Computer Aided Engineering Design
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Lecture - 27

Good morning, we continue with our discussion on B-spline curves and segments previously, we had discussed the partition of unity property.

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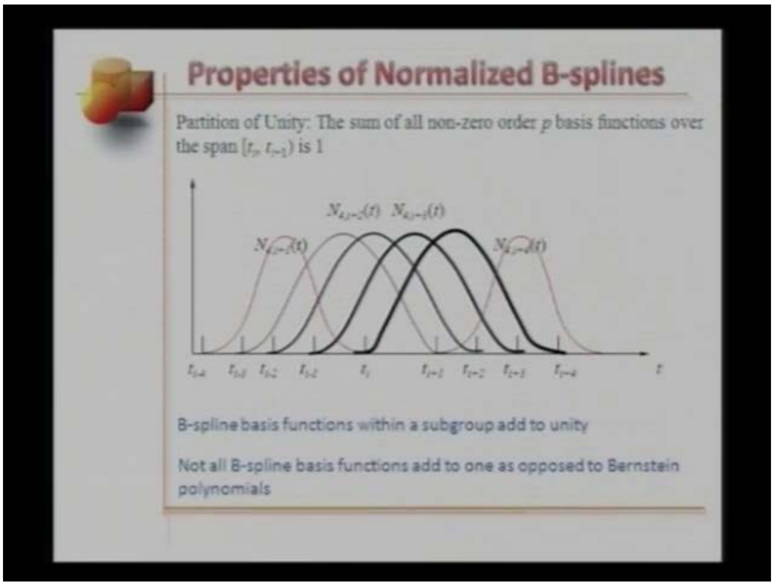

The property said, that the sum of all non zero order p basis functions, over the knot span t i, t i plus 1, t i is the closed end and t i plus 1 is the open end is 1. So, I had showed on board, that the sum of N 4 i plus 1 N 4 i plus 2 N 4 i plus 3 and N 4 i plus 4 for values of t in between t i and t i plus 1 will be 1. In fact, you would notice that, these are the only four non zero basis spline functions over it is knot span. Any other basis spline function for example, this red one over here or this red one over here will be 0 over it is knot span. So, in a sense, B-spline basis functions within a subgroup add to unity and not all B-spline basis functions add to 1, as we have seen in case of Bernstein polynomials, let me mention the property again.

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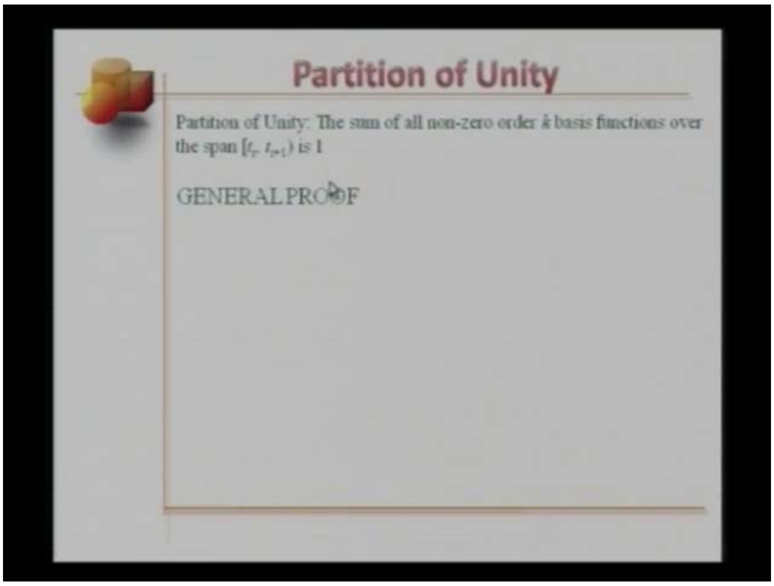

The sum of all non zero order k basis functions over the span t i t i plus 1 is 1, today I will try to give you a general proof.

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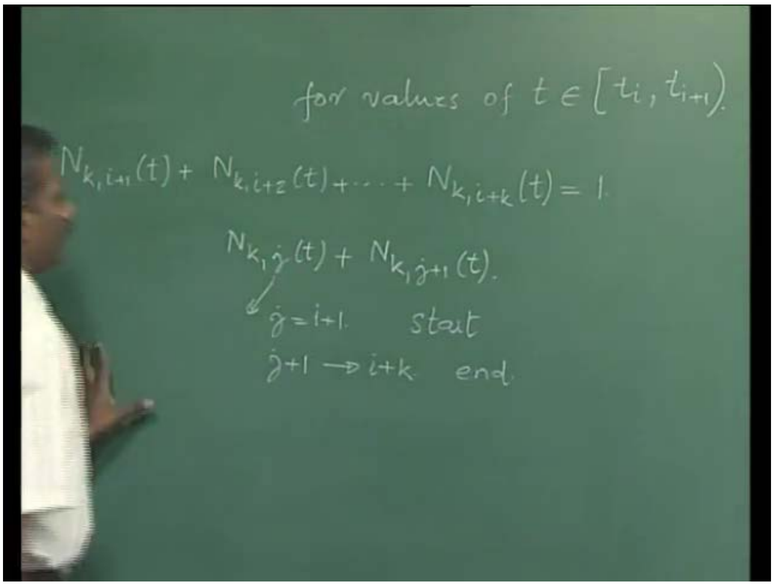

I will try to show, that the sum of k order k basis splines, which are N k i plus 1 t plus N k i plus 2 t plus, plus N k i plus k t is equal to 1 for values of t N closed t i t i plus 1 open. What we will try to do is, we first try to add two consecutive B-spline basis functions, which are N k j t and N k j plus 1. Now, if you notice, index j starts from i plus 1 and index j plus 1 would end at i plus k so, this is the start and this is the end. We will use the recursion relation to sum this consecutive basis splines and then go further from there.

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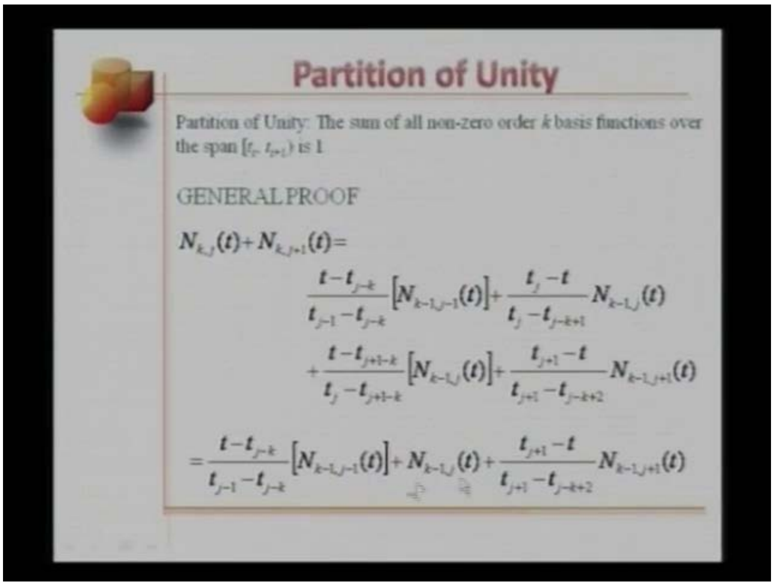

So, N sub k sub j of t plus N sub k sub j plus 1 of t is equal to t minus t j minus k over t j minus 1 minus t j minus k, times N k minus 1 j minus 1 of t plus t j minus t over t j minus t j minus k plus 1, times N k minus 1 j of t. This is the recursion relation corresponding to N k j and then all we need to do is, index j to j plus 1 to get another recursion relation. So, we have plus t minus t j plus 1 minus k over t j minus t j plus 1 minus k times N k minus 1 j t plus t j plus 1 minus t over t j plus 1 minus t j minus k plus 2 times N k minus 1 j plus 1 of t.

Now, concentrate on this term here and this term here, what you observe, you would see that N k minus 1 j is common and also the denominator t j minus t j minus k plus 1 is common. You might want to try adding these two terms together, what will happen to the numerator in that case, the numerator will become t j minus t plus t minus t j plus 1 minus k. If you realize, that is the same as the denominator, t j minus t j minus k plus 1 and they would cancel out.

What would be left will be N k minus 1 j t, which is right here so, we have the first term t minus t j minus k over t j minus 1 minus t j minus k times N k minus 1 j minus 1 of t plus, these two terms will add to give N k minus 1 j t plus the last term here, t j plus 1 minus t over t j plus 1 minus t j minus k plus 2 times N k minus 1 j plus 1 of t. Now, this is a result of two consecutive splines N k j and N k j plus 1 added together. We still have to sum those k basis spline functions, let me copy this result in the next slide like here.

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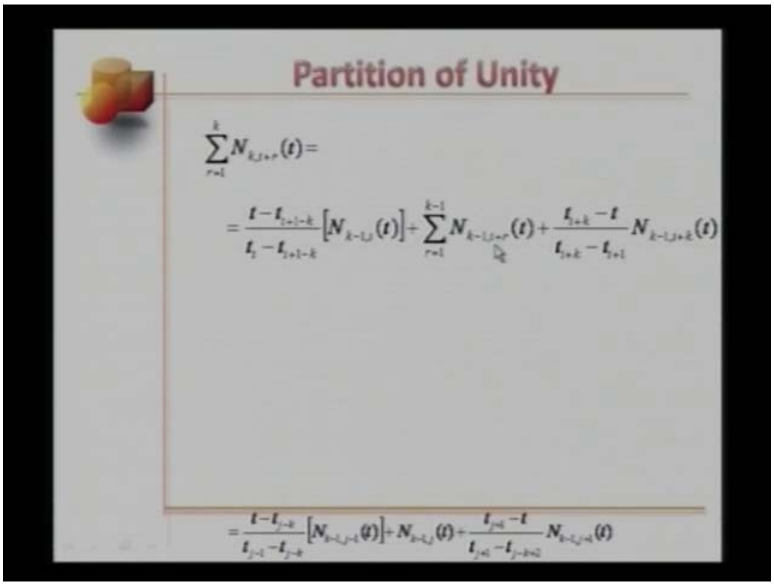

Now, if I perform the summation of all the k basis splines, what do I have, I have summation r going from 1 to k of N k i plus r of t and that would be equal to t minus t i plus 1 minus k over t i minus t i plus 1 minus k times N k minus 1 i of t plus summation r going from 1 to k minus 1 N k minus 1 i plus r of t plus t i plus k minus t over t i plus k minus t i plus 1 times N k minus 1 i plus k of t. This might look a little confusing, realize that the index j would start from i plus 1 and the index j plus 1 would end at i plus t. So, we will have the first term corresponding to j equals i, the last term corresponding to j plus 1, equals i plus k and then you will have k minus 1 terms getting added together, let me go back.

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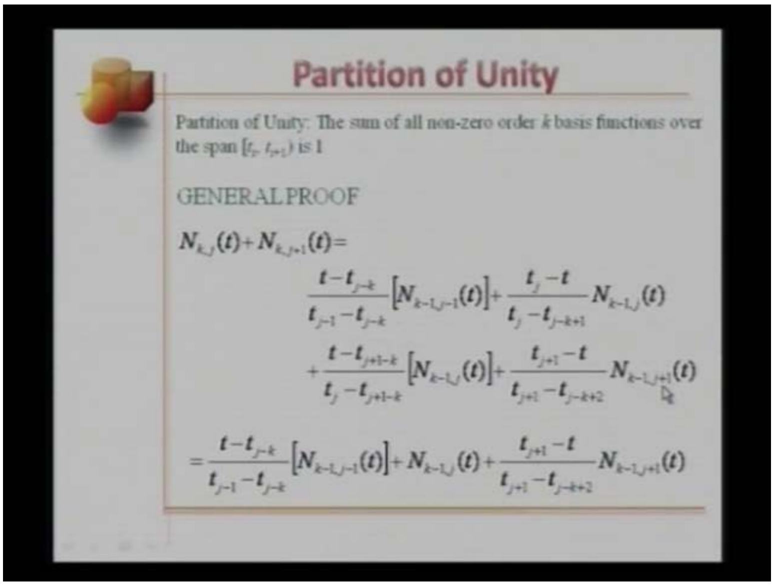

If you look at this expression over here, in the summation N k j and N k j plus 1, N k minus 1 j get added together. If I have let us say, N k j plus 2 again getting added over here correspondingly, like in the case of N k minus 1 j, we will have something very similar in case of N k minus 1 j plus 1 and so on.

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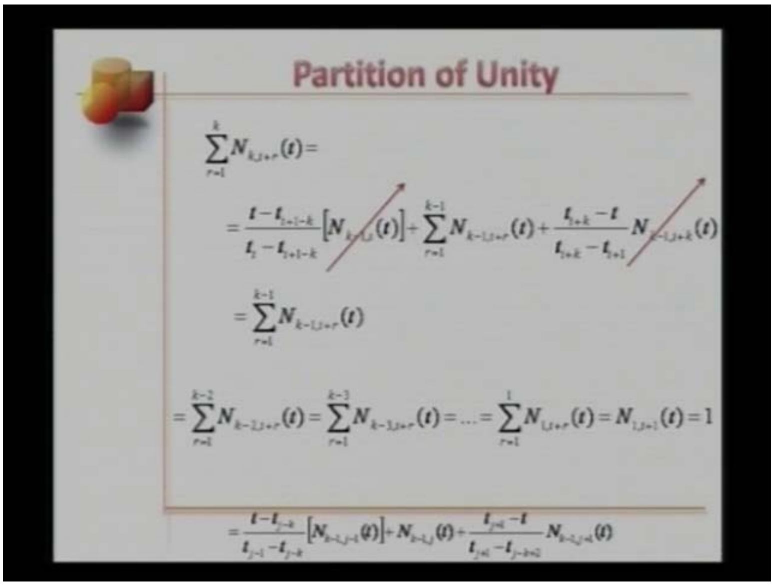

And that is why, these k minus 1 terms, they get added together now, look at this term here, N k minus 1 i, what would be it is value for t lying in between t i and t i plus 1. You get surprised it will be 0, because if you notice N k minus 1 i has the last knot as t i likewise, what would be the value of N k minus 1 i plus k for t again in between t i and t i plus 1. Notice that the last knot is t i plus k and the first knot is t i plus k minus k plus 1, which would mean the first knot is t i plus 1.

Again in the interval, t i t i plus 1, this basis spline of order k minus 1 will be 0 so, we are left with summation r going from 1 to k of N k i plus r of t, as summation r going from 1 to k minus 1 now, N k minus 1 i plus r of t. In a sense, the summation of k order k basis spline functions gets reduced to the summation of k minus 1 order k minus 1 basis spline functions. We can use this observation to further reduce this result to summation r going from 1 to now k minus 2 N k minus 2 i plus r of t.

We do not lose the generality, when we reduce this summation from this one to this one and we can keep on doing this further, summation r going from 1 to k minus 3 of order k minus 3 basis spline functions and so on up till we get summation r going from 1 to 1 of order 1 basis spline function. With the last knot, as t i plus r clearly, r equals 1 here so, this is equal to N 1 i plus 1 of t and you would know that, this order 1 basis spline function has the value 1 in the interval t i t i plus 1. So initially, you must have thought, that proving the partition of unity property would be difficult, that is not so.

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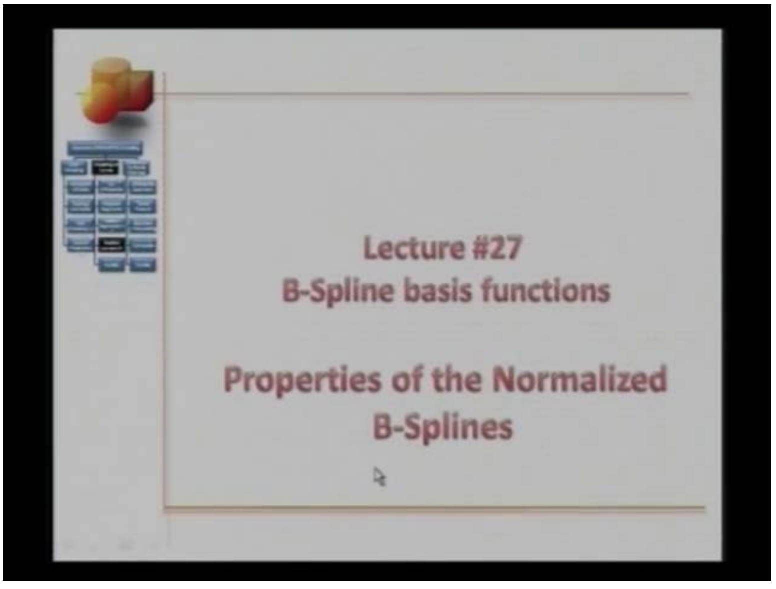

Let me continue with this lecture, which is number 27 and further, discuss properties of normalized basis splines.

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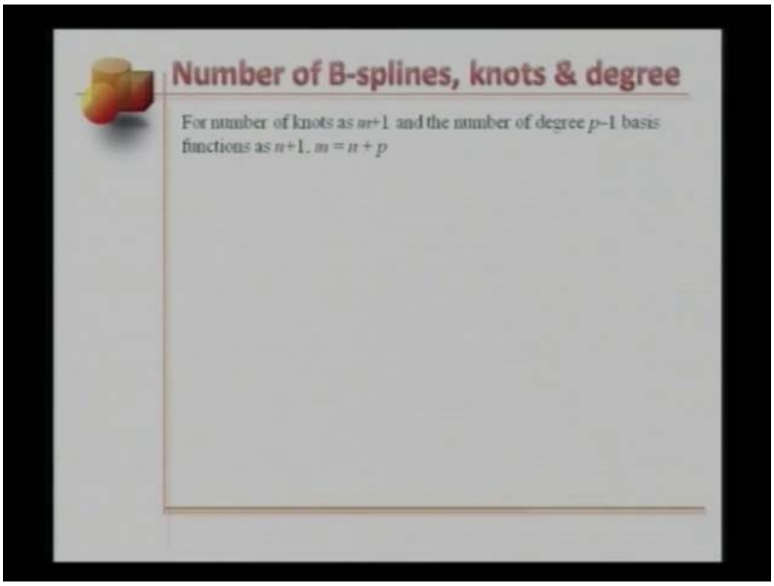

This one pertains to the number of basis spline functions, the knots and the degree for number of knots as m plus 1 and the number of degree p minus 1 basis functions, as n plus 1, m equals n plus p. In English what does it mean, it means that, if the number of order p basis spline functions is known then the number of knots gets automatically known. Let me work this thing out for you on board, I had mentioned earlier that, you should be comfortable sketching B-spline basis functions to be able to understand and workout these properties so, here we go.

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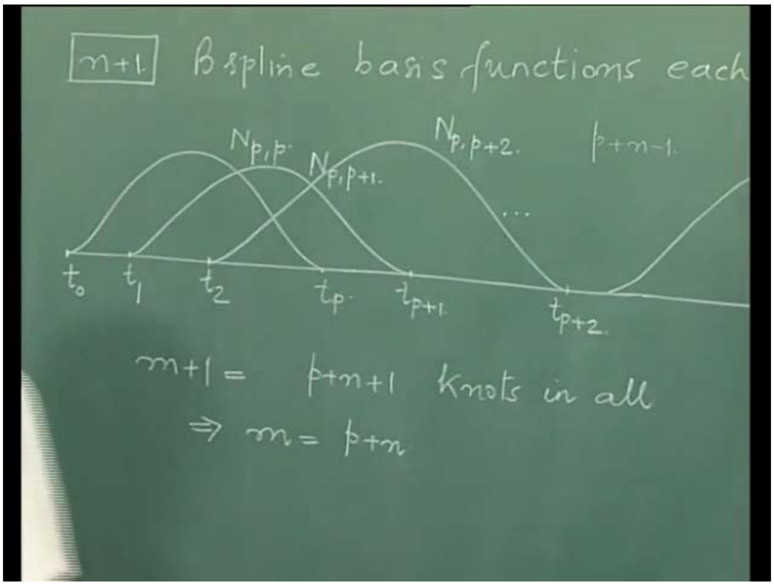

So, we have say n plus 1 B-spline basis functions each of order p, let us try to sketch them, we start with the knot t 0, the first one we sketch will be nomenclated as N p p. So, this knot here will be t p, the second one will start from t 1 will end at t p plus 1, this will be nomenclated as N p p plus 1. Likewise the third one, starts from t 2 ends at t p plus 2 and has the name N p p plus 2, I can keep doing this.

Remember, that I have to have n plus 1 basis functions, these are 3 basis functions, the last knot corresponding to the third basis function is t p plus 2. Can you guess, have the n plus 1 basis function will be standing over this parameter axis, let me sketch this thing for you arbitrarily. Let me first nomenclate so that, I am sure this is the n plus 1 basis spline, this would be N order p, p plus n plus 1, is that the case.

Now, let us analyze the second, for the first B-spline function, I have the last index is p, for the second I have p plus 1, for the third I have p plus 2, for the n th basis spline function I should have the index p plus n minus 1. And therefore, for the n plus 1, I should have the index p plus n, I have to be very careful in working with these indices. Now, what would be the last knot, over which N p p plus n will be standing, you get surprised again this is t p plus n. Start counting the number of knots t 0, t 1, t 2 up till t p plus n, these will be p plus n plus 1 knots in all. And so, if there are m plus 1 knots, they have to be equal to p plus n plus 1, which would imply that m equals p plus n, this one is the slightly different argument for the same property.

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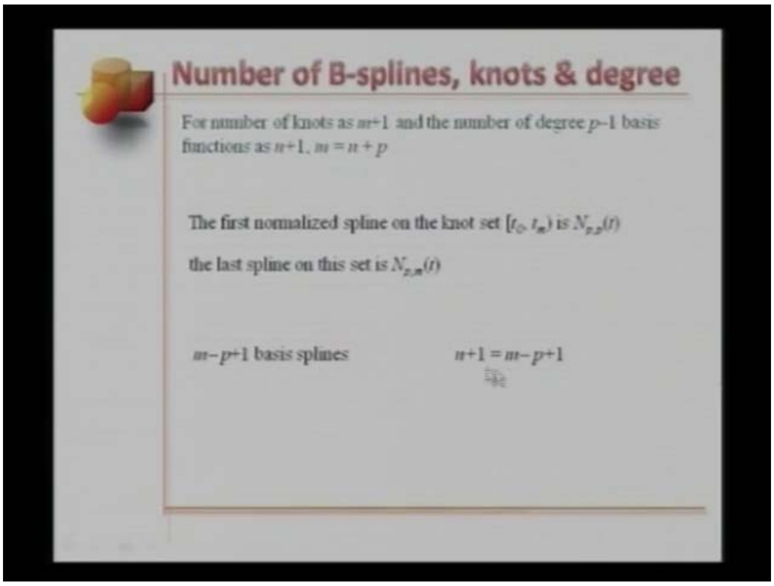

Now, if you are looking at the knots span t 0 closed t m open, the first normalized spline on this knot set is N p p of t, the last spline on this set has to be N p n of t clearly, because t sub m is the last knot in this span. Now, these if you count will be m minus p plus 1 basis splines now, since we have n plus 1 basis splines already, n plus 1 has to be equal to m minus p plus 1, which would give us the result m equals n plus p like here.

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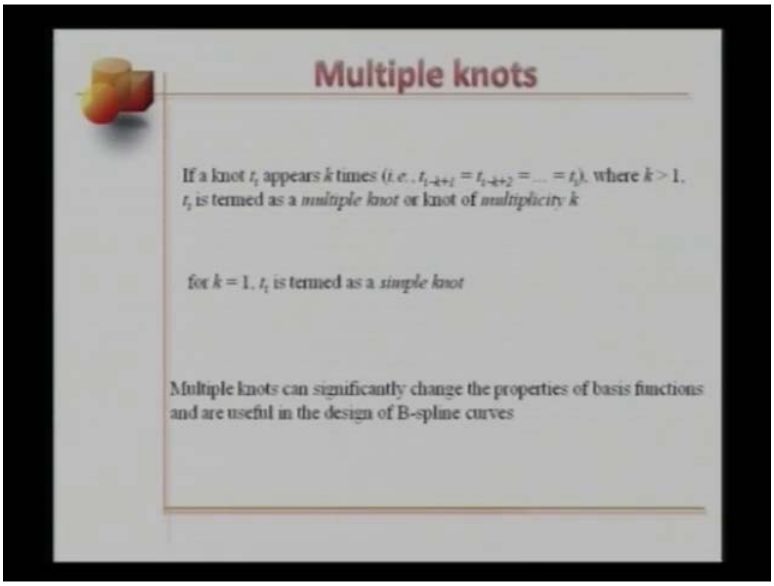

Let us now discuss multiple knots, it is a very interesting concept and today, we are going to be discussing this concept at a bit of length. The concept helps designing B-spline curves in a variety of it is, I will come to that may be in the next lecture or so. First what is the multiple knot, if a knot t i appears k times that is, if t i minus k plus 1 equals t i minus k plus 2 equals and so on up till t i where, k is greater than 1 then t i is termed as a multiple knot or to be more precise, a knot of multiplicity k. For k equals 1, t i is termed as a simple knot, as I mentioned earlier multiple knots can significantly change the properties of basis functions and are useful in the design of B-spline curves, we will see this later.

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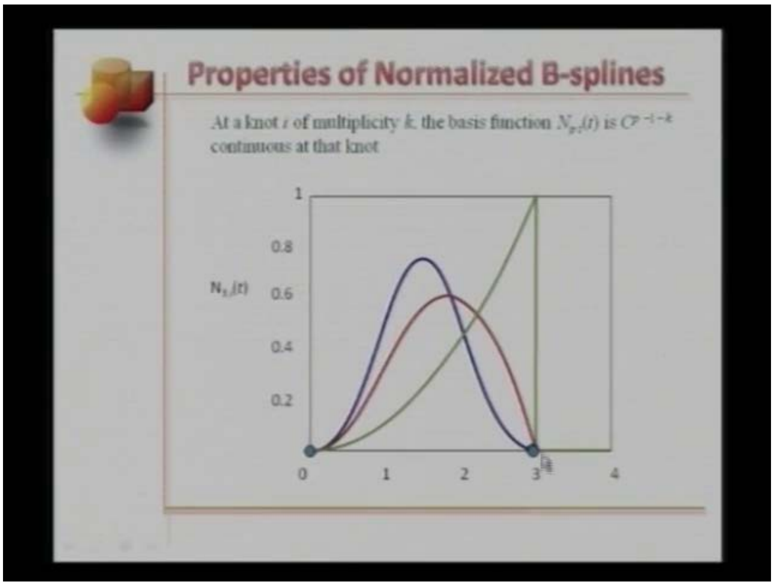

First let us look at the graphics, as to what happens if I use multiple knots, this is an important observation. At a knot i of multiplicity k, the basis function N p i of t is C p minus 1 minus k, continuous at that knot. Now, let us pause for a while and try to understand this, for simple knots that is, for k equals 1 and order p, basis spline function has to be C p minus 2 continuous everywhere. I could plug in k equals 1 here, we get the result C p minus 1 minus 1, which is c p minus 2 continuous and that is for simple knots.

Now, if I have 2 knots at the same position on the t axis, what would this statement basically mean, it would mean that I would lose the continuity of the curve at that junction point by 1. For example, if k equals 2 at a knot then the basis spline C p minus 3 continuous, if k equals 3 then at that knot which has multiplicity 3 now, the function will be C p minus 4 continuous. So, I will keep on losing the continuity conditions one by one, if I keep on increasing the multiplicity of an knot.

Here is an example of a quadratic basis function where, t i has a last knot, here i is equal to 3. If you notice we have 4 simple knots, in a sense 3 knot spans and this is how, N 3 3 of t looks like. If I move knot 2 to the right, what happens to the shape of the basis function, we will come to that now while. Let us first analyze or observe, that this basis function is C 1 continuous throughout, it has slope continuity here, slope continuity here and slope continuity throughout.

Now, once I have moved this knot to the right, the curve in red defects the shape of the new basis spline function, notice what happen here, I lost the slope continuity. If I further move knot 1 to the right, making the multiplicity of knot 3 as 3, the green one shows the new shape. Now, we have lost the position continuity, realize that we have a junk in position like here.

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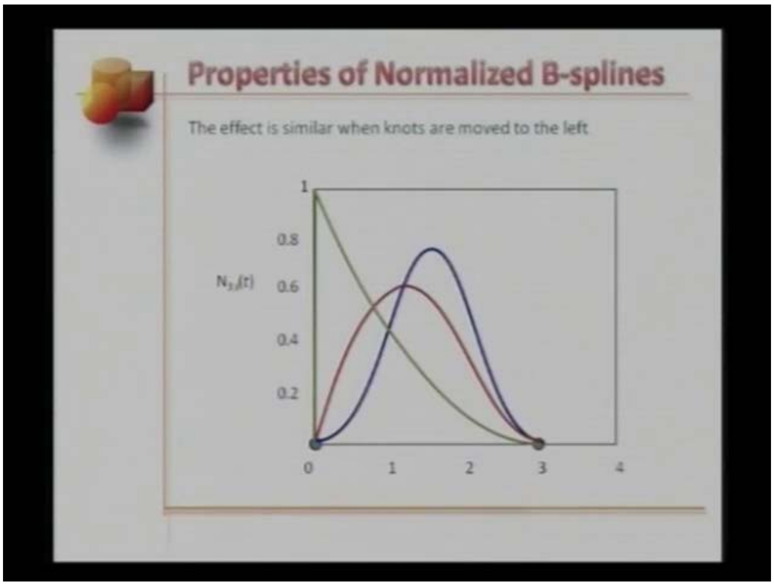

The effect is similar when knots are moved to the left, the same function in t I, t 0, t 1, t 2, t 3 as my 4 knots, this is the original basis spline function quadratic. If I move t 1 to t 0 making t 0 of multiplicity 2, I lose slope continuity here. If I further move t 2 to t 0, I lose position continuity further at this knot, why do you think this would happen, why is it. That, if we increase the multiplicity of a knot, we tend to reduce the continuity conditions at that knot one by one, we will discuss this in the later part of this lecture.

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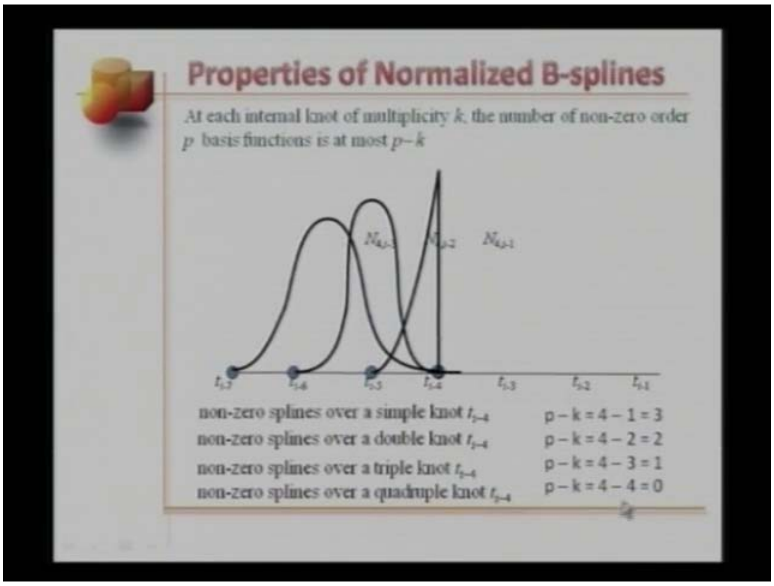

For now, at each internal knot of multiplicity k, the number of non zero order p basis functions is utmost p minus k. We can see this graphically, we have knots t i minus 7, t i minus 6, t i minus 5, t i minus 4, t i minus 3, t i minus 2 and t i minus 1. Let us concentrate on this knot here, t i minus 4 and let us try to build B-spline basis functions around it. Can you guess how you would name this B-spline basis function, write N 4 i minus 1 because it is standing over 4 knot spans with the last knot as t i minus 1.

Now, I have sketched N 4 i minus 1 in such a way that, this is non zero over t i minus 4, let us try to sketch the other spline functions. This one here is N 4 i minus 2, this one here is N 4 i minus 3 now, this example is a particular one of this generalized state here, p equals 4 in our case. If you realize for t i minus 4 to be a simple knot, non zero splines over this knot is equals to p minus k, which is 4 minus 1, which is 3, N 4 i minus 1, N 4 i minus 2 and N 4 i minus 3.

If I move t i minus 3 to t i minus 4, one of them now become 0 over t i minus 4 now, t i minus 4 having multiplicity 2, we have p minus k equals 4 minus 2 equals 2 non zero splines over t i minus 4. If we move t i minus 2 to t i minus 4, another one disappears we are left with only one non zero basis spline. And finally if I move t i minus 1 to t i minus 4, we are left with 0 basis spline functions, which are non zero over t i minus 4, you would notice that the multiplicity of this knot is 4.

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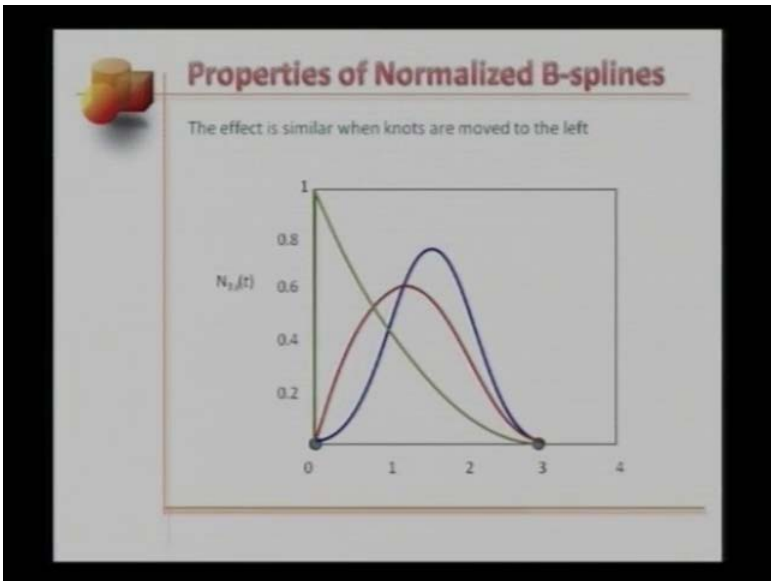

Now, let us come back and try to understand, why do we lose continuity conditions one by one at multiple knots. I will use the board to discuss this, I would like you to recall our discussion on polynomial splines, how did we construct a B-spline basis function using polynomial splines.

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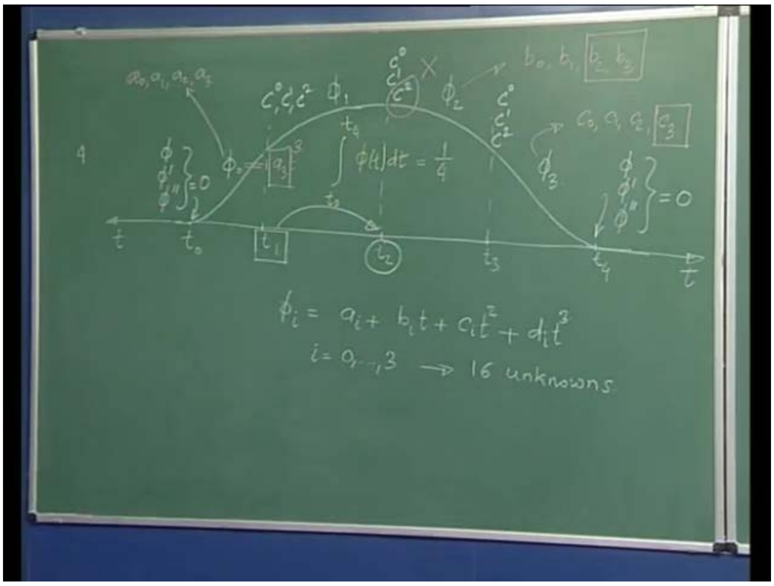

Let us try to construct an order 4 basis spline function, let me start with t 0, t 1, t 2, t 3, t 4 I would need 4 knot spans, this is the parameter axis t to the right and to the left. My B-spline basis function will start from t 0 and end at t 4, again I will have to be careful with regard to the continuity conditions, both at t 4 and t 0. Let me name the segment phi 0, phi 1, phi 2 and phi 3, each phi is a pure cubic polynomial.

I can write phi sub i as a i plus b i t plus c i t square plus d i t cube, I would go from 0 to 3 if you realize in all, we have 16 unknowns, 4 coefficients in each cubic polynomial. Now, if you remember, we started of with some conditions here and some conditions here specifically, we said, that phi, the slope and the second derivative are 0 here. Likewise, the position, the slope and the second this is slope here, the second derivative are 0 at t 4.

Also recall, that we had mentioned that the area under this curve that is, integral from t 0 to t 4 of phi t d t, phi of t is a set of phi 0, phi 1, phi 2 and phi 3, this is equal to 1 over 4, 4 is the order of this spline function. So, these are 3 plus 3 plus 1, 7 conditions, how about the rest of the conditions to be able to determine all 16 unknowns. Recall, that we had used position, slope and second derivative, continuity conditions at this junction point, at this point here and at this point here.

So, these are additional 3 plus 3 plus 3, 9 conditions, 9 plus 7 would be 16 conditions required to determine 16 unknowns overall. Now, this is something that you have known from 4, try to now guess what will happen if I move from knot t 1 to precisely layover knot t 2, making t 2 a multiple knot of multiplicity 2, there is another way I can argue. If you realize, you would need n plus 4 conditions, is this n plus 4 or n plus 3 you might want to check it out, n plus 3 conditions to completely an uniquely determine a spline function.

Here, n would be the number of knot spans, in this case it is 1, 2, 3, 4 and we have 7 conditions 3 plus 3 plus 1. Now, if I move t 1 to precisely layover knot t 2, think about it that I tend to lose 1 knot span in a sense, though initially n was equal to 4, after I have moved t 1 to t 2, n becomes equal to 3. In find now, that I would need 6 conditions, but I have 7 of them, 3 plus 3 plus 1. Clearly, I will have to lose one of the conditions, now moving t 1 to t 2, I would presume would not affect anything here around t 0, of for that matter anything here around t 4.

So, I do not think it is a good idea for me to lose any of these 3 conditions right here or any of these 3 conditions right here. I cannot let go this condition as well because this ensures non zero area under this curve, may be I am not doing this in right so, let me step back and look at this problem from the point of view of this coefficients. For phi 0 let us say, I have coefficients a 0, a 1, a 2, a 3 after I have snapped this knot span, I would not know what phi 1 is and let me not worry about it at this time.

Let me rather jump to phi 2 and say, the coefficients of phi 2 are b 0, b 1, b 2 and b 3 for phi 3 say, the coefficients are c 0, c 1, c 2 and c 3. In a sense now, these are my 12 unknowns, let me start from t 0, these 3 conditions will let us or allow us to express any of these 3 coefficients in terms of the fourth one. In fact, phi 0 will be equal to a 3 t cube, let me ignore phi 1 for now and jump to phi 2, I have c 0, c 1 and c 2 continuity conditions here.

Initially, if I use these continuity conditions, I would be able to express three of these in terms of say, b 3 and a 3. So, by the time I come here, I will have 2 unknowns. Now, if I work with phi 3 further, by the time I come here, I will have 3 unknowns c 0, c 1, c 2 using these 3 conditions, can be expressed in terms of say c 3, b 3 and a 3. So, by the time I reach t 4, I will have a 3, b 3 and c 3 as unknowns and I will have 3 conditions to greet them.

How about this condition, I will not be able to use it let us go back, at this junction point say, if I let go of c 2 continuity condition, if I do not use it, will I be able to compute my basis spline function. If I think about it, may be yes I start with a 3 as unknown here, a 0, a 1, a 2 are all z0, I come to t 2. With only these 2 conditions, I will be able to express 2 coefficients in terms of the other 2 coefficients in a sense, two of these coefficients will be unknowns. Let us say, b 2 and b 3, here a 3 is unknown and if I go on further and if I use all 3 conditions, I will have one more unknown by the time I come to t 4.

Now, we have 2, 3, 4 unknowns and 3 conditions here, the fourth condition here which is solvable. Now, the question you would want to ask your self is, why is it that I have decided to let go of the c 2 continuity condition at this junction point where, we have knot multiplicity of 2. I will ask you to keep an open mind here, although I had suggested, that we can let go of this c 2 continuity condition at this junction point. But, may be you can also think of letting go one of these conditions, may be these conditions or may be one of these conditions, it is a matter.

Mathematically, all we need to do is ensure that, you have the number of conditions the same as the number of unknowns, to be able to solve for this B-spline uniquely. Let me pose an open question for you to (( )), why is it that make sense for us only to let go of this condition and why not any other. We will discuss this in the next lecture but before I leave, let me give you an assignment.

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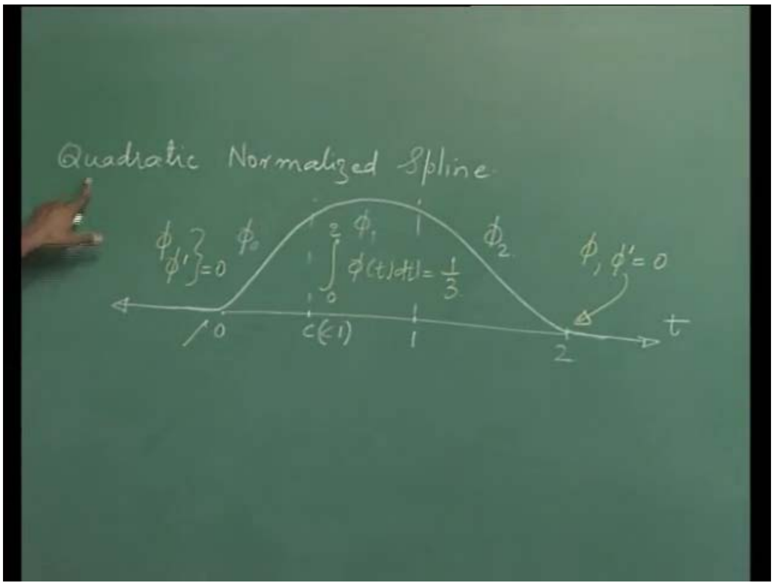

Let us construct a quadratic normalized spline over this knot span 0, c and this valuable is small than 1, 1 and say 2. We have t extending to both right and left, let us say this is a quadratic normalized spline or spline in general, does need to be normalized, divide this into 3 segments phi 0, phi 1, phi 2. Now, since this is quadratic, you need to be a little careful, uniquely ensure that position and slope are 0 at the starting knot and position and slope are both 0 at the any knot.

Now, these are 2 plus 2, 4 conditions and 5 th one is integration from 0 to 2 phi t d t equals 1 over 3, 3 is the order of quadratic spline so, you have 5 conditions. So, this is what the idea is, you are trying to do an experiment, you are trying to construct quadratic normalized spline with arbitrary value of this knot. Here, c which is smaller than 1 so, that c happens to be a simple knot and these are simple knot spans. And once we get the expressions for phi 0, phi 1 and phi 2, we will let c tend to 1 so that, this knot here becomes a multiple knot. And observe what happens to this quadratic spline overall, let us meet next will the solution to this problem.