Computer Aided Engineering Design
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Lecture - 28

Good morning and welcome again. This is lecture twenty eight and we will primarily be discussing multiple knots.

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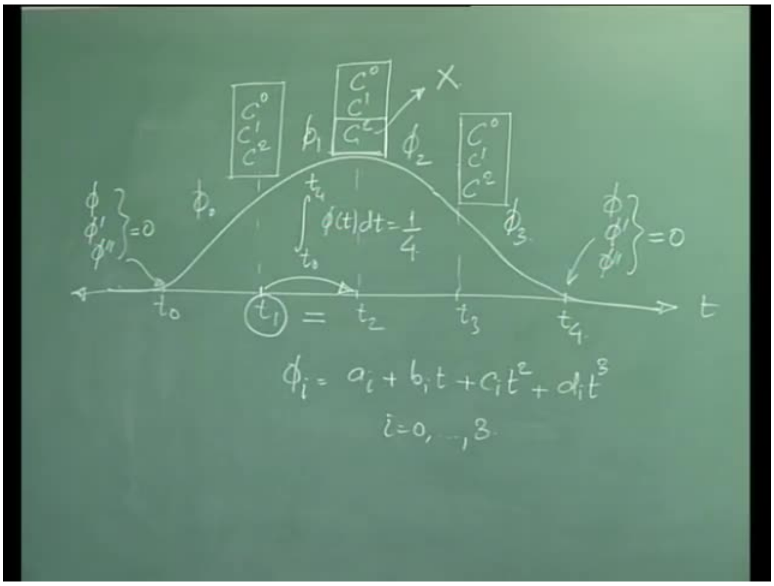

Last time I had posed two questions. The first one pertaining to the order of four normalized basis function. In particular, I asked you why is it that we will have to let go of this C 2 cautionary condition, if I move knot t 1 to knot t 2 making t 2 a multiple knot of multiplicity 2. Let us try to address this slightly differently. First a little recap, let us look at the number of unknowns and the number of the conditions available to us for an order 4 B spline basis function for simple knots t 0, t 1, t 2, t 3, t 4. We got three conditions here. Three conditions here making it 6, this is the seventh one plus 3 10 plus 3 13 plus 3 16 conditions here.

And if you look at each of these individual cubic polynomials phi 0, phi 1, phi 2 and phi 3 in general phi i being given as a i plus b i times t plus c i times t square plus d i times t cube. i going from 0 3, we have one two three four times four 16 unknowns, 16 conditions, 16 unknowns this system is solvable. Now, if I move t 1 to t 2 I let go of this non span and because of the which I probably have to let go of one of these conditions.

As I mentioned most likely it is going to be c 2 cautionary condition at this junction point where we have multiple knots t 1 equals t 2. Let us try to formally determine that it is only c 2 condition that if you let go would make sense to us. Now, let us look at these phi's one by one.

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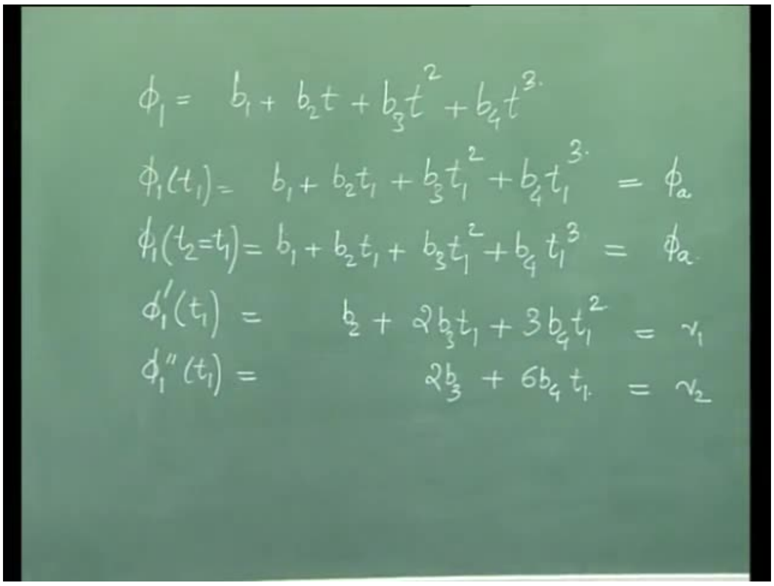

Phi 1, let me write thing is as a 0 plus a 1 t plus a 2 t square plus a 3 t cube, 3 because of these three conditions. I will have a 0, a 1 and a 2 all of them to be equal to 0. So, phi 1 will be equal to a 3 t cube. Even if I have moved t 1 to t 2, I will not be letting go of the c zero or position information here. In the sense, the position information will still be valid. When I move t 1 to t 2, I would not to know what phi 1 is. Let me ignore this for now as a straight away come phi 3.

Phi 3, if write this thing as d 0 plus d 1 t plus d 2 t square plus d 3 t cube, phi 3 at t 4 is not here will be greeted by these three conditions. And remember nothing has happened in the vicinity of t 3 in a sense the position, slope and curvature continuity conditions still exist at this junction point. So, I would say that phi 3 can be uniquely determined because of these three conditions under fourth condition over here I would say that even phi 0 is known to us.

Now, let us come back to phi 2. I write phi 2 as c 0 plus c 1 t plus c 2 t square plus c 3 t cube. t 3 is again a simple knot. These three conditions are available to us and what will happen when t 1 moves to t 2 is that this junction point will move to this one, in a sense the position information at these two junction points will be identical. So, we have three conditions and the position condition here which would mean that even phi 2 will be known to us.

Now, let us try to figure what happens to phi 1? There is a minor typo here, this thing should be phi 0 and not phi 1. Coming back to phi 1, let me express phi 1 as a cubic polynomial since this is an order four B spline basis function. So, phi 1 is b 1 plus b 2 t plus b 3 t square plus b 4 t cube, does not matter what choices of coefficients you take b 0 to b 3 or b 1 to b 4.

Now, let us try to figure the conditions which are available for phi 1. Remember, after having moved t 1 to t 2 the position information is the same. Let us write it. So, phi 1 at t 1 equals b 1 plus b 2 t 1 plus b 3 t 1 square plus b 4 t 1 cube. phi 1 at t 2 which is the same as t 1 will have the same position and the same equation. How about the slope? phi 1 prime at t 1 equals b 2 plus 2 times b 3 t 1 plus 3 times b 4 t 1 square. This is the slope and the second derivative phi 1 double prime t 1 will be equal to 2 b 3 plus 6 b 4 at t 1. Let us say this has some value v 1, this has another value let say v 2. As I mentioned earlier the position here and here that means the coordinates they will be the same and let me represent those coordinates by phi a.

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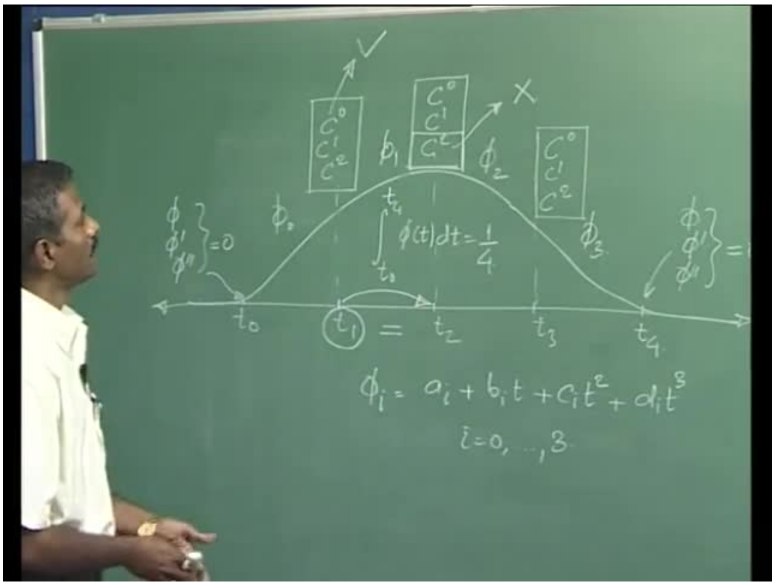

Now, if you notice of these four equations the first two are identical. What would that mean? This would mean that I will have to let go of one of these b's either b 1 or b 2 or b

3 or b 4. Now, which one do I let go of is something that is not clear to me as yet. I cannot presume beforehand that phi 1 will be quadratic or phi 1 would be cubic or anything like that. But one thing is for sure, I will have to let go of one of these unknowns. What would that mean? That would mean that I will have to let go of one of the three conditions at this junction point.

Now, which one do I let go of? I will have to retain c 0 condition because that is used to determine phi 0 completely. Now, I have to either let go to c 1 or c 2 condition. Amongst the two it has to be the c 2 condition that I will have to get rid of simply because if I say that spline is not c 1 continuous over here, and it is c 2 continuous over here, that does not make sense for a function to be c 2 continuous at any point it has to be c 1 continuous at that point. So, between these two I will have to retain c 1 continuity, I will have to let go of this c 2 continuity condition here. That answers the first question in the previous lecture.

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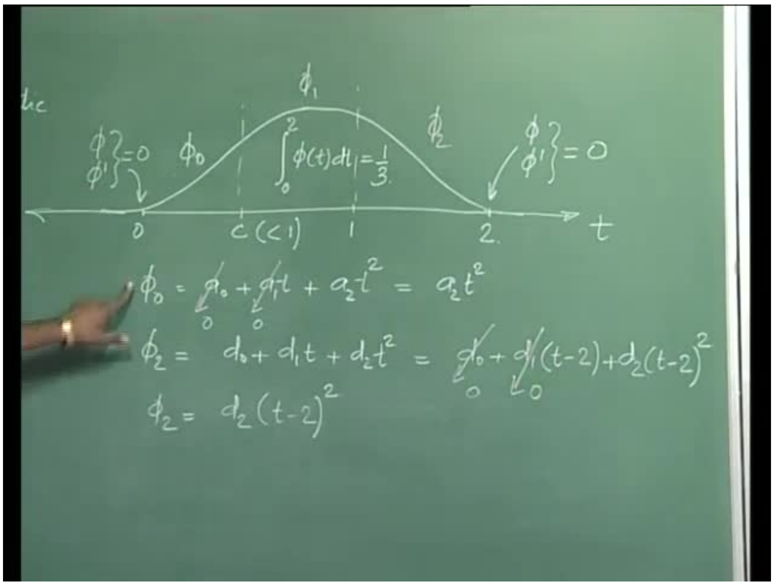

Now let us investigate multiple knots further using a simple example, an example which was an assignment problem to you in the previous lecture. So, this is a quadratic spline that we are trying to construct for the knots given to us 0, c this value at this time is in between 0 and 1. The third knot 1 and fourth knot 2. It stands over three knot spans. So, clearly it will have three segments phi 0, phi 1, phi 2 and each of these segments will be quadratic, pure quadratic polynomials. How about the condition that we will be needing. We will be needing phi and phi 1 both are 0 here.

Phi and phi 1 again both are 0 here. These are four conditions and the fifth one will be integration from 0 2 2 phi t d t will be equal to 1 over 3. 3 is the order this B spline function. This investigation is about what happens when c becomes 1, in the sense what happens this knot moves to this position over here. We will try to figure thing out mathematically. Let us start constructing phi 0, phi 1 and phi 2. First phi 0, I write phi 0 as a 0 plus a 1 t plus a 2 t square. I will be skipping some steps here. So, these two conditions where imply that a 0, a 0 and a 1 is 0. So, phi 0 will be a 2 t square. Let us ignore phi 1 for now and jump to phi 2. Now, let us be a little smart.

I can write phi 2 as say d 0 plus d 1 t plus d 2 t square or realizing that two of these conditions, the potion the slope above 0 over here at equals to I can replace this quadratic polynomial without any loss of generality. I can replace my 2 as d 0 plus d 1 t minus 2 plus d 2 t minus 2 the whole square. Such that the 3 d's over here will be different from the three d's over here. Let me use this expression and imply these conditions to find that d 0 will be 0 and d 1 also be 0. So, that phi 2 is equal to d 2 t minus 2 the whole square. Let me save this result for us to use later. Phi 0 is a 2 t square phi 2 is d 2 t minus 2 square. Let us come now 2 phi 1.

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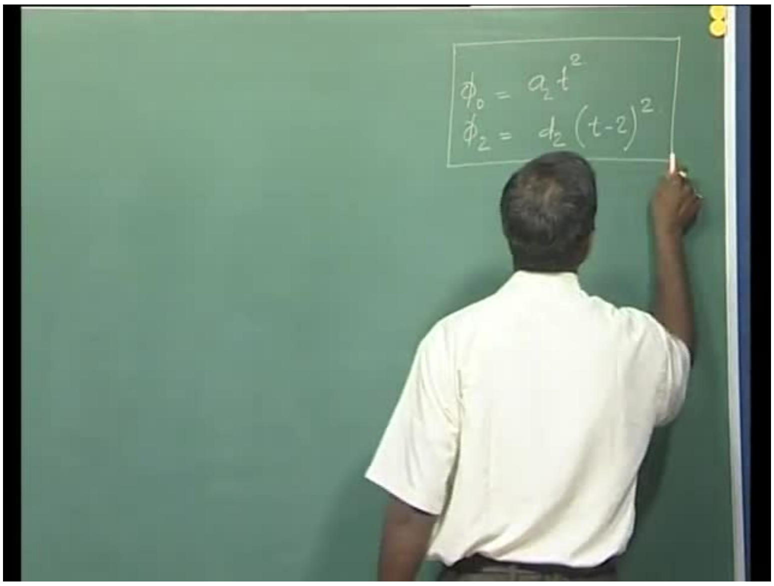

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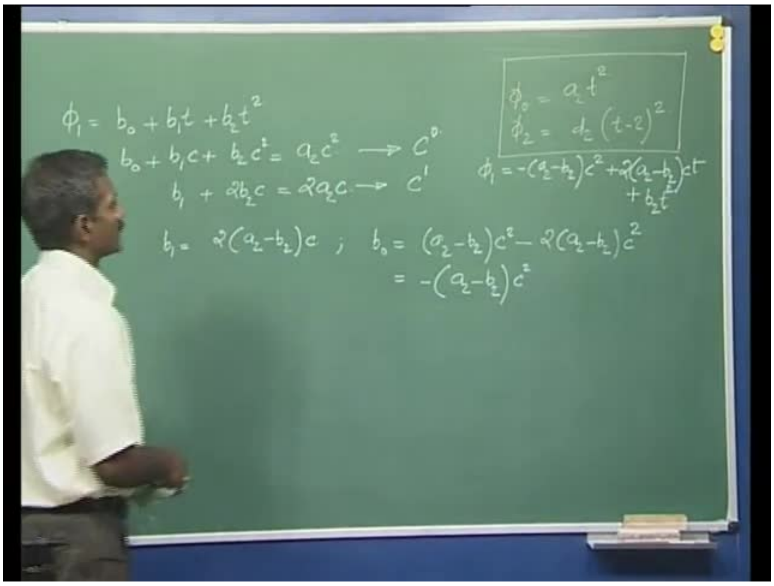

Phi 1 equal c 0 plus c 1 t plus c 2 t square. Phi 1 has to be a quadratic polynomial again. Let us try to fine what c 0, c 1 and c 2 are. By the way, these symbols may not to be good for us to use because we are using symbol c to represent one of the knots. So, there might be some confusion later on. So, let me replace phi 1 to b 0 plus b 1 t plus b 2 t square. Since, c at this time is a simple knot I can use position and slope continuity at this junction point. Remember, this is a quadratic B spline, order is 3. So, at any point over it this spline has to be c 3 minus 2 that is c 1 continues.

So, these are the two conditions available here and there would be a position condition available over here. Three conditions, three unknowns b 0, b 1, b 2 I should be able to uniquely find phi 1. Let me represent this junction point by symbol B and this one by symbol X. So, at B we have b 0 plus b 1. Now, the t value is c plus b 2 c square equals phi 0 at c which is a 2 c square. This corresponds to position continuity and for slope continuity I have b 1 plus 2 b 2 evaluated at c will be equal to two times a 2 evaluated again at c. So, this is the position continuity condition at junction b and this is the slope continuity condition at the same junction.

Now, let us express b 1 and b 0 in terms of b 2. So, b 1 is two times a 2 minus b 2 times c. Substitute for b 1 here, you will get b 0 equals a 2 minus b 2 c squared minus b 1 times c, b 1 is 2 times a 2 minus b 2 c and one c comes from here. So, we will have c squared. This simplifies to minus a 2 minus b 2 c square. Let me write phi 1 here now. This would be minus of a 2 minus b 2 c squared plus 2 times a 2 minus b 2 c t plus b 2 t square. Now, overall how many unknowns do we have? We have a 2, we have d 2 and we have b 2. What are the conditions that we may have missed? We have missed the c 0 and c 1 continuity conditions at this junction point X. Let us work with them.

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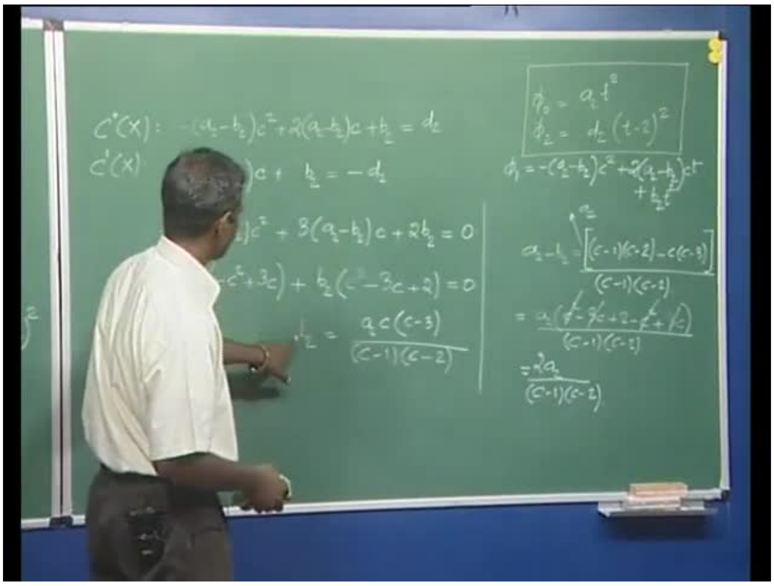

Now, c 0 at X would give us from phi 1, we will have minus a 2 minus b 2 c squared plus 2 a 2 minus b 2 c the t value is 1 plus b 2; again the t value is 1 and this would give us d 2, if I substitute t as 1 likewise c 1 continuity at X will give us… I will have to differentiate phi 1 and phi 2 both ones with respect to t and evaluate both at t equals 1. When I do that I will get 2 times a 2 minus b 2 c, this t goes off plus 2 b 2 which is 1 and this would be equal to 2 times d 2 t minus 2 for t equals 1 t minus 2 will be minus 1, I let go of this with a minus sign. You would realize 2 is common.

So, I let go of 2 from these terms. The idea here is to express any two of three unknowns in terms of one unknown using these two conditions. Now, let us add these two equations together which would eliminate d 2. So, when I do that I have minus a 2 minus b 2 c square. If I add these two guys together I will get 3 times a 2 minus b 2 times c plus 2 b 2 equals 0. This is an equation in terms of a 2 and b 2. Let us start clubbing the coefficients pertaining to a 2 and b 2 separately. So, we have a 2 times minus c square plus 3 c and corresponding to b 2 we have c squared minus 3 c plus 2.

This would be equal to 0, I would be able to write b 2 now in terms of a 2. You can realize this expression can be factorized. So, b 2 is a 2 times c times c minus 3, this thing is going to the right over let me write the thing in the factor form c minus 1 times c minus 2. So, I have obtained b 2 in terms of a 2. If I substitute for b 2 here and here I will get d 2 in terms of a 2. I would need the term a 2 minus b 2 as well. Let us work both of them out.

First let me work on what a 2 minus b 2 is. So, a 2 minus b 2 equals c minus 1 c minus 2 minus c c minus 3 over c minus 1 c minus 2 with a 2 common doubt; let me simplify this. This would be equal to a 2 c squared minus 3 c plus 2 minus c squared plus 3 c over c minus 1 c minus 2. So, this would be a 2, these terms cancel out. 3 c cancels with minus 3 c as well. All I will be left with in the numerator will be 2 over c minus 1 c minus 2. I would need this result and this result to get d 2.

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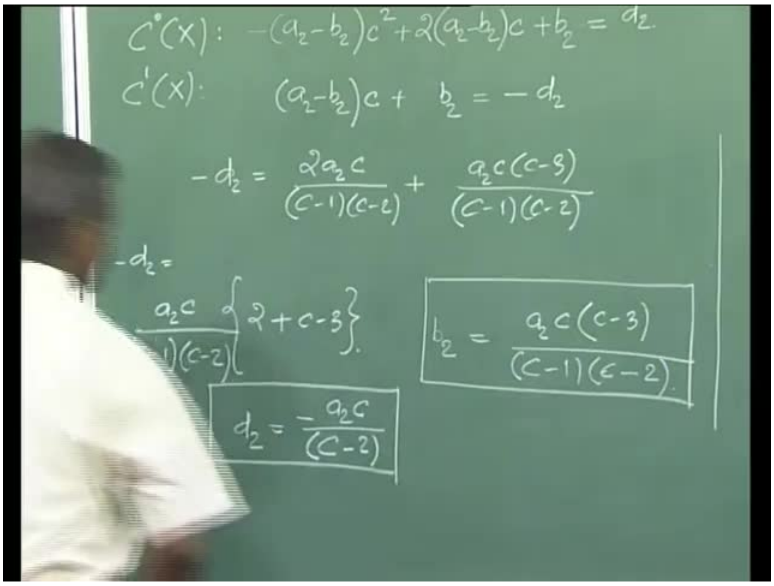

So, now d 2 equals a 2 minus b 2 which is 2 a 2 times c over c minus 1 c minus 2, I will have minus sign here plus b 2 which is a 2 c c minus 3 over c minus 1 c minus 2. Let me block b 2 for now so that I do not disturb it and let me try to simplify for d 2 over here. So, minus d 2 equals a 2 c are common and so it is the denominator. And I am left with 2 plus c minus 3. Now, this is c minus 1 which cancels out to this one here. So, d 2 will be equal to minus a 2 c over c minus 2. Let me block this result also. Now, that we have d 2 and b 2 in terms of a 2 let me write phi 0 phi 1 and phi 2 in terms of single unknown.

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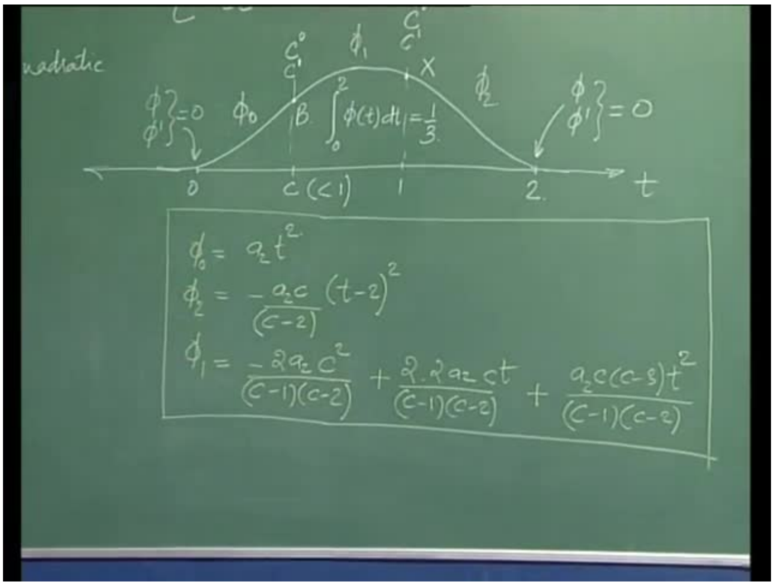

So, phi 0 equals a 2 t square phi 2 equals d 2 which is minus a 2 c over c minus 2 t minus 2 the whole square and now phi 1 equals minus of a 2 minus b 2 c square. We have what a 2 minus b 2 is so this is minus of 2 a 2 over c minus 1 c minus 2, we have a c squared term here plus 2 times a 2 minus b 2 which is again 2 times a 2 over c minus 1 c minus 2 times c t. Let me see if I got all terms over here. So, 2 times c t and then a 2 minus b 2 is 2 a 2 over c minus 1 over c minus 2, seems okay plus b 2, we have b 2 right here. a 2 c c minus 3 t squared over c minus 1 c minus 2. So, far so good, we have all phi's which by themselves are pure quadratic polynomials in t in terms of a single unknown a 2.

How do I find a 2? We use this condition that the area under this function over the knot span over which it stands is equal to 1 over 3 essentially non zero. To compute this integration we will need three results. One, integration of phi 0 over this interval. Second, integration of phi 1 over this interval and third integration of phi 2 over this interval.

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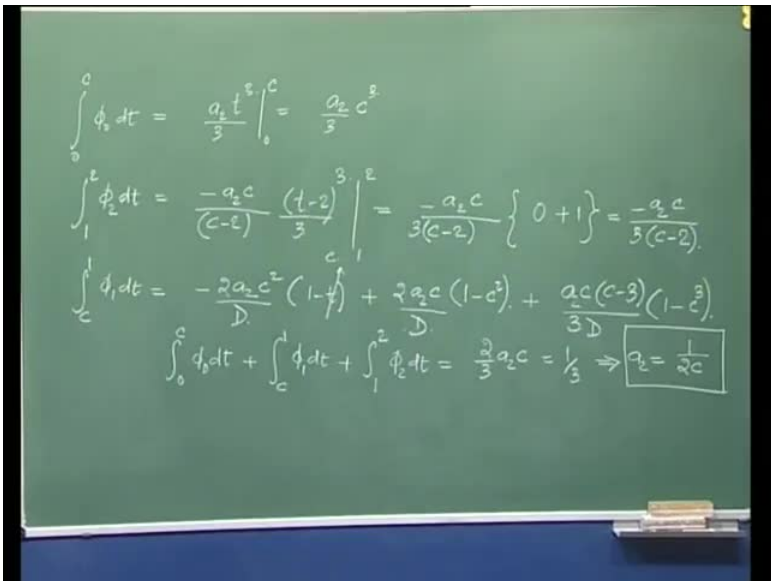

So, integral phi 0 d t from 0 to c equals a 2 t cube over 3 0 to c which is a 2 over 3 c cube. Let us work for phi 2 now. Integral 1 to 2 phi 2 d t equals minus a to c over c minus 2 integral of this. This would be t minus 2 the whole cube over 3 evaluated at 1 and 2. This would be equal to minus a to c over c minus 2. Let me have 1 over 3 here for t equals 2, this is 0 minus for t equals 1 this is minus 1. So, this is equal to minus a 2 c over 3 c minus 2. Now, this one would be a little complicated.

Let us find out. Integral from c to 1 of phi 1 d t equals let me denote this denominator by capital D to make my algebra a little easier. So, I have minus 2 a 2 c squared over D, I will introduce t here because of integration and because of these limits c to 1 there will be 1 minus t plus 4 a to c over D which is this term here. This t will become t squared over 2 and t square if I evaluate pertaining to these limits will become 1 minus c square, 4 over 2 is 2 about the third term will have a 2 c c minus 3 over D t cube over 3 and t cube when evaluated at these two limits will give us 1 minus c cube.

By the way this is c here. Now, let me have you do the algebra from here on and let me skip a few steps to just tell you that integral 0 c phi 1 d t, I am sorry phi 0 d t plus integral c to 1 phi 1 d t plus integral 1 to 2 phi 2 d t equals 2 over 3 a 2 c. It turns out to be a very simple expression and this equal to 1 over 3 from this function which would imply that a 2 equals 1 over 2 c. Let me block this. Now, try substituting for a 2 in these three expressions. What do we get?

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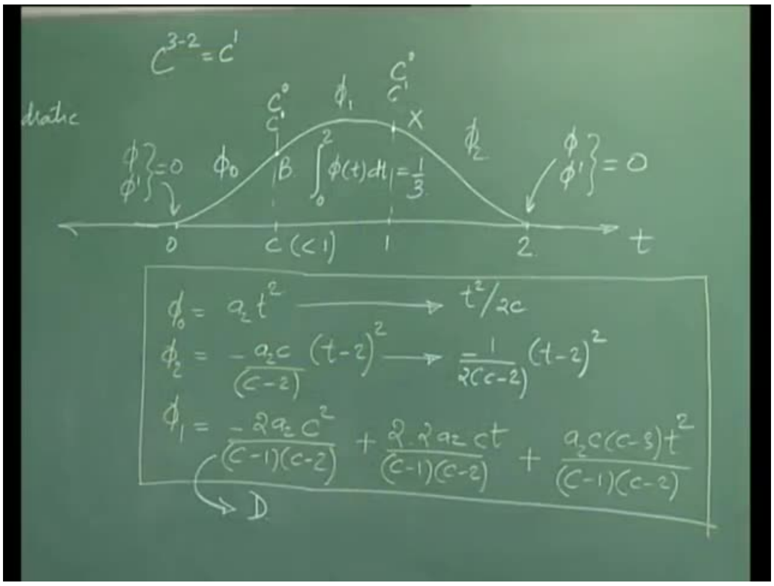

Phi 0 becomes t square over 2 c phi 2 becomes minus 1 over 2 c minus 2 t minus 2 the whole square. a 2 is there is half so minus half c minus 2 c minus 2 square we are okay and I will have you the figure out what happens to phi 1 if you substitute for a 2 in this expression. Now, let us observe the behavior of these three pure quadratic polynomials. For values of c not equal to 1 and for values of c equal to 1 at a later stage. Let us see how this behaves, we will work with a small matlab code.

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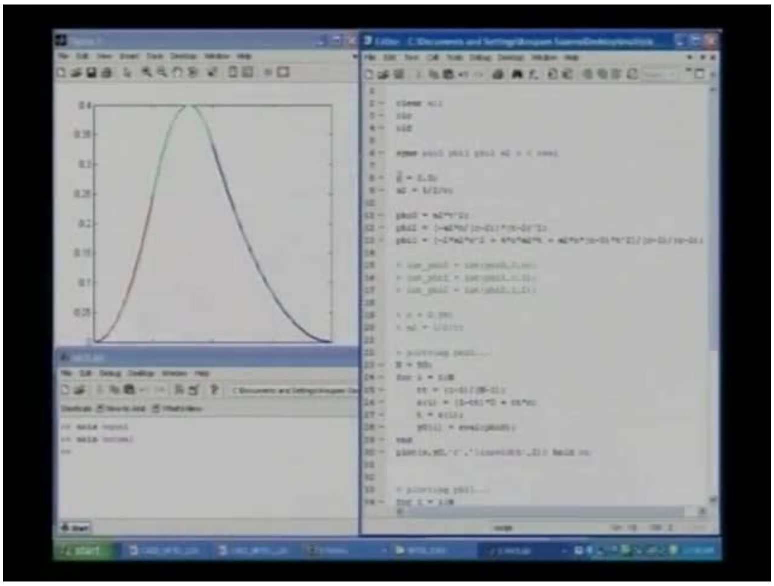

Now, this here on the right is a matlab code that I have written. You can see the expressions for phi 0 phi 2 and phi 1 here. I have initially chosen the knot value c as 0.5 and a 2 is given as 1 over 2 over c. Let us see how phi 0 phi 1 and phi 2 behave for this knot phi. You will reserve a nice c 1 continues B spline curve in red is phi 0, the curve in green is phi 1 and the curve in blue is phi 2. The knot value 0 equal 0.5, the knot value 1 at the knot value 2. Now, let me increase this so that c becomes closer to 1. Let me use a value 0.7 and execute the code again.

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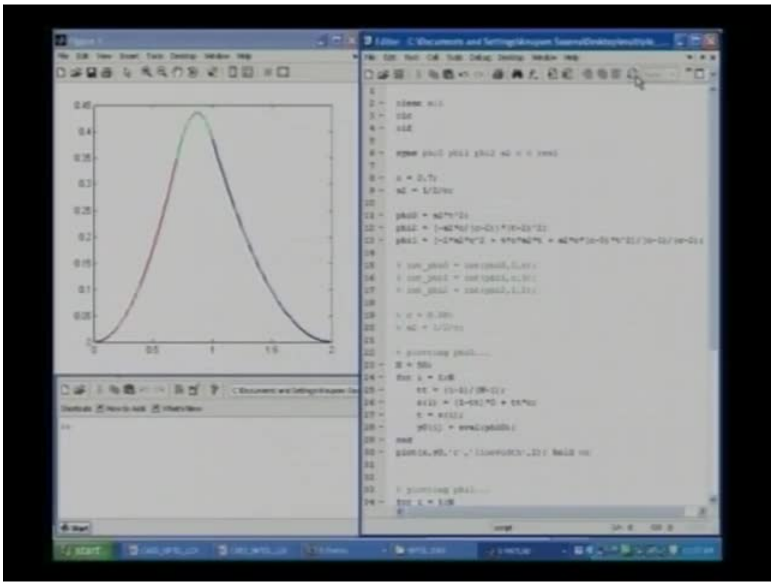

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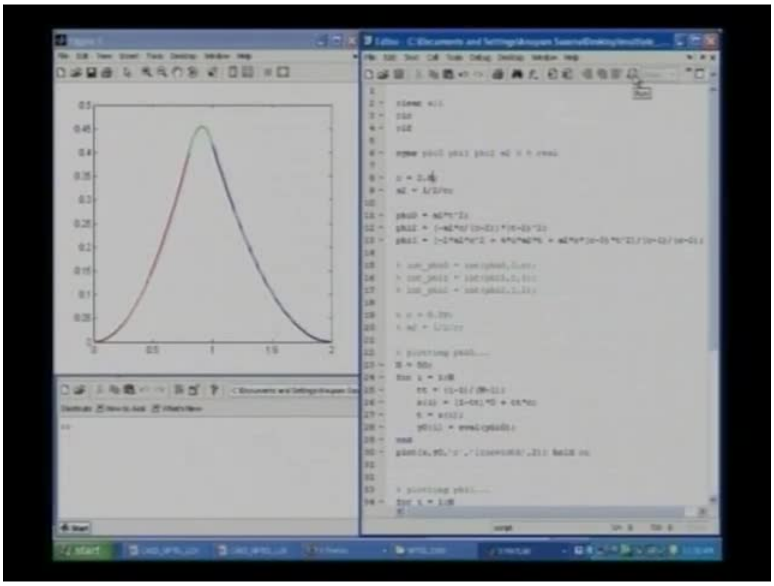

You would observe the change in the shape of this B spline function.This junction point here corresponds to c value equal 0.7. Let me further increase this value. Now, this junction point here is getting closer to this one and you would observe this portion is now trying to develop a peak. Now, that c equals 0.9.

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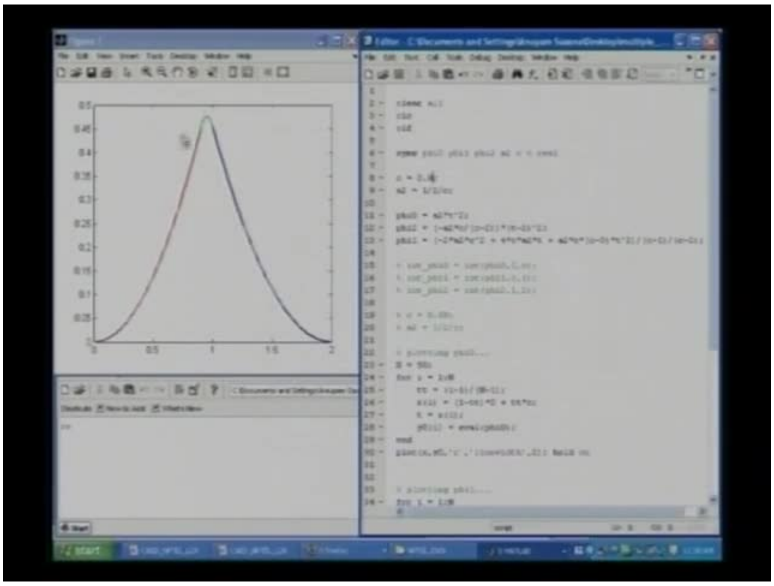

The peak is trying to get little more dominant here. May be c equal 0.99, it is very close to 1.

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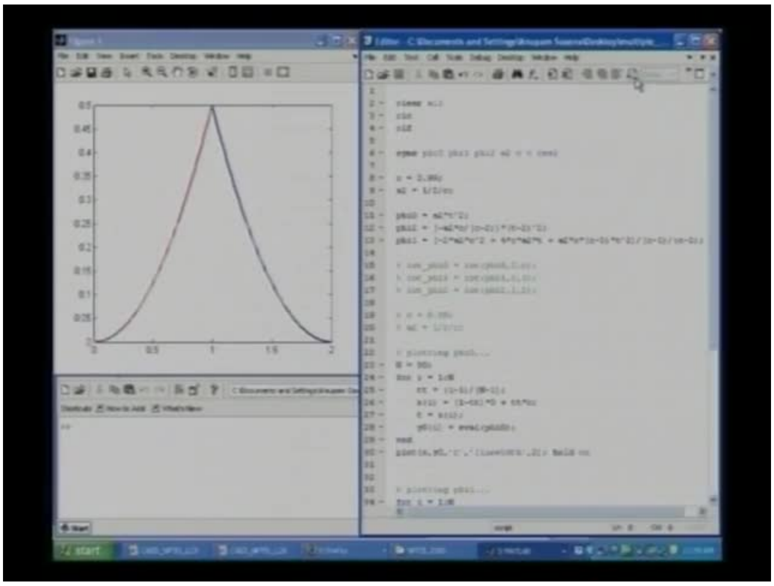

Now, you see a clear peak. Not quite though, you still if you observe closely you will see a little green patch here. Now, for this value let us try to get the expression for phi 0, phi 1 and phi 2. Phi 0 looks like 50 over 99 times t square, phi 1 looks like some complicated coefficience, but it is still quadratic of course, phi 2 would be quadratic as well. Let us try to now make this c value very close to 1, but not quite 1. You see a clearly defined peak here at this junction point initially you would observe that the B spline basis function was c 1 continuous.

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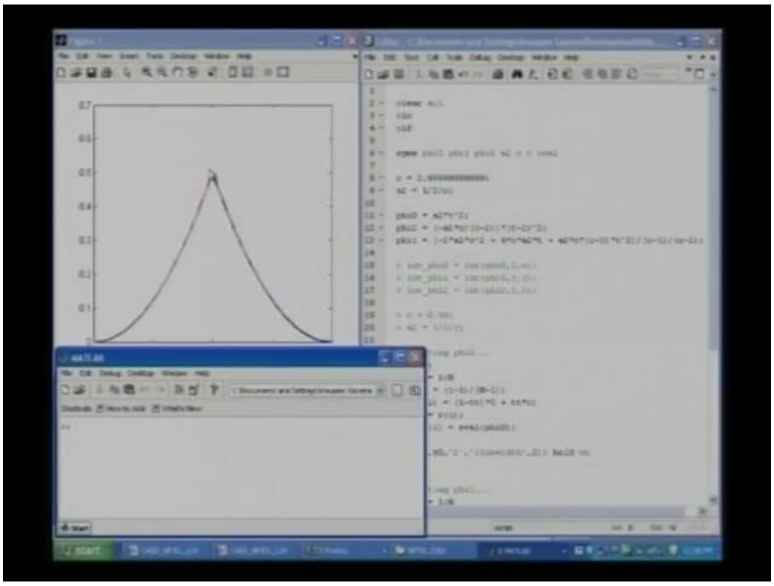

Now, you would see or clearly observe slope it is continuing here, but position continuity is still maintained. You would also observe that the green segment has vanished. So, only phi 0 and phi 2 that remain. Well has the green segment vanished? Let us find out. So, we have phi 0 again as quadratic, phi 2 again as quadratic and let us see worth of values for phi 1. I am storing the value of phi 1 in the array by 1, it is so happens that all values of phi 1 are equal to 0.5. It seems like a constant function but if you look at the expression for phi 1 it was still carry the term t square and if I plug in C equals 1 straight out making t equals 1 a multiple knot and if I execute the code.

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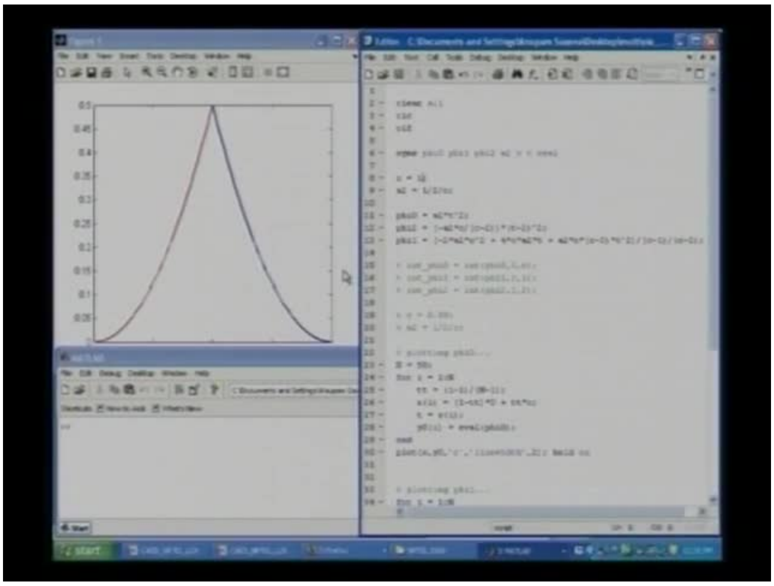

We clearly see slope discontinuity here. The expression of phi 0 is half of t square, that of phi 2 is half of t minus 2 squared and that of phi 1 is minus of infinity, is that so? You might want to check what happens to phi 1 at this junction point.

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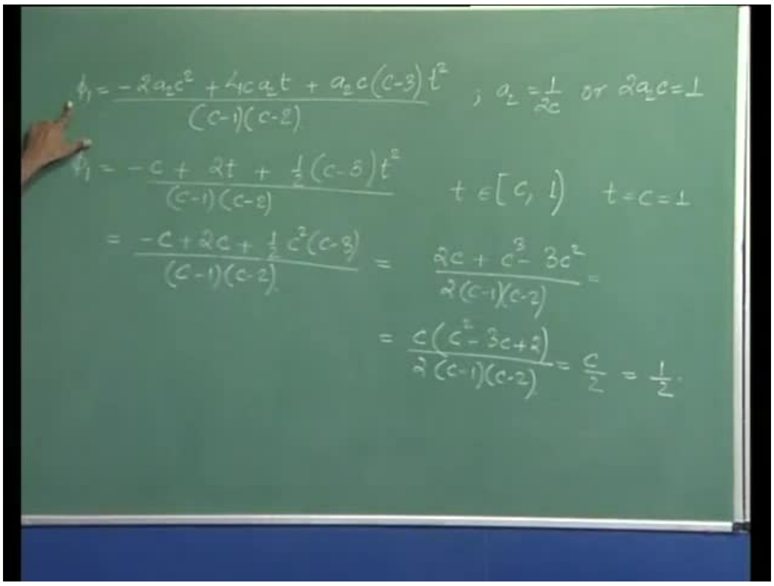

This is the expression for phi 1 here. Phi 1 equals minus 2 a 2 times c squared plus 4 times c times a 2 times t plus a 2 times c times c minus 3 times t squared entire thing over c minus 1 times c minus 2. Here a 2 equals 1 over 2 c or twice a 2 c equals 1. We would like to see what happens to phi 1 when c is equal to 1 and when t is also equal to c which is equal to 1. At this time you would observe that this factor c minus 1 would offer a singularity to phi 1.

Let us simplify this expression further. If we substitute for twice a 2 times c equal 1 we have phi 1 equals minus c 2 a 2 c is 1 plus I have 2 a 2 c is 1 t plus this become half c minus 3 t square over c minus 1 c minus 2. Now, first we should know that the values of t are permitted to be in between c and 1. This is the closed interval on the left, open interval on the right. When c is equal to 1, all this parameter t can take is the value equals c which is 1. So, let us substitute t equals c here.

This is minus c plus twice of c plus half c square times c minus 3 over c minus 1 c minus 2. This is equal to… This would becomes c times 2, 2 c plus c cube minus 3 c square over twice c minus 1 c minus 2, this would be equal to… I can take one c common. This is c square minus 3 c plus 2 over twice c minus 1 c minus 2. Now, if you notice a numerator here is nothing but c minus 1 times c minus 2. So, phi 1 evaluated at t equals c is equal to c over 2. And when c is 1 this would be 1 over 2. So, even though the matlab code was giving phi 1 as minus infinity when c was equal to 1. This analysis shows that that is not quite so. Phi 1 at t equals c equals 1 is finite and is equal to half.

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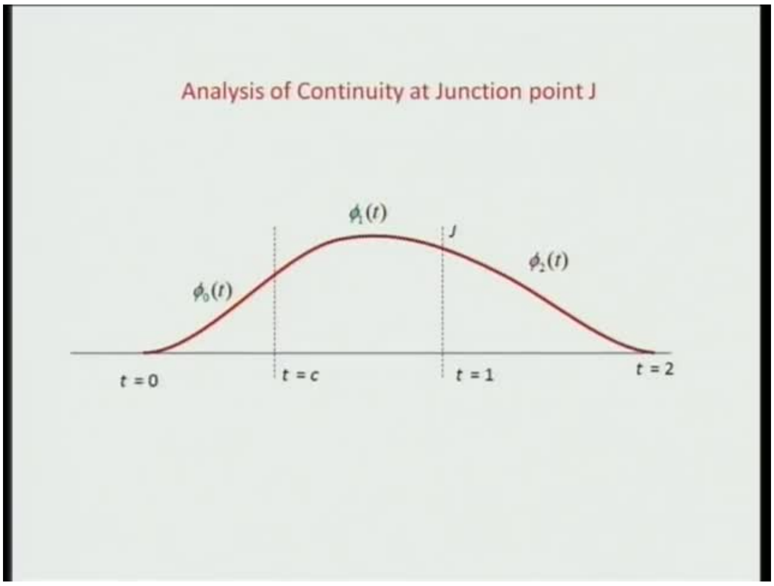

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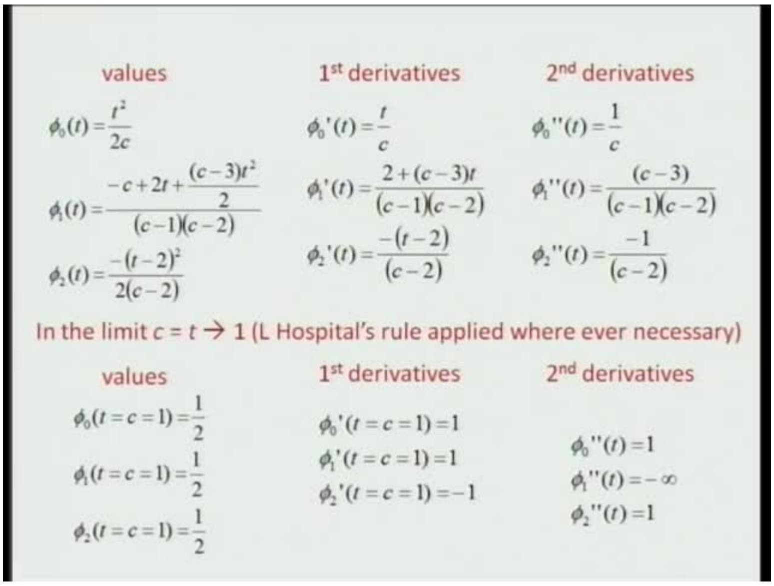

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