Computer Aided Engineering Design
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Lecture - 26

Good morning, we will continue with our lectures on computer aided engineering design. We have been discussing B spline segments and curves. This is lecture number 26 we will discuss normalized basis splines and their properties today.

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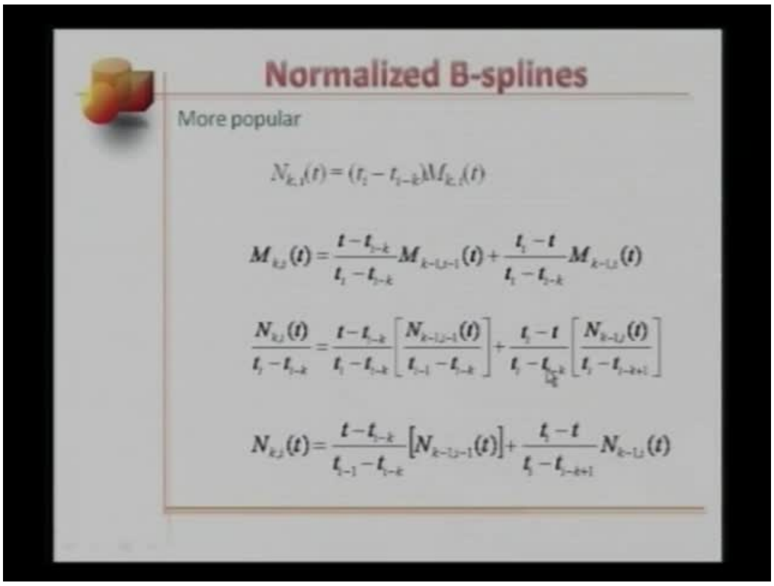

Normalized B splines, they are more popular than regular B spline basis functions. A normalized B spline basis function is given by N sub k sub i of function of t which is equal to t i minus t i minus k times M k i t. In the sense, if we multiply the regular B spline basis function by its entire knot span over which it is spanning t i minus t i minus k, we get the normalized basis function. It is very easy as I mentioned before for me to remember the expression for M sub k sub i of t. So, M k i t is equal to t minus t i minus k over t i minus t i minus k times M k minus 1 i minus 1 a function of t plus t i minus t over t i minus t i minus k times M k minus 1 i t.

Now, let us substitute for these M’s over here. So, M k i of t will be N k i of t over t i minus t i minus k. This relation comes from here. Likewise, M k minus 1 i minus 1 t will be N k minus 1 i minus 1 t over t i minus 1 minus t i minus k. t i minus k is i minus 1 minus k minus 1 and then M k minus 1 i is N k minus 1 i over t i corresponding to the last knot minus t i plus k plus 1. This is i minus k plus 1.

We can simplify this relation to get N k i t equals t minus t i minus k over t i minus 1 minus t i minus k times N k minus 1 i minus 1 of t plus t i minus t over t i minus t i minus k plus 1 times N k minus 1 i of t. What we have done here is if this factor comes in the numerator here on the right hand side, this factor gets cancelled with this factor and this factor here.

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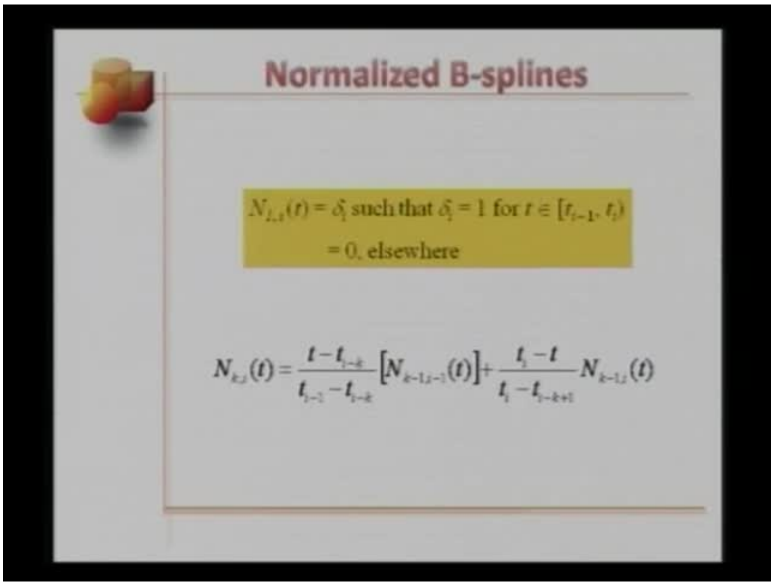

So finally, normalized basis splines can be defined as follows. N 1 i which are the order one normalized splines of function of t is equal to delta i such that delta i is equal to 1 for values of t belonging to closed t i minus 1 comma t i open. This essentially means that all values of t will include the value t i minus 1, but will not include the value t i. For all other values of t not belonging to this interval here N 1 i is equal to 0, this is for order one normalized basis splines.

For order k normalized basis splines where k is greater than 1 we have the recursion relation that we have seen before. N k i t equals t minus t i minus k over t i minus 1 minus t i minus k times N k minus 1 i minus 1 t plus t i minus t over t i minus t i minus k plus 1 times N k minus 1 i of t. Now, this normalized basis spline as you would know is non zero for values of t in between t i and t i minus k. Here I am going from right to left, for all other values of t not belonging to this knot span over which N k i stands this normalized spline is 0.

In some sense the normalized basis spline will derive all properties of your general basis spline function M k i t, just that the only difference between the two would be that of the factor t i minus t i minus k. Let us look at some properties of normalized basis spline functions.

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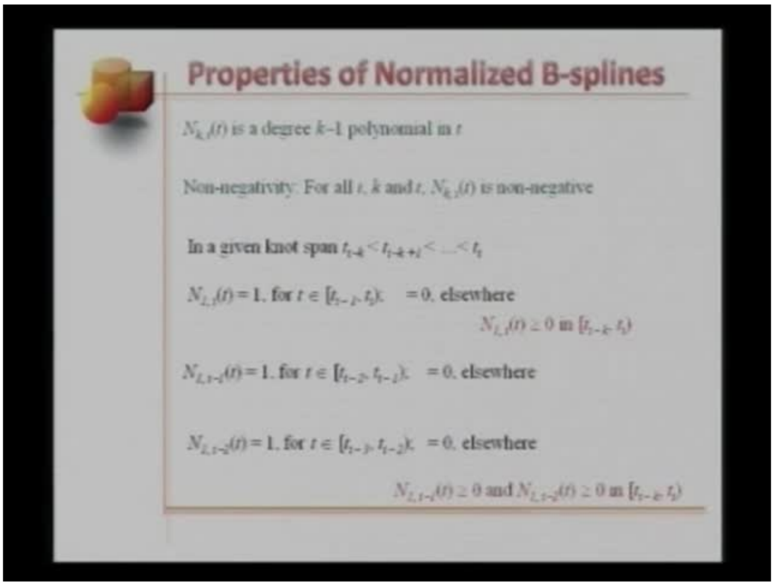

So, in the previous lecture I had mentioned that it is important for you and me to be comfortable in sketching these basis spline functions and normalized basis spline functions. If you are comfortable in sketching them you do not need to remember these properties. All you would need to do is simply work them out and we will see that today.

The first one N k i of t is a degree k minus 1 polynomial in t. k is the order of the normalized B spline basis function. So, naturally k minus 1 will be the degree of this polynomial. You would also note that N k i of t is not a pure polynomial, rather it is a piecewise continuous composite polynomial. I would want to take you back to the definition of B spline curves or segments. A B spline curve of order k has to be c k minus 2 continuous throughout.

This is the second property, non negativity. For all i k and t N k i t is non negative. This is something we have not seen before, but the recursion relation that we had derived in the previous lectures allows us to visualize this property. Let us try to give an informal proof. So, in a given knot span t i minus k t i minus k plus 1 and so on up till t i notice that these knots are arranged in ascending order. We would realize that N 1 i of t is equal to 1 for values of t belonging to closed t i minus 1 open t i interval and N 1 i of t is equal to 0 for all other values of t.

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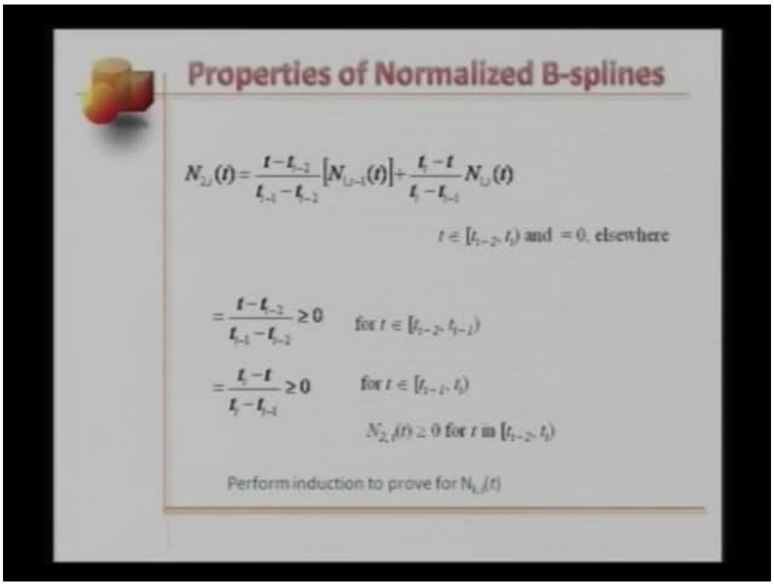

In essence N 1 i of t is equal to or greater than 0 in t i minus k t i which is appearing in the interval here. If that is so then we can say something very similar about this order one normalized basis function, N 1 i minus 1 of t is equal to 1 for t and t i minus 2 t i minus 1 and N 1 i minus 1 is equal to 0 elsewhere. Likewise N 1 i minus 2 of t is equal to 1, for values of t belonging to the interval t i minus 3 t i minus 2. t i minus 3 included, t i minus 2 not included and N 1 i minus 2 is equal to 0 for all other values of t here. These two statements lead us to the following observation. N 1 i minus 1 of t is equal to or greater than 0 and likewise N 1 i minus 2 t is also equal to or greater than 0 in the period knot span t i minus k t i. Where are we heading with this?

Let us see N 2 i as you would know from the recursion relation will be a weighted linear combination of N 1 i minus 1 and N 1 i. To be more precise N 2 i of t will be equal to t minus t i minus 2 over t i minus 1 minus t i minus 2 times N 1 i minus 1 t plus t i minus t over t i minus t i minus 1 times N 1 i of t. Now, we have seen previously that N 1 i minus 1 is equal to or greater than 0 in the period interval t i minus k t i and likewise N 1 i also behaves in a similar fashion. Now, let us look at these weight functions here which are linear as you would realize. Of course, N 2 i is defined to be non zero in the interval t i minus 2 t i, i is going to be 0 for all other values of t.

Now, if you look at these weight functions here. t minus t i minus 2 over t i minus 1 minus t i minus 2 will be equal to or greater than 0. Why? Because I have already mentioned before that these knots are arranged in ascending order. So, t i minus 1 is greater than t i minus 2 and values of t lie between t i minus 2 and t i. So, the numerator is going to be greater than 0 and the denominator as well will be greater than 0. Again, if you look at this factor here t i minus t over t i minus t i minus 1 will be greater than or equal to 0 again. Why is that? t is smaller than t i maybe t can also be equal to t i and t i is of course, greater than t i minus 1. So, this would make this factor greater or equal to 0 for value of t belonging to the interval t i minus 1 t i.

Now, when I say that this would be equal to 0 I mean in the limiting sense. Notice, that we are using the open interval here. The value of t can be as close to t i as possible, but not equal to t i. We would want to keep that in mind. Well what do we observe? So, we have seen that for values of t in between t i minus 2 n t i and 1 i minus 1 is greater than 0 or equal to 0 and 1 i is greater than or equal to 0 and these weight factors are also equal to or greater than 0. Naturally, N 2 i will be equal to or greater than 0 for t lying in the interval t i minus 2 t i. We can use similar arguments and show using mathematical induction that N k i of t is equal to or greater than 0 for values of t in between t i minus k and t i. I would want to take you back to our previous lectures.

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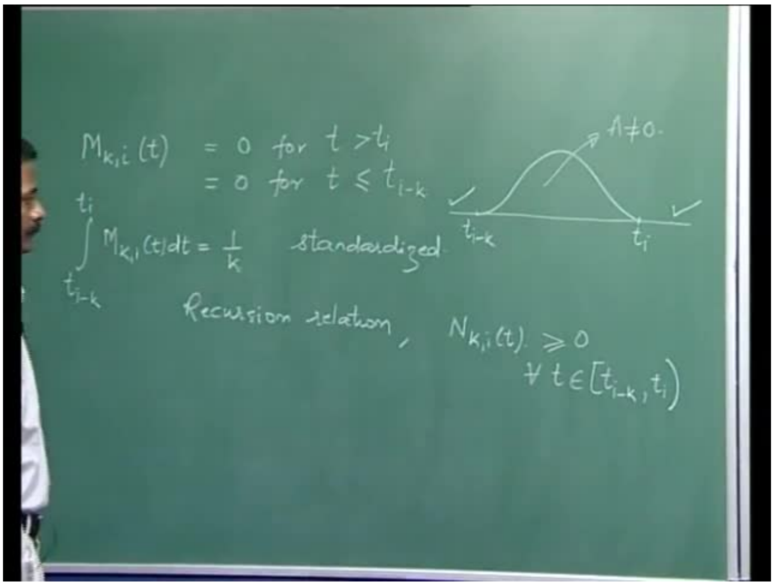

Recall, we had made a few comments about M k I of t. We had said that this is equal to 0 for t greater than t i, this is equal to 0 again for t smaller than or equal to t i minus k and integral t i minus k to t i m k i of t d t was equal to 1 over k, because M k i t d t was designed to be standardized. So, if you look at the bell shaped characteristic of this function we had worked on this part for values of t greater than t I, we had worked on this part for values of t smaller than t i minus k and we had shown that the area is non zero.

If you recall we had never talked about the non negativity of this function for values of t in between t i minus k and t i. It is the recursion relation that allows us to see that M k i of t or for that matter N k i of t is non negative in this range of t. In the sense N k i of t is greater than or equal to 0 for all t belonging to closed t i minus k t i open.

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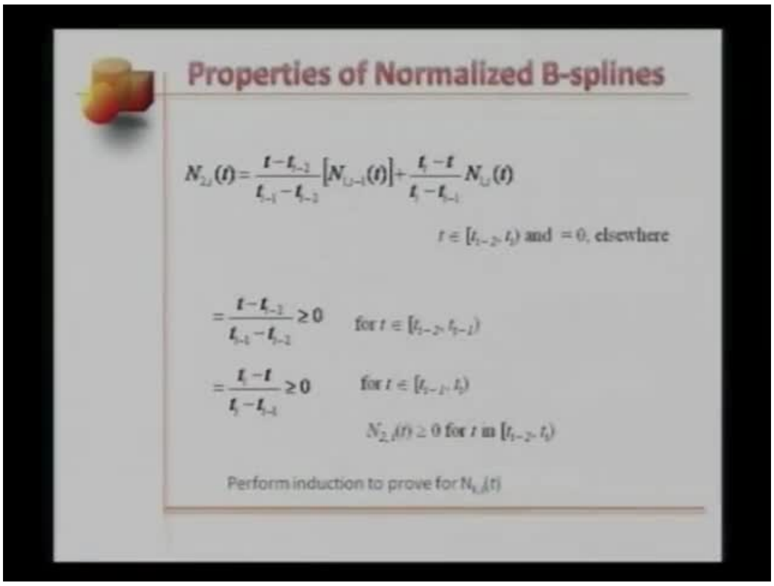

Let this be an exercise for you to show that indeed the normalized basis spline function N k I of t is equal to or greater than 0 for in fact all values of t ranging from minus infinity to plus infinity. In particular for values of t in between t i minus k and t i this basis spline function is non negative.

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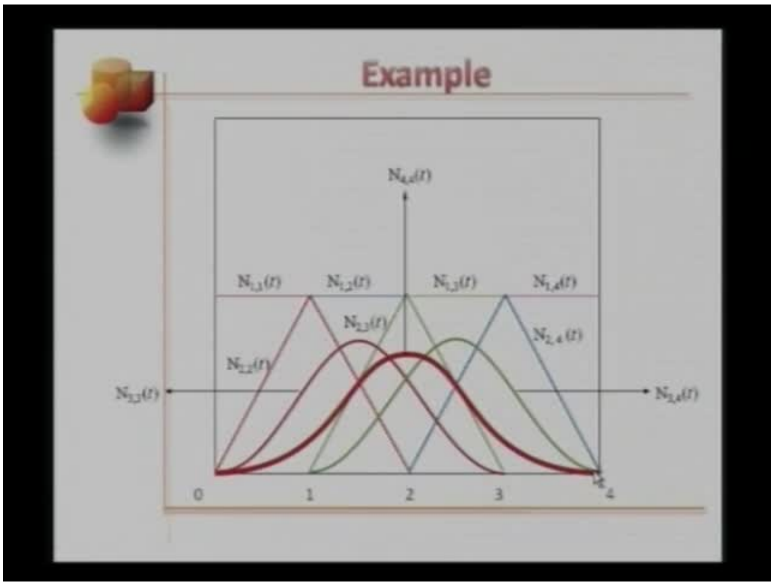

Just take a look at an example. Let us work with uniform knot span. t 0 is 0, t 1 is 1, t 2 is 2, t 3 is 3 and t 4 is 4. This spline here would correspond to N 1 1, the last knot is t 1 and this of order 1 or degree 0. It has to be a constant likewise this blue line here corresponds to N 1 2 of t. The green line here corresponds to N 1 3 of t and the magenta line here is N 1 4 of t, last knot is t 4 the first knot is t 3. All these normalized basis functions, they stand over a single respective knot span as you would observe.

Now, this triangle corresponds to N 2 2 of t, it is a linear interpolation between N 1 1 and N 1 2. It satisfies the definition of a spline, it is of order two meaning it is of degree 1 that would imply that it has to have position continuity throughout. So, there would be three junction points, junction 1 2 and 3. If you observe N 2 2 will be 0 here. There would be a position continuity maintained here, it will be non zero in fact it will be increasing in the first knot span between t 0 and t 1.

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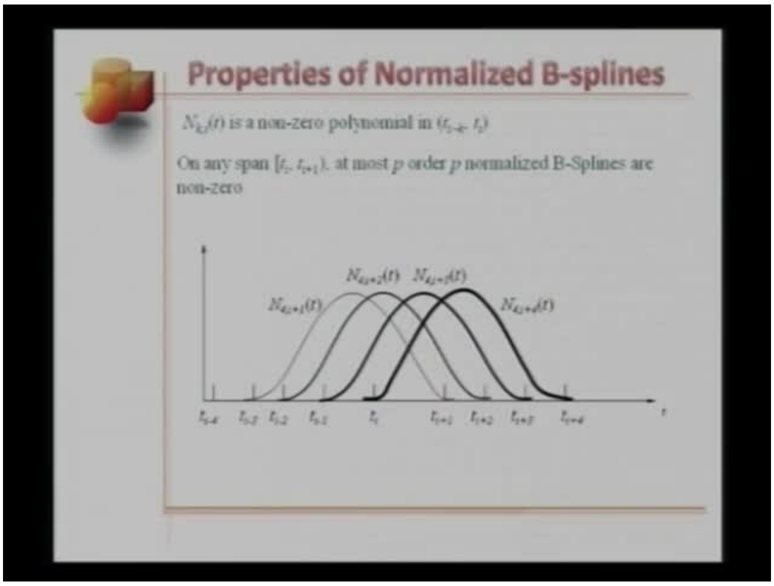

There will be position continuity here. Notice, that slope is discontinuous and then N 2 2 will be decreasing in the knot span t 1 t 2 and finally, at this junction point or rather from this junction point N 2 2 will be 0 for t greater than 2. This here is N 2 3 t. A linear combination between N 1 2 t and N 1 3 t and this here will be N 2 4 t, combination of N 1 3 t and N 1 4 t. This is a quadratic normalized basis function N 3 3 of t that combines N

2 2 of t and N 2 3 of t. This one here is N 3 4 of t. This combines N 2 3 of t and N 2 4 of t and finally, this normalized basis function that stands at the knot span t 0 t 4 is N 4 4 of t.

We continue with the properties of normalized basis B spline functions. N k i of t is a non zero polynomial in t i minus k t i. This should be closed interval and this should be open interval. This is very clear. This comes from the standardization condition that the area under the basis function N k i of t is non zero. The next one and this is an important one. I would want your attention here.

On any span closed t i t i plus 1 at most p order p normalized B splines are non zero. Let me show you an example and then discuss a general case on the board. We start with a case which has the knot span t i minus 4 t i minus 3 t i minus 2 t i minus 1 t i t i plus 1 t i plus 2 t i plus 3 and t i plus 4 all arranged in ascending order and an interval of importance is this one here t i t i plus 1.

Let us sketch an order four B spline basis function, the last knot will be t i plus 1, the first knot will be t i minus 3, this is N 4 i plus 1 of t. As you would see this function is non zero in this interval. What is this function? This stands over 4 knot spans 1 2 3 and 4 and hence at the knot t i plus 2 it has to be N 4 i plus 2 of t. How about this one? Again, standing over 4 knot spans and ending at knot t i plus 3. This is N 4 i plus 3 of t and finally, we have N 4 i plus 4 of t.

Now, notice that all these normalized basis splines are non zero over this knot interval. If I wish to sketch any other normalized basis function involving this interval here t i t i plus 1 I will have to either end that basis function at t i or will have to start a basis function from t i plus 1. In a sense any other basis function that I draw will have to be 0 over this knot span t i t i plus 1. We see here in case of an order four basis spline function that about one two three and four, order four basis spline functions are non zero over the knot span t i t i plus 1.

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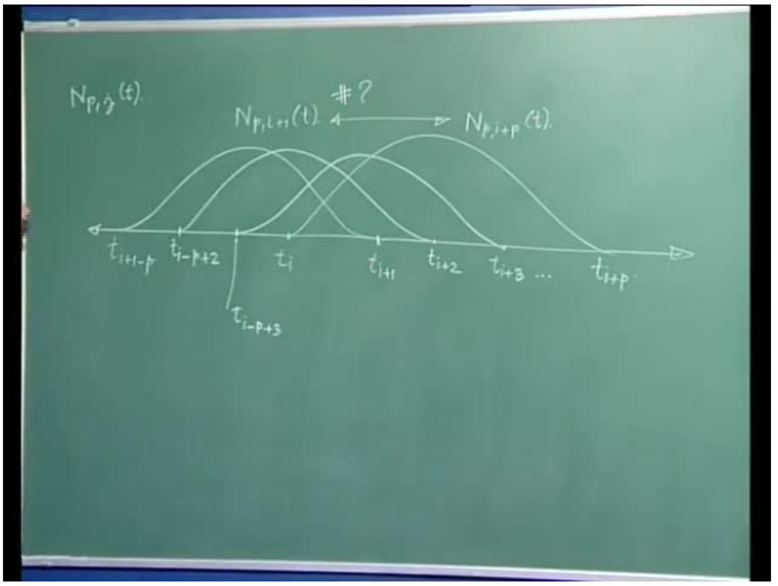

Let me now explain a general case on the board for you. So, let me have this interval of importance t i t i plus 1, let me extend this to the right and to the left. And let me work with order p normalized basis functions, j would be any knot p j at which N p j would end. If I start with knot t i and sketch a normalized basis function, this would end at t i plus p and this B spline here will be nomenclated as N p i plus p of t, of course, N p i plus p of t will be non zero over this span here.

Now, if I sketch another B spline basis function normalized of course, in a manner that it ends at t i plus 1. So, I will have to sketch this thing backwards. The starting knot will be t i plus 1 minus p and this knot here will be N p i plus 1 of t. Now, let us try sketching a few more of these normalized basis functions of order p for example, t i minus p plus 2, if I start from here I will end at t i plus 2, t i minus p plus 3 if I start from here I will end at t i plus 3 and so on. All of these will be clearly non zero over this knot span. So, in a sense all order p normalized basis splines starting from N p i plus 1 of t and ending at N p i plus p of t will be non zero. You might want to count how many of these are?

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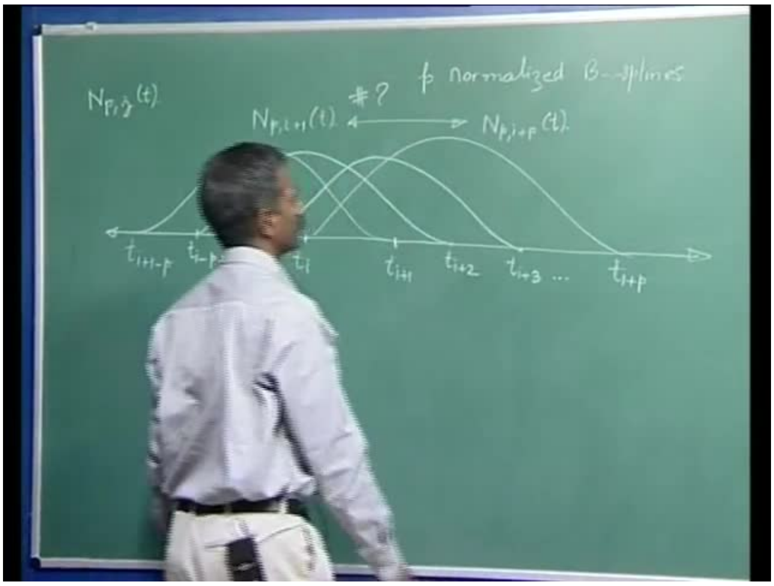

N p i plus 1 N p i plus 2 N p i plus 3 up till N p i plus p you would realize that these are p normalized B splines.

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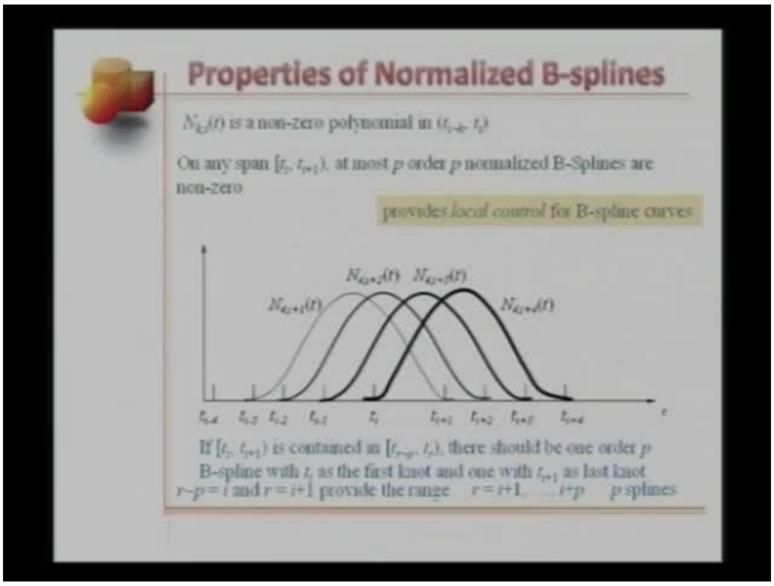

In general for any r N p r of p is greater than or equal to 0 in the knot span closed t r minus p up till t i. If closed t i t i plus 1 open is contained in closed t r minus p open t r, it should be 1 order p B spline with t i as the first knot and 1 with t i plus 1 as the last knot. We have seen this before. So, r minus p equals i and r equals i plus 1 would provide a range for such normalized basis spline functions, implying that r would go from i plus 1 up till i plus p and they would be p normalized B spline basis functions. Now, this is one of the properties that would provide local control specifically local shape control for B spline curves. Let us look at the other one now.

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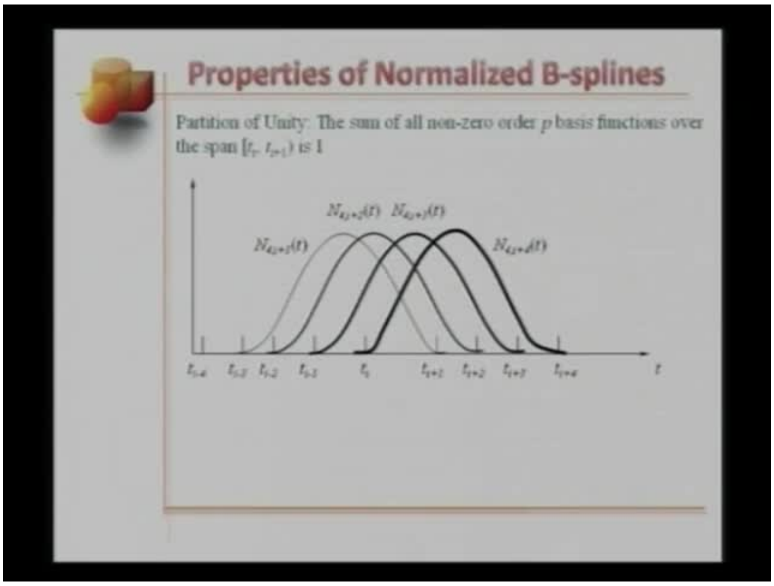

This is again important and I would want your attention on this one. Partition of unity, the sum of all non zero order p basis functions over the span closed t i open t i plus 1 is 1 no matter what the t value over the parameter value is. Let us revisit the example in the previous slide. So, we had sketched four of the order four normalized basis functions N 4 i plus 1 N 4 i plus 2 N 4 i plus 3 and N 4 i plus 4, all functions of t and we had seen that these basis functions were non zero over the knot span t i t i plus 1. What this property of partition of unity says is that all these four basis splines will further sum to 1 for values of t in between t i and t i plus 1. Let me work this out on the board for you.

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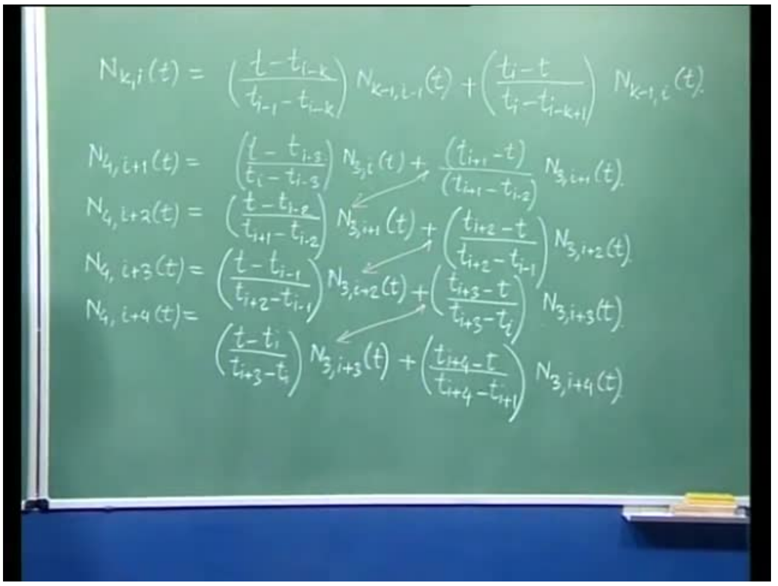

We have the general recursion relation for N k i of t and that is equal to t minus t i minus k over t i minus 1 minus t i minus k times N k minus 1 i minus 1 of t plus t i minus t over t i minus t i minus k plus 1 times N k minus 1 i of t. We are interested in adding these four normalized basis functions each of order four, N 4 i plus 1 N 4 i plus 2 N 4 i plus 3 and N 4 i plus 4. Let us use this recursion relation to figure what N 4 i plus 1 is first. So, I have t minus t.

Now, for i have i plus 1, for k i have 4. So, this would be i plus 1 minus 4 this is i minus 3 over t minus t i minus 3 times N 3 k equals 4 in our case and i equals i plus 1 in our case here. So, this would be i of t plus t i plus 1 minus t over t i plus 1 minus t i plus 1 plus 1 which is i plus 2 minus 4 i minus 2. This is N 3 i plus 1 of t. How about this point here? An easier way for us will be to start indexing i by i plus 1 because here i is getting incremented by one each. So, I can copy this relation from top to bottom with i as i plus 1. Let us see how it works.

So, this one here will look like t minus t i minus 2 over t i plus 1 minus t i minus 2. I will have to be a little careful here. This is N 3 i plus 1 of t plus t i plus 2 minus t over t i plus 2 minus t i minus 1 n 3 i plus 2 t. Let us see if this is correct, i minus 3 to i minus 2, i to i plus 1, i minus 3 to i minus 2, i to i plus 1, i plus 1 to i plus 2, i plus 1 to i plus 2, i minus

2 to i minus 1, i plus 1 to i plus 2 seems okay. How about the third one? t minus t i minus 1 t i plus 2 minus t i minus 1 N 3 i plus 2 of t plus t i plus 3 minus t over t i plus 3 minus t i times N 3 i plus 3 of t and finally, we have t minus t i over t i plus 3 minus t i times N 3 i plus 3 t plus t i plus 4 minus t over t i plus 4 minus t i plus 1, I have to increment this also by 1 times N 3 i plus 4 of t.

Now, let us be careful in observing something here. There are some common terms N 3 i plus 1 N 3 i plus 1 and look at the denominator here t i plus 1 minus t i minus 2 t i plus 1 minus t i minus 2. I can club these two terms together likewise N 3 i plus 2 is common here and the denominator t i plus 2 minus t i minus 1 is also common. So, I can club these two terms together and if you see here also a similar thing happens. So, I can club these two terms together. I will try not to make a mess of this as much as I can because I will be erasing a few terms and adding those parts here. Let us see how good I am at that. So, let me first add this term here and then erase it.

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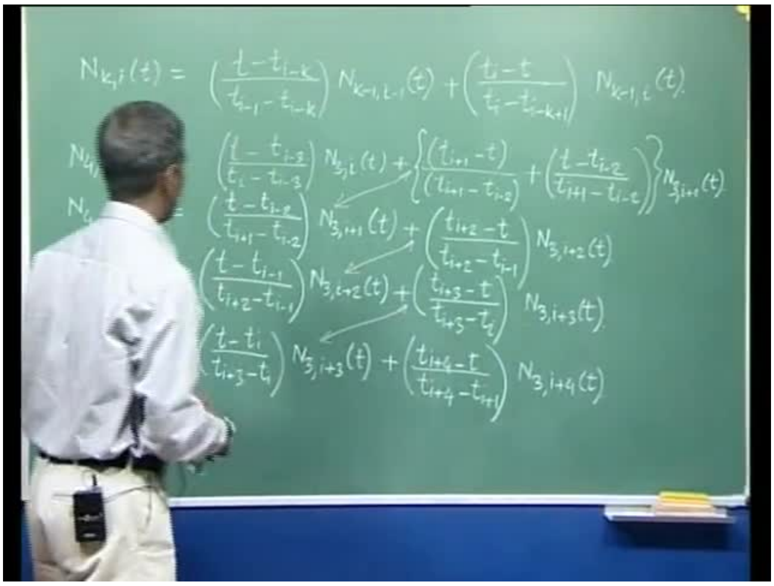

So, I will have plus t minus t i minus 2 over t i plus 1 minus t i minus 2 and this is N 3 i plus 1 of t and I am erasing this now. Now, I am going to be clubbing this term with this one here. Let me add this thing on the left.

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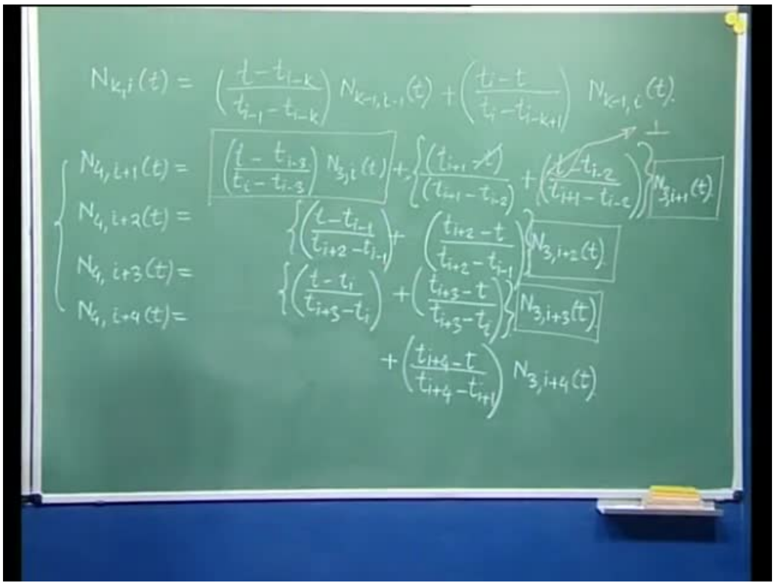

I will have plus, this would be t minus t i minus 1 over t i plus 2 minus t i minus 1 and curly parenthesis here. So, I let go of this term and let me take this term here. This would be t minus t i over t i plus 3 minus t i again curly parenthesis and I let go of this term here. Let me remind you that we are adding these four normalized basis functions. Now, let us look at these terms here. What do we have? Denominator is common and this t cancels with this t here.

And if you see t i plus 1 minus t i minus 2 is the same as the denominator or in fact this term will be 1. Likewise what do we have here? This t cancels with this t and we have t i plus 2 minus t i minus 1 again, the same as the denominator. So, this term here is again 1. How about this term here? Something very similar, this t cancels with this t t i plus 3 minus t i is the same as the denominator. So, this term again is 1. So, in the entire addition here we have this term plus N 3 i plus 1 of t plus N 3 i plus 2 of t plus N 3 i plus 3 of t plus this term here. Now, let me clear the board and express the sedation once again.

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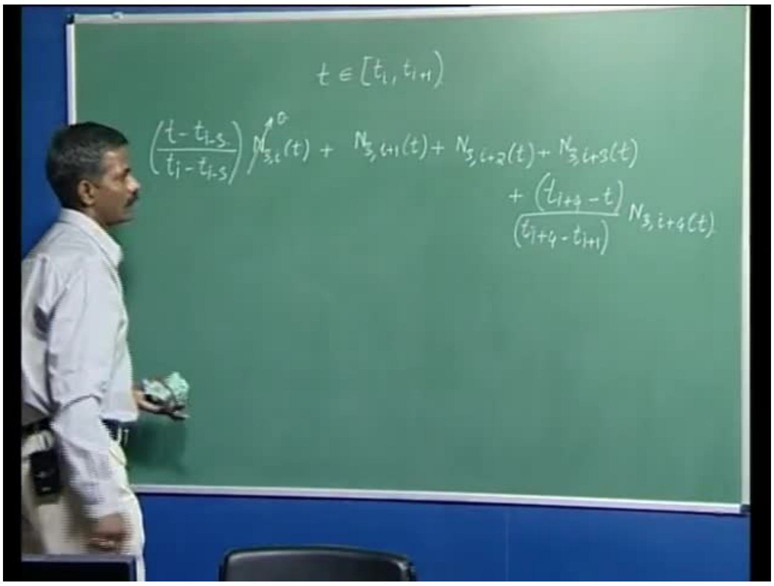

Now, let me remind you that we are looking for values of t in between t i t i plus 1. With that said notice what would happen to N 3 i. N 3 i is only non zero for values of t in between t i and t i minus 3. So, this here is 0. How about this expression here? Now, N 3 i plus 4 is non zero for values of t in between t i plus 1 and t i plus 4.

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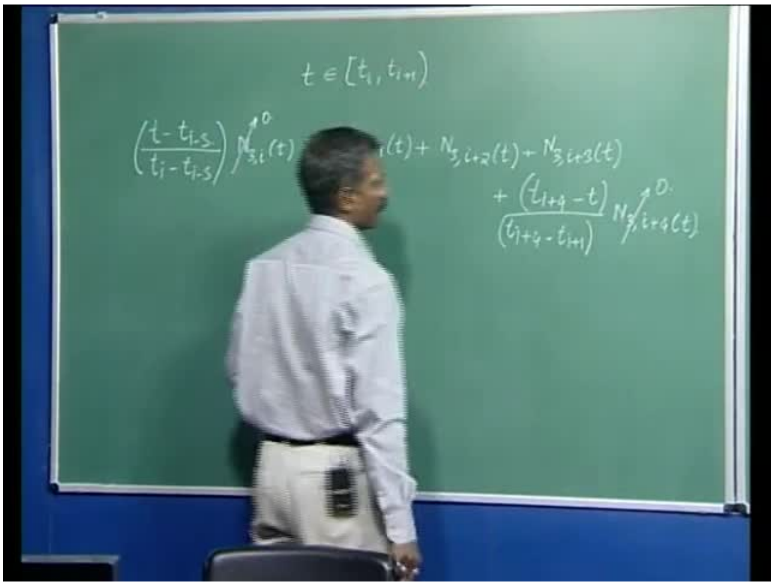

Now, for those between t i and t i plus 1 this will again have to be 0. So, we can erase the two terms and the result will be N 3 i plus 1 of t plus N 3 i plus 2 of t plus N 3 i plus 3 of t.

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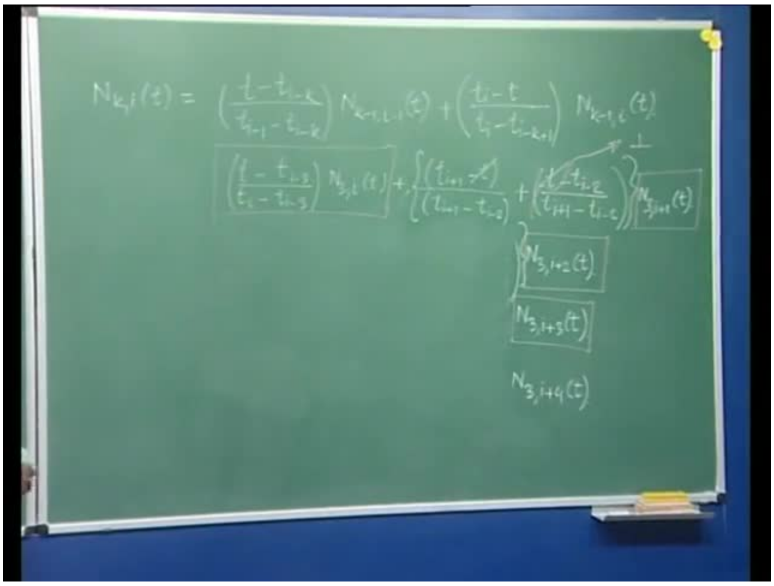

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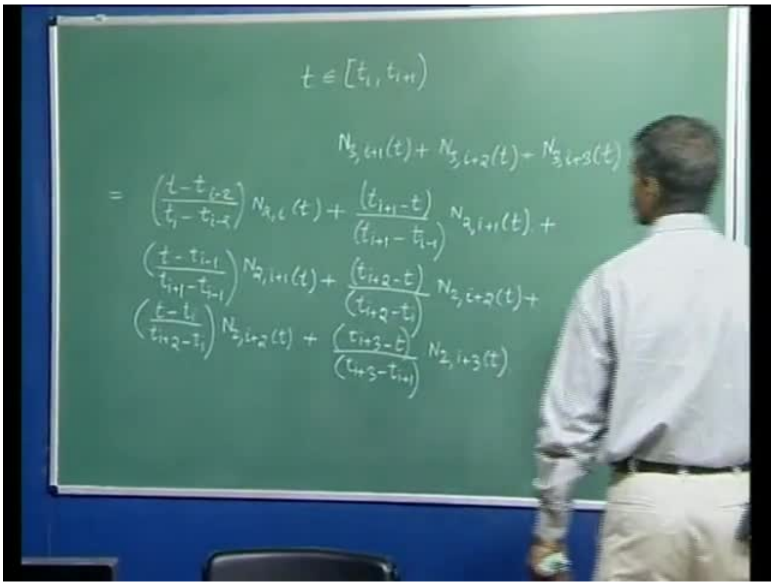

Can we use the same recursion relation to convert all N three’s into N two’s. Let us see.

So, N 3 i plus 1 is t minus t i plus 1 minus 3 is i minus 2 over t i minus t i minus 2. Then we have N 2 i of t plus t i plus 1 minus t over t i plus 1 minus t i plus 1 minus 3 which is i minus 2 plus 1 which is i minus 1 of N k. So, this is N 2 comma i plus 1 of t plus… Now, let us see if this is correct. So, t minus t i plus 1 minus 3, this is i minus 2 and then this is t i minus 1. So, i plus 1 minus 1 is t i minus t i minus 2, this is N 2 i plus 1 minus 1 into i plus t i plus 1 minus t over t i plus 1 minus t i plus 1 minus 3 which is i minus 2 plus 1 which is i minus 1 and then this is N 2 i plus 1 of t and then keep on incrementing the i in this expression.

So, we will have t minus t i minus 1 over t i plus 1 minus t i minus 1, I will have to be careful here this is N 2 i plus 1 t plus t i plus 2 minus t over t i plus 2 minus t I, this is N 2 i plus 2 t plus t minus t minus t i over t i plus 2 minus t i into i plus 2 t plus t i plus 3 minus t over t i plus 3 minus t i plus 1 into i plus 3 of t. Once, again we have this term common here and this denominator common here and then we have N 2 i plus 2 common here and once again the denominator common here. Let us simplify this further. So, if you look at these two terms the coefficient will be 1 of N 2 i plus 1 and likewise of N 2 i plus 2 again the coefficient will be 1.

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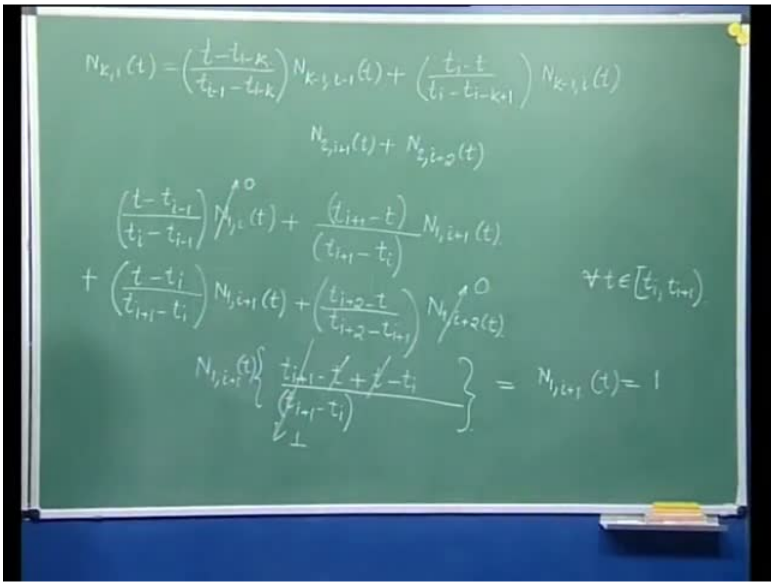

So, we will have t minus t i minus 2 over t i minus t i minus 2 N 2 i of t plus N 2 i plus 1 of t plus N 2 i plus 2 of t. So, we have N 2 i plus 1 of t, we have N 2 i plus 2 of t and then we will have this term here which is t i plus 3 minus t over t i plus 3 minus t i plus 1 times N 2 i plus 3 of t. Once, again let me remind you that we are looking for values in between t i and t i plus 1. Now, N 2 i is non zero in between the values t i and t i minus 2, otherwise this is going to be 0. So, in this interval N 2 i will be 0.

Likewise, N 2 i plus 3 is non zero in between the interval t i plus 1 and t i plus 3. So, clearly here N 2 i plus 3 will be 0. Let me get rid of these two terms to finally have in this summation N 2 i plus 1 of t plus N 2 i plus… Now, let us convert these N twos into n ones using this recursion relation. So, N 2 i plus 1 is t minus t i plus 1 minus 2 which is i minus 1 over t i plus 1 minus 1 which is t i minus t i minus k which is t i minus 1 and then we will have N 1 i plus 1 minus 1 is i of t plus t i plus 1 minus t over t i plus 1 minus t i plus 1 minus 2 which is i minus 1 plus 1 is t i.

We will verify if this is correct and N 1 i plus 1 of t. So, once again i minus k is i plus 1 minus 2 which is i minus 1, t i minus 1 would be t i plus 1 minus 1 which is t i minus t i plus 1 minus 2 is t i minus 1. This is N 1 i plus 1 minus 1 which is an i. Here we have t i plus 1 minus t over t i plus 1 minus t i plus 1 minus 2 which is t i minus 1 plus 1 which is t i N 1 i plus 1 of t. And then simply increment the i here to get N 2 i plus 2 t. So, here we have plus t minus t i over t i plus 1 minus t i and 1 i plus 1 of t plus t i plus 2 minus t i sorry t i plus 2 minus t over t i plus 2 minus t i plus 1 incrementing this by 1 times N 1 i plus 2 of t.

Now, this is 0 for t in between t i and t i plus 1. This is 0 for values of t in between t i and t i plus 1 again. Note that this is non zero for t in between t i plus 1 and t i plus 2. So, this basis function does not bother us. We are left with N 1 i plus 1 here, N 1 i plus 1 here. Denominator is common t i plus 1 minus t i, t i plus 1 minus t i we can club these two terms together to get N 1 i plus 1 times t i plus 1 minus t i. This is t i plus 1 minus t plus t minus t i. So, these t’s cancel, the numerator t i plus 1 minus t i is the same as the denominator.

So, this would be 1 and so we are left with N 1 i plus 1 and by definition this should be a function of t, by definition N 1 i plus 1 is equal to 1 for all t belonging to t i comma t i plus 1. It should be possible for you to show this for a generic case when you add p order p normalized basis spline functions.