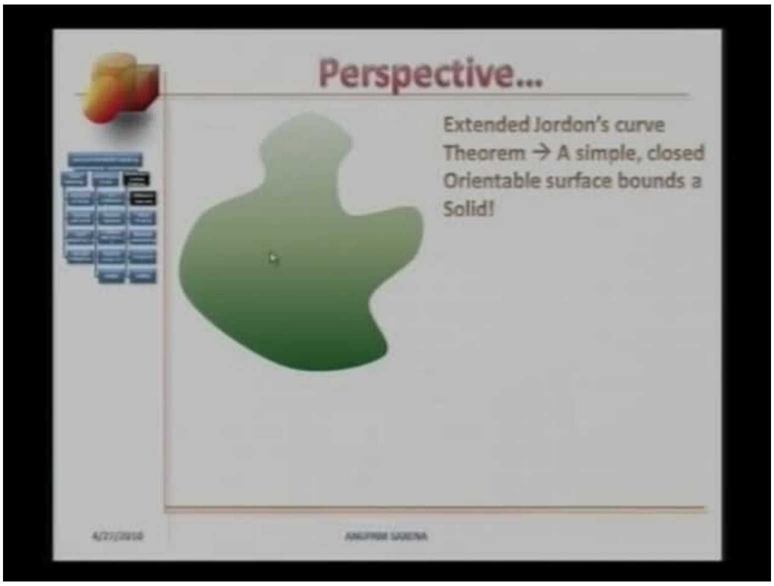
Computer Aided Engineering Design
Prof. Anupam Saxena
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 33
Differential Geometry of Surfaces

(Refer Slide Time: 00:25)



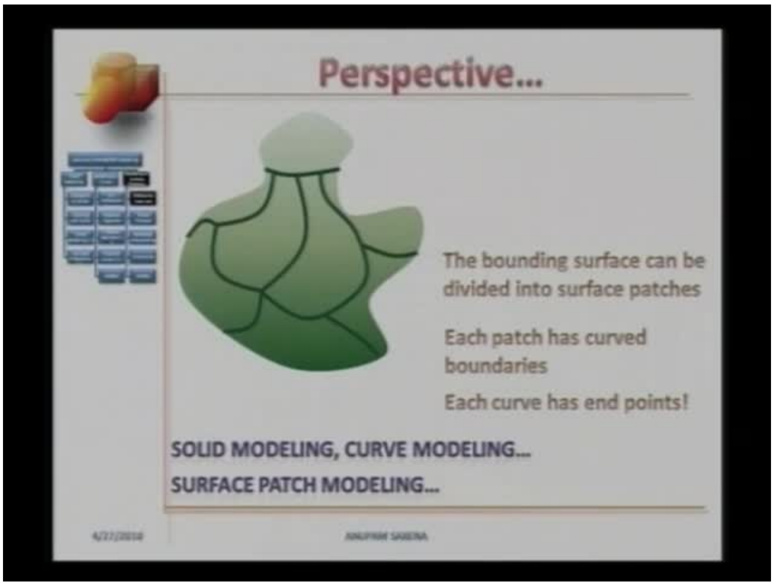
(Refer Slide Time: 00:44)



Good morning and welcome back again. Let us graduate now from curve modeling and move on to the third column of our layout, and start with discussions on surface patches.

Before I do that let me put things in perspective. So, we have this typical solid here, and we have based our course on the extended Jordon curve theorem, which says that a simple close and orientable surface bounds a solid.

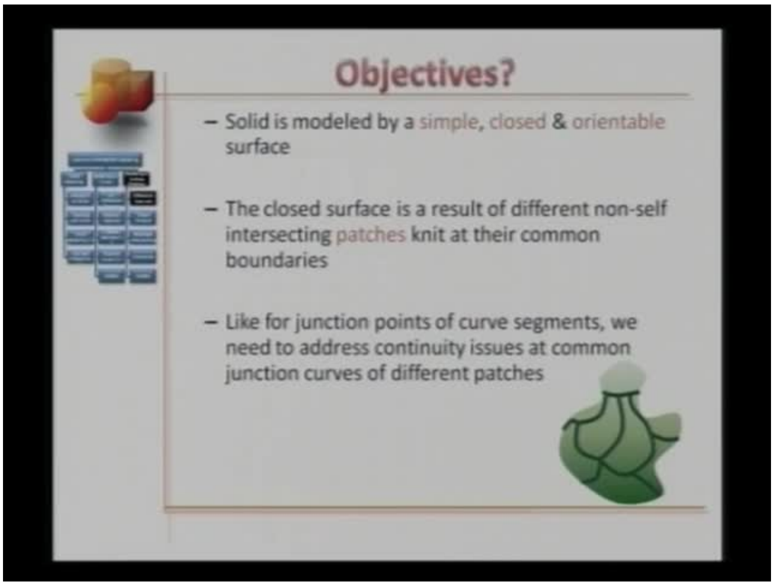
(Refer Slide Time: 01:08)



You can divide this bounding surface into a set of patches, each of which will have curve boundaries. Each of the curve would have its respective end points. We have devoted a lot of time on solid modeling and different methods on curve modeling. A quick recap in solid modeling, we discussed topology, we discussed three different methods of solid modeling namely the wireframe approach, the b-rep or the boundary representation method and constructive solid geometry. In curve modeling we discussed primarily three models, those are the Ferguson curves or segments Bezier segments or curves and more importantly B-spline models. We devoted quite a bit of time on B-spline segments. It is time for us to move on to surface patch model. This is lecture number 33, we start with differential geometry of surfaces.

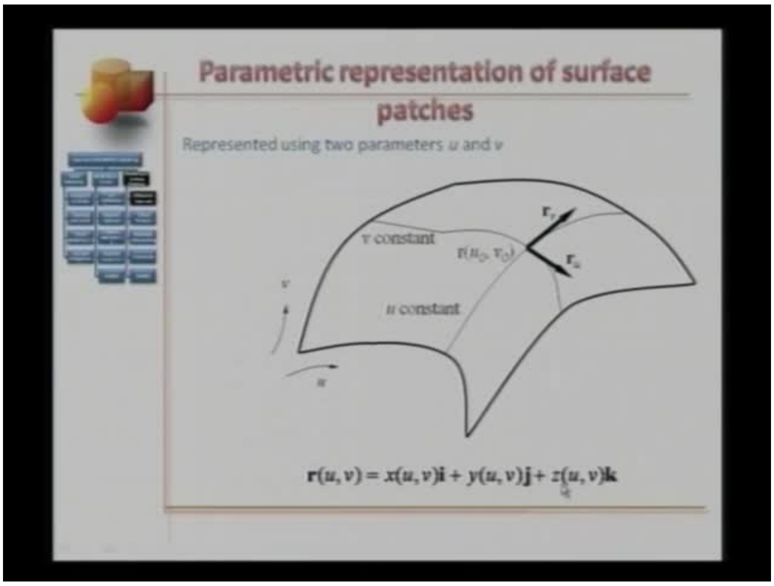
Before we start what would be our objectives? We know that a solid can be modeled by a simple closed and orientable bounding surface. The closed surface is a result of different non-self intersecting patches, knit at their common boundaries. Like for junction points of curve segments, we need to address the continuity conditions at common junction curves of different patches. What do I mean here?

(Refer Slide Time: 02:47)



Well let us take a look at this solid, each of these are surface patches and these are curves, which are common between different surface patches. For example, this curve or this segment here is common between these two patches. We need to worry about the continuity between these two patches around this curve. By around I mean both along like so and a cross like so. For that we need to study different differential properties of surface patches. We will start with some of them today.

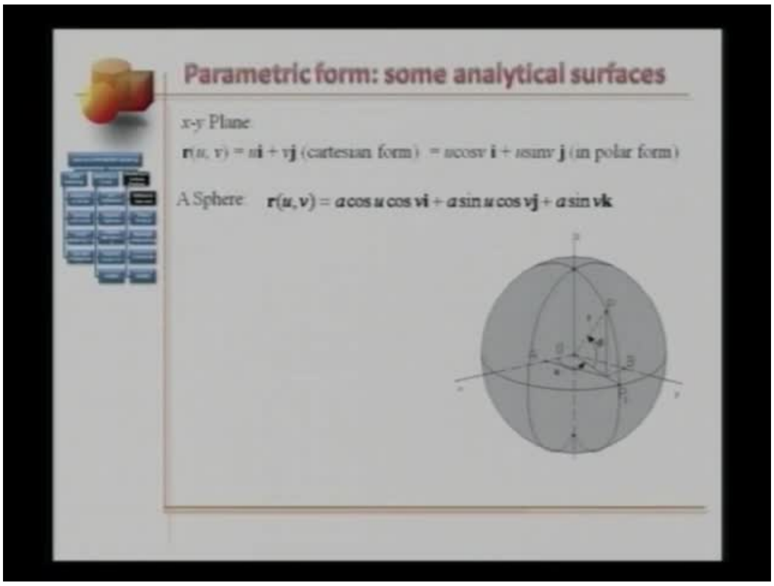
(Refer Slide Time: 04:22)



Let us first try to understand, how we can mathematically represent a surface patch? If you recall, we have been working with parametric equations. A curve segment as you would know is represented using a single parameter, but a surface patch needs to be represented using two parameters u and v. This is a typical surface patch, you can draw a set of curves on this patch. If you think about it, we will have four bounding curves along two curves. One can think of using parameter u and along the two other curves. This one and this one, one can think of using the parameter v.

So, if we take a look at this curve, we can think of parameter u as being constant. Likewise along this curve parameter v is constant, this point here is an intersection between these two segments and represented as r of u subzero v subzero. So, along this curve u is say u subzero and along this curve v is let us say v subzero. You can think of constructing tangents, along these curves at this point along this curve. Let us say we have this tangent represented by r sub v because it is along the v direction.

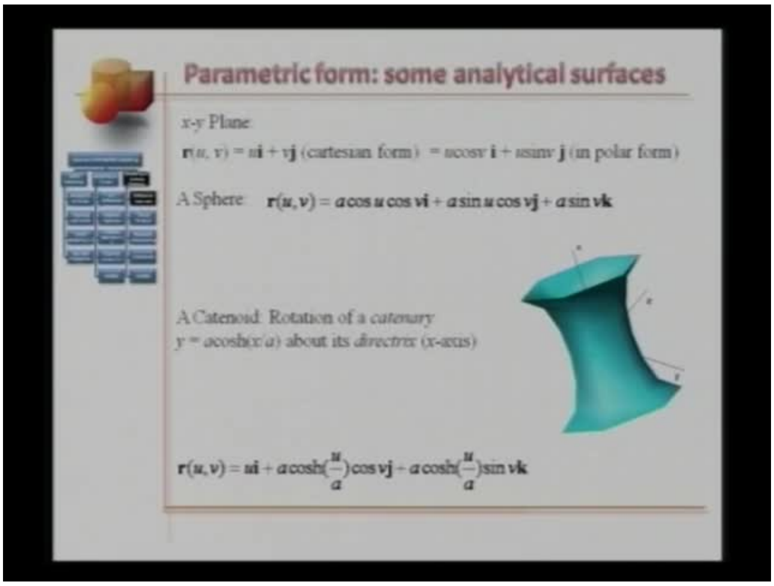
(Refer Slide Time: 08:01)



Likewise we can have a tangent along this curve at this point represented by r sub u, which is along the u direction. The parametric equation of the surface patch can be written as r as a function of u and v, which is this scalar free x function of u and v i plus scale of v y of u and v times j plus the third scale of v z in terms of u and v times k. These are unit vectors along three mutually orthogonal Cartesian directions. So, this scale of u will represent the x coordinate of the point on the patch, this would represent the y coordinate and this would represent the z coordinate. Let us take a look at some analytical surfaces.

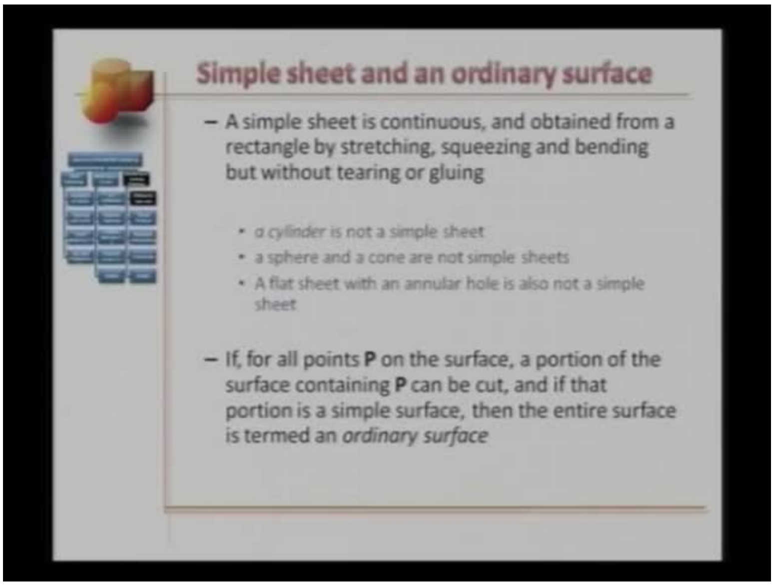
An x-y plane in its parametric representation in Cartesian form is r of u and v, which is equal to u times i plus v times j. In polar form a plane patch r u and v can be written as u times cosine of v times i plus u times sine of v times j. A sphere for example, another analytical surface can be expressed in terms of parameters u and v as r of u and v, and that is equal to a times cosine of u times cosine of v times i plus a times sine of u times cosine of v times j plus a times sine of v times k. Here a is another parameter, this is how a sphere looks, you know that.

(Refer Slide Time: 09:22)



As a third example, we have a catenoid, which can be constructed via a rotation of a catenary, which is given by y equals a hyperbolic cosine of x over a. The rotation is about its directrix, say the x-axis. This is how a catenoid would look like, whose equation in parametric form is given by r of u and v equals u times i plus a hyperbolic cosine of u over a times cosine of v times j plus a times hyperbolic cosine of u over a times sine v times k. Likewise we can construct many other parametric forms of different surface patches, whether analytic or free point. Let us try to understand some concepts now pertaining to surface patches, simple sheet and an ordinary surface.

(Refer Slide Time: 10:38)

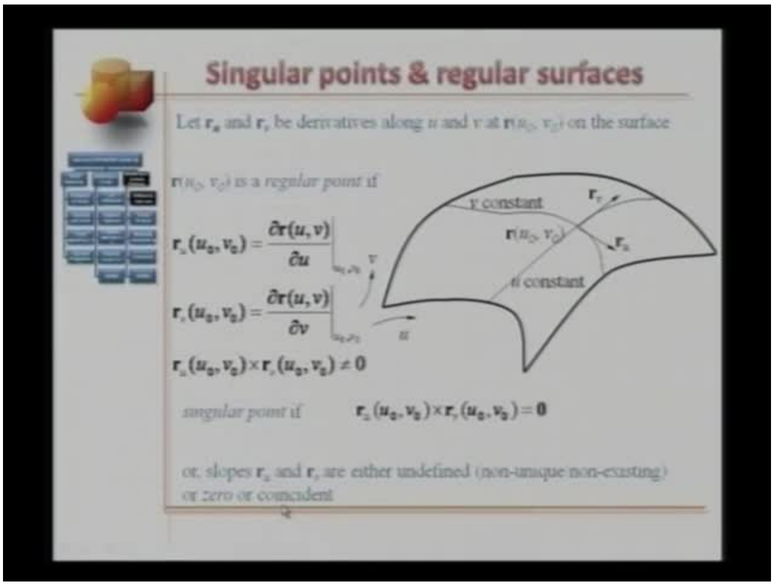


So what is a simple sheet? A simple sheet is continuous and obtained from a rectangle by stretching squeezing and bending, but without tearing or gluing. So, these are some topological operations that we had studied way back, while discussing solid modeling. A cylinder will not be a simple sheet, why is that so? Because we will have to correct to be able to get a rectangle.

Likewise a sphere and a cone are not simple sheets because there are cutting operations involved a flat sheet with an annular hole is also not a simple sheet. If for all points p on the surface a portion of the surface containing p can be cut. If that portion is a simple surface, then the entire surface is termed as an ordinary surface. So, let us say we have this arbitrary surface here, it can have any shape. This surface is constituted by a set of points p, let me take any portion of the surface and cut it.

So, this is the portion I have now and I realize that this portion is homeomorphic to a rectangle. In other words, I can perform any topologically cumbersome operations on the rectangle to get this shape. Since, this is homeomorphic to a rectangle; I would say this is a simple sheet. If I can perform this operation for any portion of such a surface, then I would say that this surface is an ordinary surface. Let us take a look at some singular points and regular points.

(Refer Slide Time: 14:01)

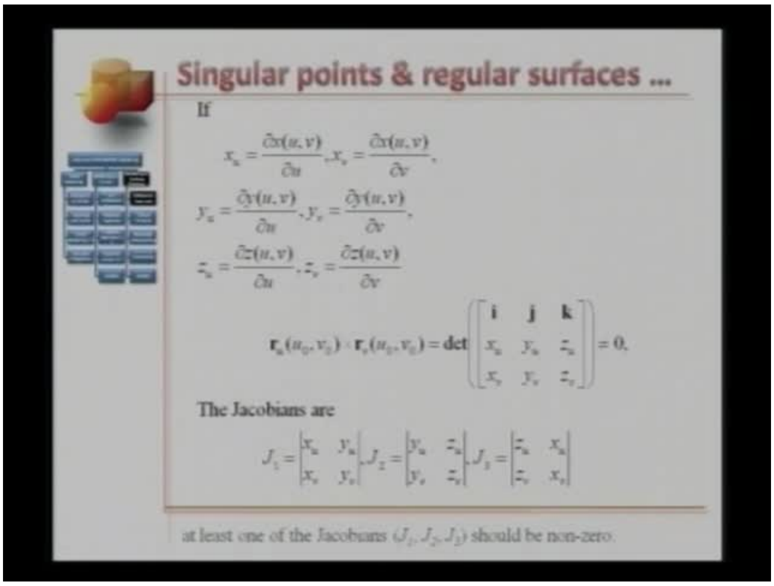


Let r sub u and r sub v be derivatives along u and v at the point r u subzero v subzero on the surface. So, we have this surface patch, we have the parametric directions identified u and v. We have curves corresponding to constant parameter u and constant parameter value v. These intersect at r u subzero v subzero and we have at this point tangents r sub u r sub v.

So, this point r u 0 v 0 is a regular point. If these conditions are satisfied r sub u is given by partial r over partial u evaluated at u subzero v subzero r of v is given by partial of r over partial of v again evaluated at u subzero. The cross product between these two directions is not equal to 0. Once again, what we are saying is that point r u subzero v subzero will be a regular point, if these directions exist mathematically. That the cross product between these two tangents is not equal to 0, this is what I said just now.

The slopes are sub u and r sub v should both exist at a regular point and that the cross product between these slopes should be not 0. A surface patch will be a regular surface patch, if all points on it are regular points one would get a singular point. If either of the three conditions does not hold, that is if either the cross product is 0 or any other slopes r u and r v are either undefined. They may not be unique or they may not exist or any of these directions is 0 or they are coincident.

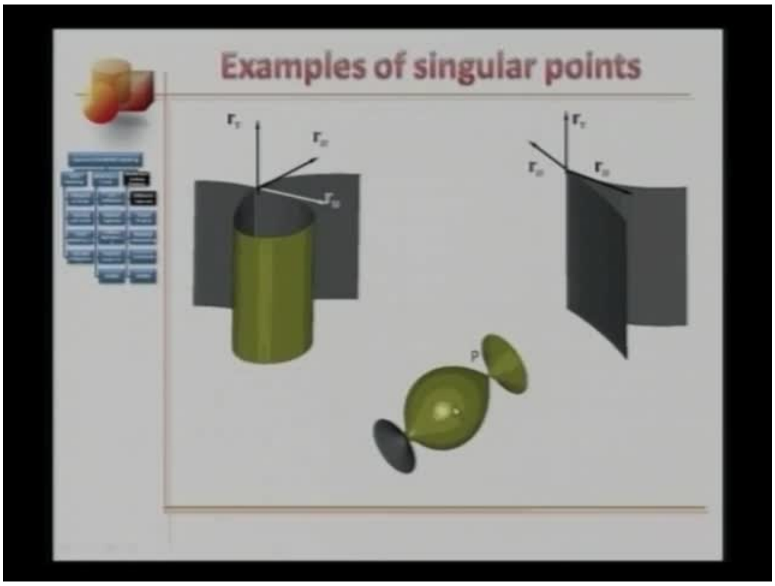
(Refer Slide Time: 17:39)



If x sub u is given by partial x over partial u x here is a scalar field in parameters u and v giving the x coordinate of the parametric surface patch, if x v is defined as partial x over partial v. Likewise for the y coordinate or the patch, if y u is defined as partial y over partial u, if y v is defined as partial y over partial v. If z u as partial z over partial u and z v as partial z over partial v, then one can compute the cross product between the two tangents r sub u and r sub v.

You would know that the cross product is given by the determinant of this 2 by 2 matrix. The first row contains the x, y and z direction unit vectors, the second row will contain x sub u, y sub u and z sub u. And the third row will have x sub v, y v and z v and for the cross product to be not 0. This is what the condition is one can compute the three Jacobians for this determinant. The first one is the determinant of this 2 by 2 sub matrix x u, y u, x v, y v represented by j sub one. The second Jacobian j 2 is y u, z u, y v, z v, the determinant of this 2 by 2 matrix, which is this one. Likewise the third Jacobian j 3 is given by the determinant of the 2 by 2 sub matrix defined by these columns z u, z v, x u and x v. So, for this cross product to be not 0 at least one of the jacobians j 1, j 2 or j 3 should be non zero

(Refer Slide Time: 20:42)

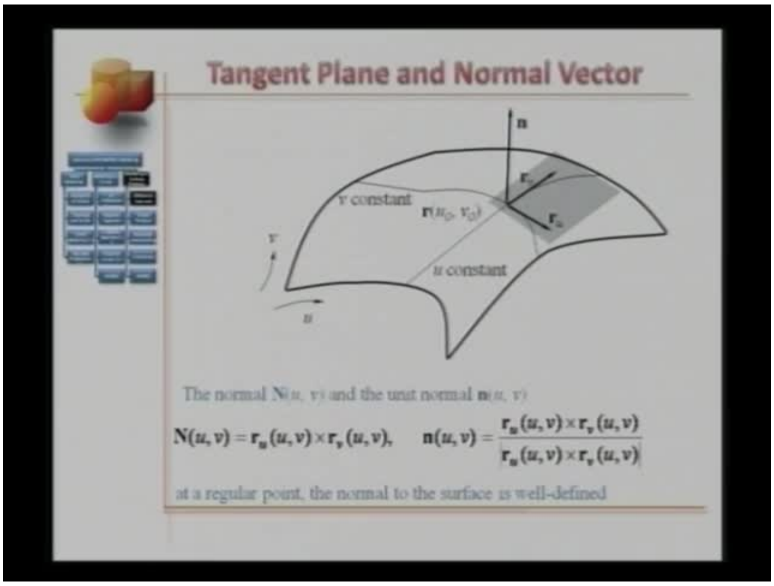


Let us look at some examples where we see singular points. This is the first one, the patch is a self intersecting patch that looks like this. Try to identify the two parametric directions, I will possibly use the parametric direction u like this. Once again like this and the parametric direction v will be along this intersection line. Let us consider any of the points on this region here on this line, so at this point if you think about it, will have two directions pertaining to the tangents along the u direction.

One will be given by this here r of u and the other will be given by this r of u. There would be a unique tangent along the v direction r of v. So, since the tangent along the u direction is not unique at this point, you would say that this point is a singular point this is not a regular point. The second example, it is a patch that looks like this. So, let us say this is a curve along the parameter u and the vertical curves would correspond to parameter v. What do you have to say about the points on this edge or along this edge? Something very similar, if you look at this point, you will have non-unique tangents along the u direction. So, you will have multiple directions pertaining to r sub u while you will have a unique direction pertaining to r sub v. Once again one of the tangents is non-unique in… Therefore, all the points on this edge are singular points.

This one is a third example, if you look at this point here or maybe this point here, this is a singular point again because tangents are not uniquely defined all other points on this patch in this figure on this one, in this figure. Here are all regular points because the three conditions that we had seen before are satisfied. Let us look at the tangent plane and the normal vector.

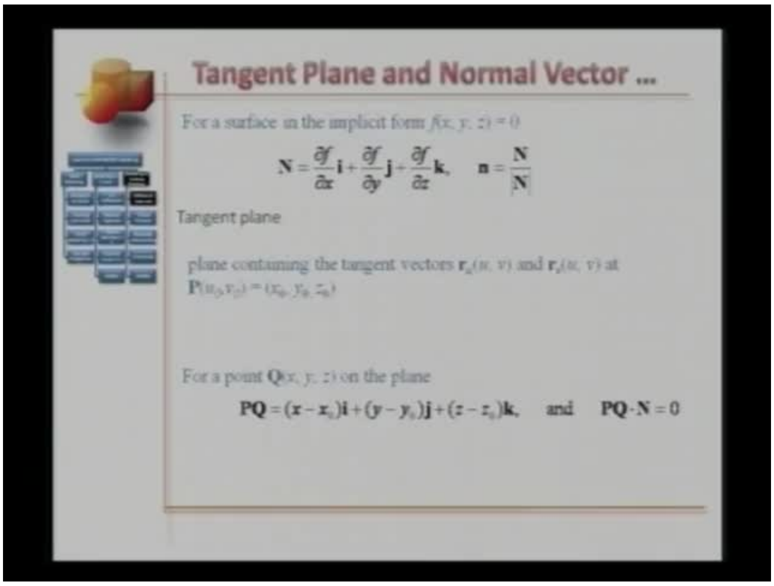
(Refer Slide Time: 25:13)



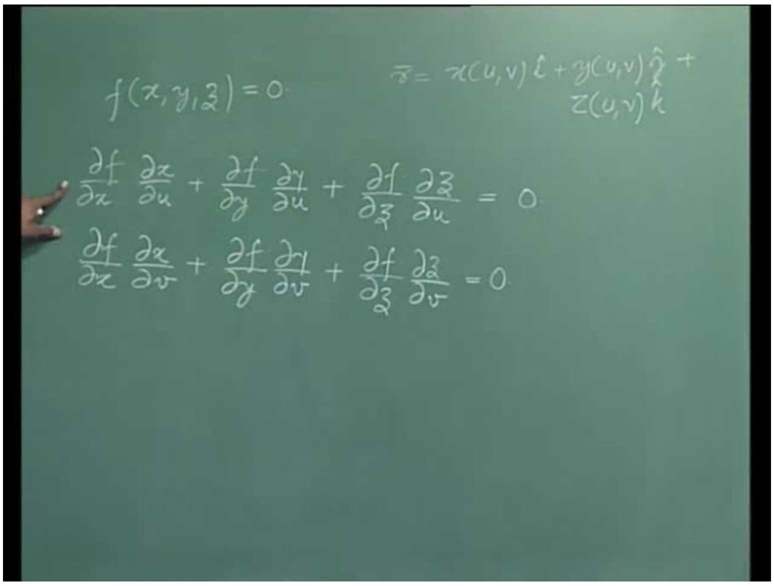
Now, you have seen this figure before let us try to concentrate in this region. This is a point of intersection between two parametric curves u equals u 0 and v equals v 0. We have the tangents r u and r v defined this plane here, that contains the two tangent vectors is called the tangent plane. The unit normal small n or the normal capital N as a function of u and v is given by the cross product of these two vectors r sub u and r sub v. Both are functions of parameters u and v, of course the unit normal will be this cross product divided by its magnitude. At a regular point the normal to the surface is well defined. Let us see what happens for surface patches given in implicit form?

In other words for equations f of x y and z equals 0. The normal direction is given by partial f over partial x times i plus partial f over partial y times j plus partial f over partial z times k. The unit normal small n is then capital N divided by its magnitude. You would want to ask or maybe you would know already as to why the normal for implicit surface patches is given in this form? Let me work out the answer for this question for you on board.

(Refer Slide Time: 26:40)



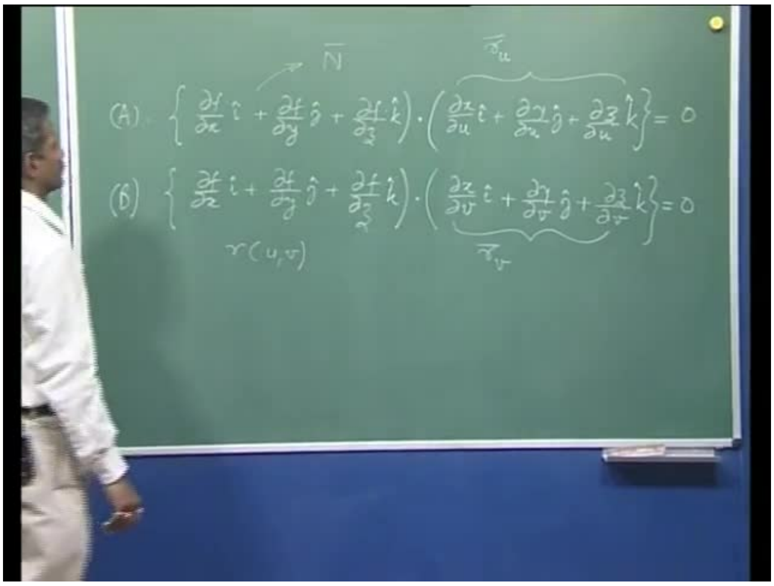
(Refer Slide Time: 27:45)



Say we have this implicit surface patch and the position vector of any point on this patch is given by r equals x u v i plus y u v j plus z u v k. That means the x y and z coordinates of this point are functions of parameters u and v. Let us differentiate this function with respect to u, we use the chain rule to get partial f over partial x partial x over partial u plus partial f over partial y partial y over partial u plus partial f over partial z partial z over partial u.

This is equal to 0 differentiation of 0 with respect to u is 0. Likewise, let us differentiate this equation with respect to the second parameter v partial f over partial x partial x over partial v plus partial f over partial y partial y over partial v plus partial f over partial z partial z over partial v is also equal to 0. Now, let us try to express the two left hand sides in a slightly different form, as a dot product between two vectors.

(Refer Slide Time: 30:22)



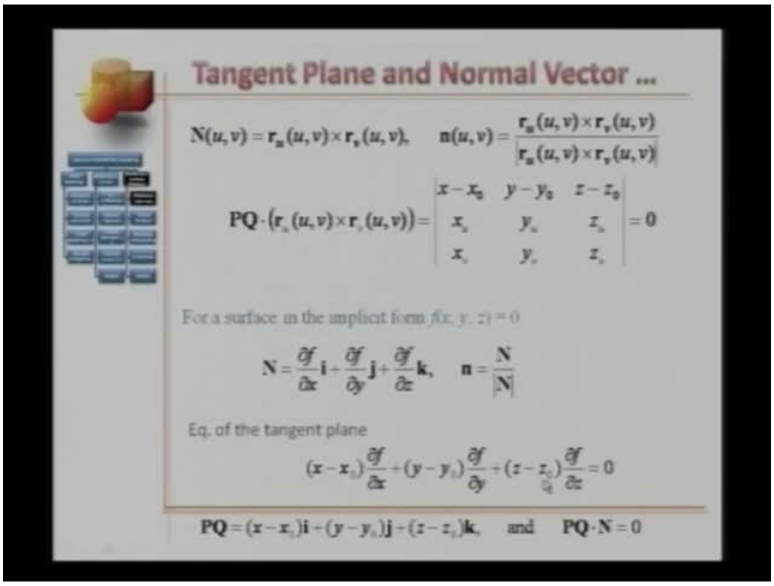
So, these are equations a and b, so I can write equation a as partial f over partial x i plus partial f over partial y unit vector j plus partial f over partial z unit vector k dotted with partial x over partial u i plus partial y over partial u j plus partial z over partial u unit vector k, this is equal to 0. Likewise, equation b is partial f over partial x i plus partial f over partial y j plus partial f over partial z k dotted with partial x over partial v i plus partial y over partial v j plus partial z over partial v k.

This is again equal to 0. Now, try to identify what this vector is? This is the tangent along the u direction. How about this one? This is the tangent along the v direction. So, what are these two equations conveying? They are conveying that this vector is orthogonal to r sub u the tangent along the u direction, because of this condition. The same vector is also orthogonal to the tangent along the v direction. Since, r u and r v constitute the tangent plane, this vector is orthogonal to the tangent plane.

Therefore, it had to be normal to the surface at point r u v. So, this vector here aligns with the normal vector. The tangent plane is a plane, containing the tangent vectors r sub u function of u and v and r sub v, which is also a function of u and v at some point p for parameter values u subzero and v subzero in Cartesian coordinates. This point is expressed as x subzero y subzero and z subzero.

Now, for a point as a matter of fact for any point q which is given by x y and z on the tangent plane. Let us say we have the free vector given by p q, which is equal to x minus x 0 times the unit vector i plus y minus y 0 times j plus z minus z 0 times k. So, for q to be lying on a tangent plane this vector on the tangent plane must be perpendicular or orthogonal to this normal vector. In other words PQ dotted with n must be equal to 0 and this would give us the equation of the tangent plane.

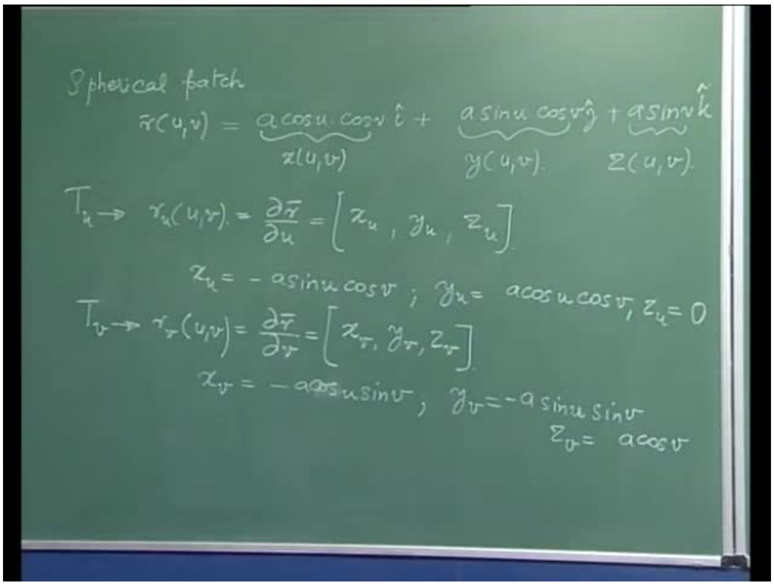
(Refer Slide Time: 35:32)



So, I have copied the previous condition here, if the normal is given by the cross product between two vectors identifying the tangent plane, namely r u and r v. If we can extract the unit normal from this information, then we have PQ dotted with r u crossed with r v, which is given by the determinant of this 3 by 3 matrix. Now, this here is a scalar triple product with elements x minus x 0 y minus y 0 z minus z 0. In the second row, we have partial derivative of x with respect to u y sub u z sub u partial derivative of x with respect to v same for y and the same for z and because of this condition it is equal to 0.

As I said this would give us the equation of a tangent plane, in case our surface patch is expressed in the implicit form f of x y and z is equal to 0. Then we will have a slightly different equation, then our normal is given by the gradient of this function partial f over partial x times i plus partial f over partial y times j plus partial f over partial z times k. We know, how to compute the unit normal? For this case, the equation of the tangent plane will be given by the dot product between PQ. The gradient that is x minus x 0 times partial f over partial x plus y minus y 0 times partial f over partial y plus z minus z 0 times partial f over partial z equals 0. Let us work an example on the board for the case of a spherical patch.

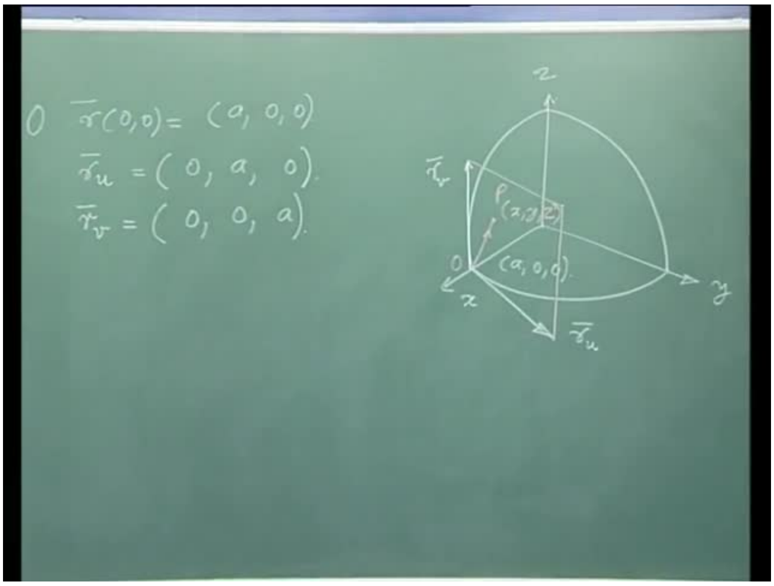
(Refer Slide Time: 38:15)



So, the patch r as a function of the two parameters u and v is given by a cosine of u times cosine of v times i plus a sin of u cosine of v times j plus a sin of v times k. So, this is x u v, this is the scalar field y u v and this one here is z u v. Let us compute the tangent at any point. So, r u of u v the tangent along the u direction is partial r over partial u in order form this is given by x sub u y sub u and z sub u partial derivative of x with respect to u partial derivative of y with respect to u, and partial derivative of z with respect to u.

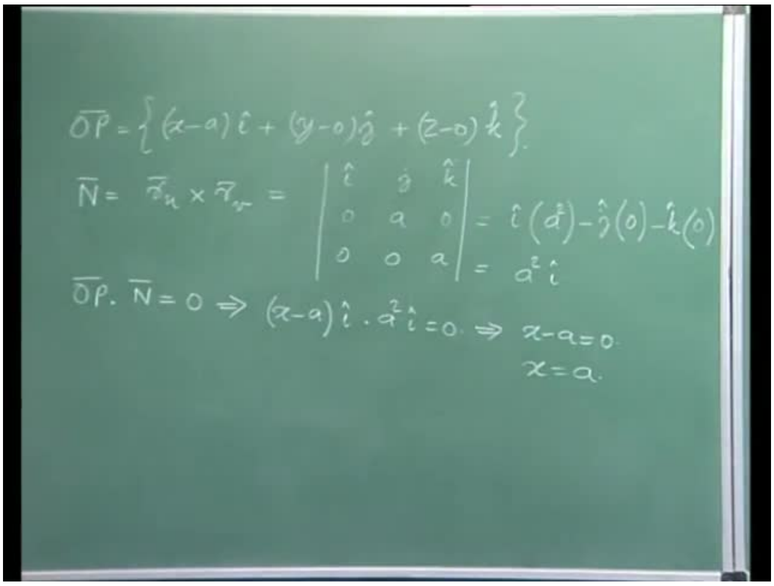
So, x sub u is minus a sine u cos v y sub u is a cosine u cosine v z sub u. There is u dependence here. So, this is 0 likewise the tangent along the v direction is given by r sub v u v. This is partial r over partial v in order form this is x v, y v, z v. So, x of v equals minus a sine u sine of v y of v equals, this should be cosine of u y of v equals a sine u sine v with a negative sign and z of v equals a cosine v. So, let us compute these two tangents at a point on a sphere. Let us say for u equals 0 and v equals 0, what do we have?

(Refer Slide Time: 42:51)



So, r for 0, 0 equals a this would be 0 and this would be 0. This is a very simple example. Let us continue with it, so r sub u tangent along the u direction t 0. 0 will be this would be 0 y of u will be a and z of u will be 0. Likewise the tangent along the v direction equals x of v is 0, y of v will be again 0 and z of v will be a. Let us get some geometric perspective here. So, this would be the portion of the sphere here and at this point a 0 0 that we are trying to compute.

(Refer Slide Time: 46:15)



The two tangents, the first tangent u here will be along the y direction 0. A 0 this is r u, the second tangent r of v will be along the z direction, r of v clearly the tangent plane will be the y z plane geometric. Say this is the tangent plane, let us have a point p here given by x y z. Let us say this is point O here, where we have define our two tangents. The coordinates of o are given by r 0 0.

So, the equation of the free vector O P is given as x minus a i plus y minus 0 j plus z minus 0 k. I can compute the cross product between these two tangents r u and r v, so the normal n is r u crossed with r v. This is given by the determinant i j k 0 a a 0 components of r u and 0 0 a components of r v. This is i times a squared minus j times, this would be 0 minus k times. This would be 0 again. So, this is equal to a squared times i. Since, OP is perpendicular to n, we have the condition OP dotted with n equals 0.

This would imply that x minus a i dotted with a squared i is 0. This gives x minus a equals 0 or x minus a. Now, if you realize this is the y z plane intersecting the x axis at x equals a, which is what we expected. This is a very simple example, geometrically (( )) that allows us to see, how we can compute the equation of a tangent plane? You can expect many more complex examples in real life.