Computer Aided Engineering Design
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Lecture - 24
B-Spline Basis Functions Recursion Relation

Hello and welcome, we are on lecture number 24 will continue to the discussion on b-spline basis functions.

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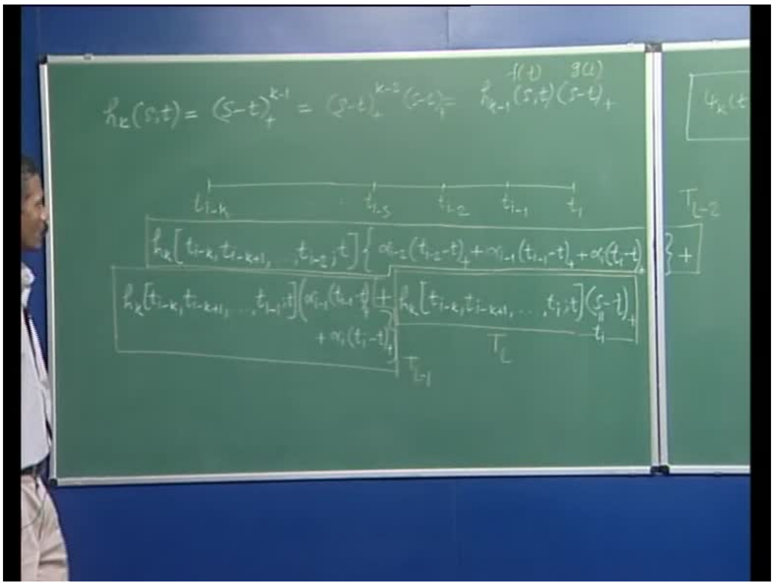

In the Leibnitz formula, let us consider the last 3 terms of the k th divide difference of h k S colon t. So, the last term will be h k t i minus k t i minus k plus 1 up to t i colon t S minus t truncated. Here S will be equal to t i the last, but 1 term will be h k t i minus k t i minus k plus 1 up to t i minus 1 colon t times, the first divided difference of S minus t truncated, that would be equal to alpha i minus 1 t i minus 1 minus t truncated plus alpha i t i minus t truncated. And of course, there would be a plus sign here.

And the third from the right hand side term correspond to h k t i minus k t i minus k plus 1 up to t i minus 2 colon t times alpha i minus 2 t i minus 2 minus t truncated plus alpha i minus 1 t i minus 1 minus t plus alpha i t i minus t truncated. So, once again the last three terms of the k th divided difference of h sub k S colon t are this expression plus this expression here plus this expression. You would know what alpha i minus 2 alpha i minus 1 alpha i would be they would relate to w prime you would know that, and I am not really bothered about what this constants are at this time.

Let us concentrate on the last term here, let me call this term T l let me call this term, here T l minus 1 and this term here, T l minus 2, l corresponds to the last term. So, let us concentrate on T sub l, all I am interested in doing is to figure, which of these terms will be 0 and which will be not 0 for values in between t i minus k and t i, if you look at this term this term is t i minus t truncated, for t greater than t i this term becomes negative because of which this term here would become 0, irrespective of whatever this is.

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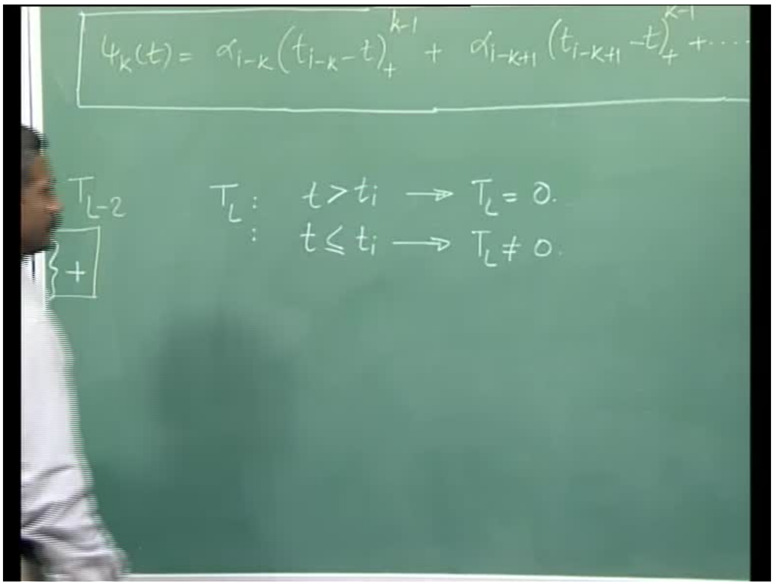

So, T l for t greater than t i is 0, how about for t smaller than equal to t i. Now, this term here becomes faster and this term here will not be 0. I would not know about this term as yet, I would assume that this term is non zero as well. So, I can say that t l is not equal to 0.

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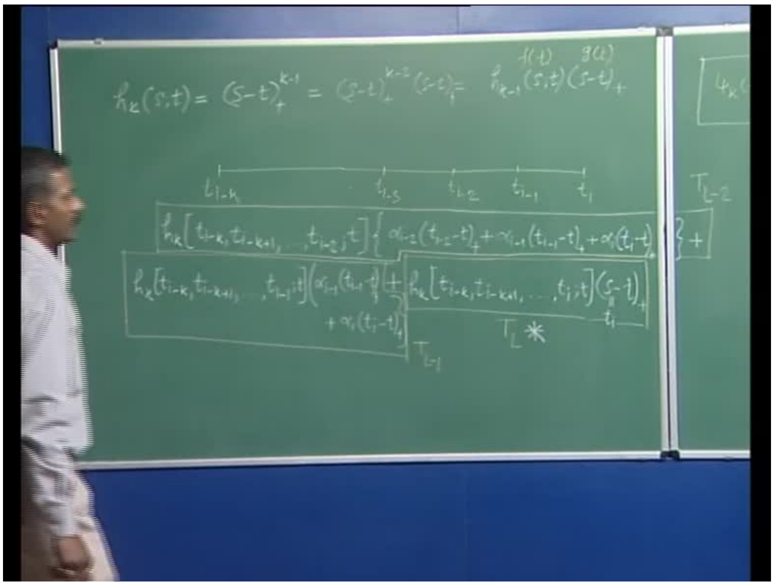

So, there is a very high like here that this term gets retain. So, this term is important to us.

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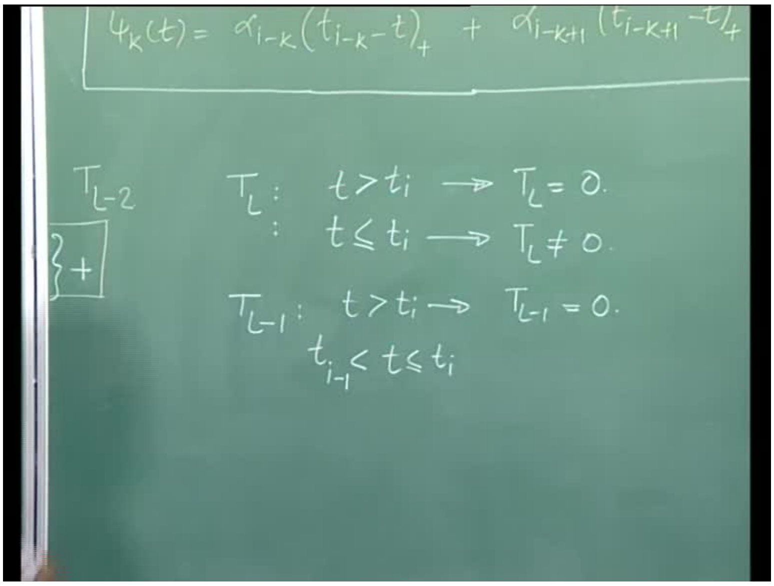

Now, let us concentrate on T l minus 1 and let us concentrate on alpha i minus 1 t i minus 1 minus t truncated plus alpha i t i minus t truncated. Clearly for t greater than t i T l minus 1 is equal to 0. Why? Because this term is negative and this term as well is negative because of this truncation sign, this term and this term they are both 0. Now, for t smaller than equal to t i and greater than t i minus 1, what happens look at this term here t i minus t truncated. Since, t is smaller than t i this term here becomes faster and so, this term is not 0. How about this t i minus 1 minus t is still negative, and because of the truncated sign this term is 0.

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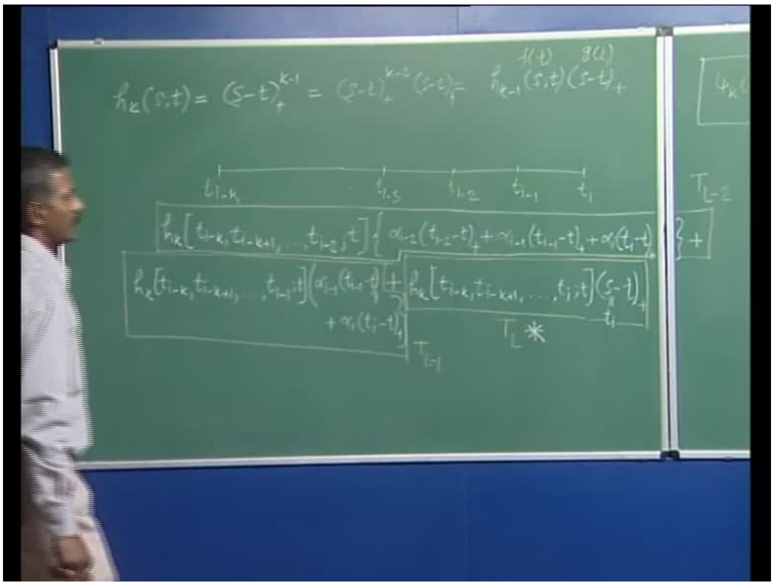

Now, let us concentrate on this term here can you figure, what this term is this is the k minus 1 divided difference of h sub k. The last number here corresponds to t i minus 1 the first number here corresponds to t i minus k. Now, for values of t greater than t i minus 1, what would you say will happen to this function would be 0, I would think so let us verify that.

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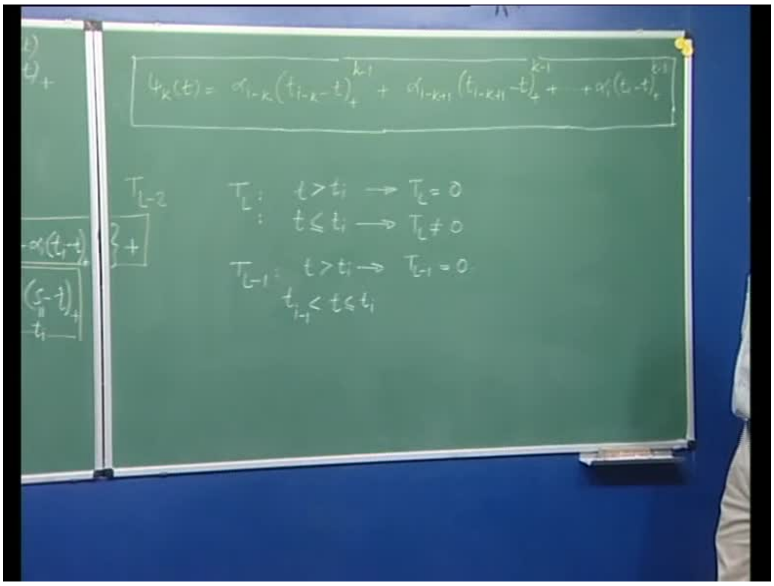

So, h sub k t i minus k t i minus k plus 1 up to t i minus 1 colon t will be very similar to this expression, will be a linear combination of terms like t i minus k minus t truncated raise to k minus 1 plus alpha i minus k plus 1 t i minus k plus minus t truncated raise to k minus 1 and so on and so for. Now, when you place the last knot as t i minus 1 this is t i minus 1 write here, change this index to i minus 1. And for while use of t greater than t i minus 1, you would notice that all this terms they will be negative and because of which this entire expression will be 0.

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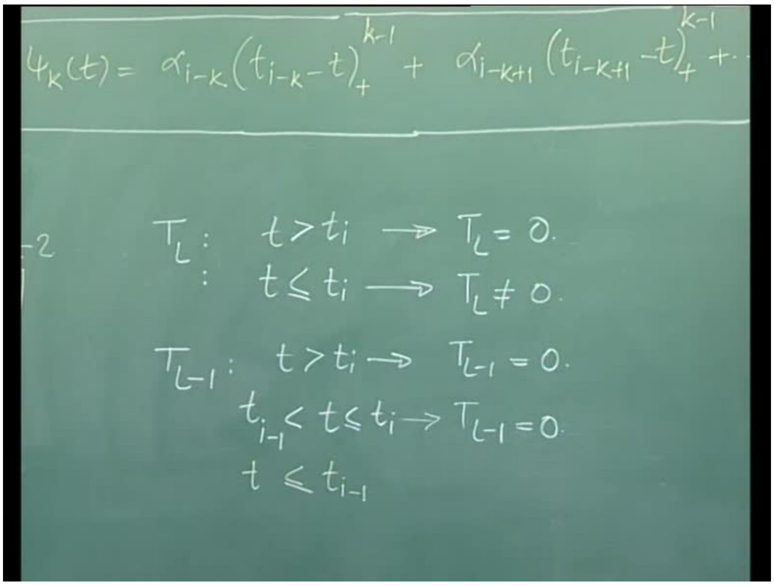

And so, h k t i minus k t i minus k plus 1 up to t i minus 1 colon t, this expression here will be 0. So, to summaries this entire expression here T l minus 1 for values of t greater than t i minus 1 would be 0, is only for values smaller than t i minus 1 that this expression here will be non zero and let us vary about those. So, for t smaller than t i minus 1, smaller than equal to t i minus 1 this is non zero here.

Now, what we have to say about these two terms. I want you to be very careful, this term here is positive this term here is positive. So, I can drop this truncated sign, the second thing and a very important thing that you want to notice is that, this term actually corresponds to the first divided difference of a linear polynomial. Once, again this term would correspond to the first divided difference of a linear polynomial, a linear shear polynomial. What would that mean? This would mean that this term will become 1.

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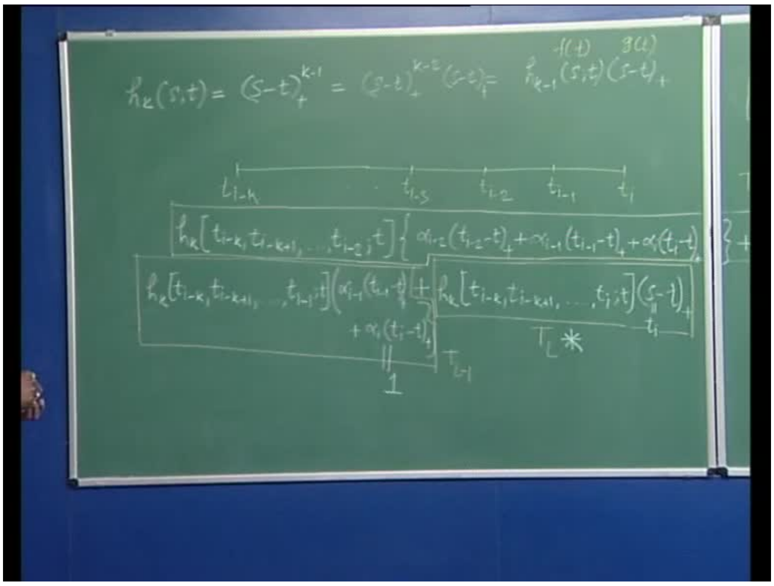

And we have seen this using examples in previous lectures, in general the k th divided difference of degree k shear polynomial is 1. So, t i minus 1 becomes important for values of t smaller than t i minus 1.

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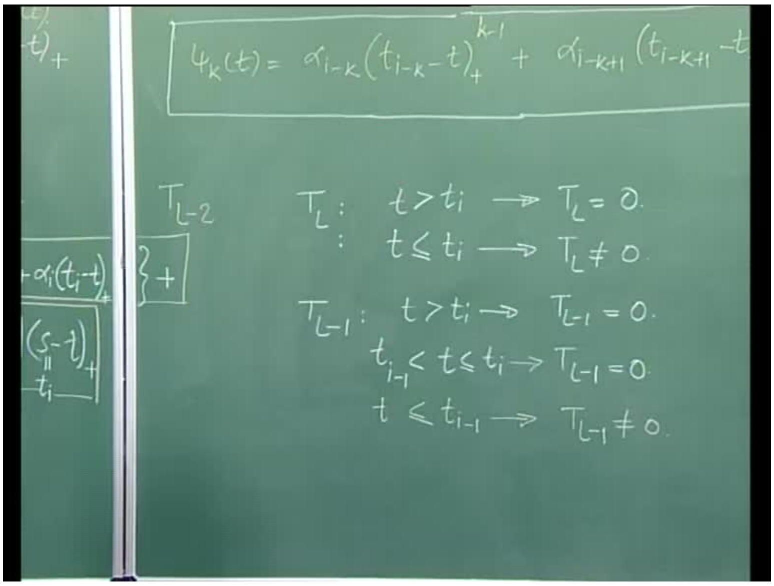

And so, I would say that this is not equal to 0. Now, let us go to the third last term here and we are going to be using similar arguments, look at this expression over here and in particular look at this last number t i minus 2. What do you have to say for values of t greater than t i minus 2, if we come back over here and instead of i if place i minus 2 will see that for values of t greater than t i minus 2, all this expressions they are become 0. What would that mean? This would mean that this term here is 0, for t greater than t i minus 2, let me write this in dark for you.

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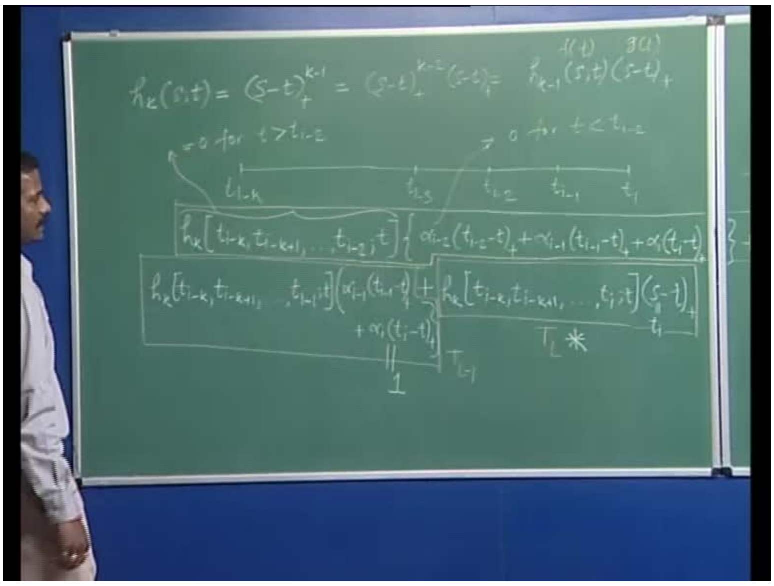

For t smaller than t i minus 2, this term will be non zero. What would happen to this term here? Alpha i minus 2 t i minus 2 minus t truncated plus alpha i minus 1 t i minus 1 minus t truncated plus alpha i t i minus t truncate, this term will be positive, this term here will be positive and this term here will be positive. So, in a way we are looking at the second divided difference of a pure linear polynomial here alpha i minus 2 alpha i minus 1 and alpha i are set in that fashion that this expression represents, the second divided difference of a degree 1 pure polynomial. And you know what that is, this is equal to 0 for t smaller than t i minus 2 and look at this combination here for t larger than t i minus 2, this is 0 for t smaller than t i minus 2 this is 0, and if you extend this argument to all the terms before this term, you would see that all those terms will be 0. So, we will be left with only T l minus 1 and T l.

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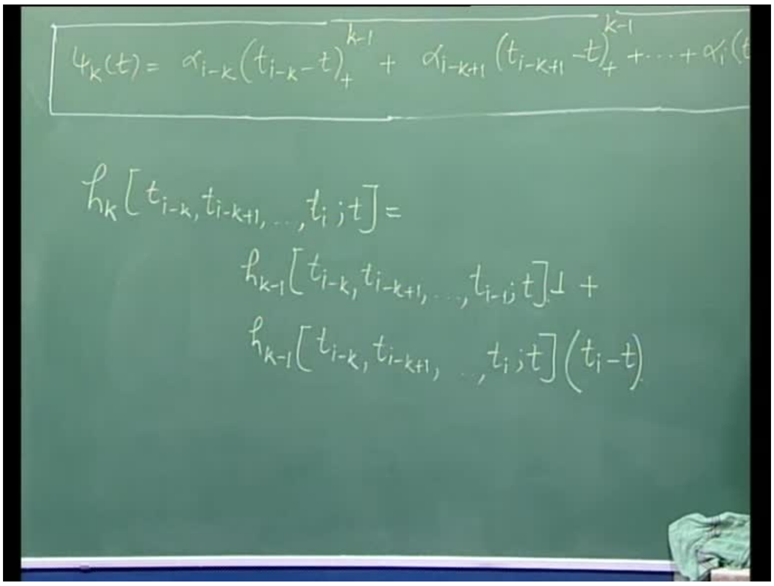

So, in summary the k th divided difference of this function here, which is h k t i minus k t i minus k plus 1 up to t i colon t will be equal to this term here, which is h k minus 1 t i minus k, t i minus k plus 1 t i minus 1 colon t times you would know that this is 1 plus this term here, this is equal to h k minus 1 t i minus k t i minus k plus 1 t i colon t times t i minus t. Since, I am consent ring values of t smaller than t i, I can drop this truncated sign, I am not going to be using this truncated sign here.

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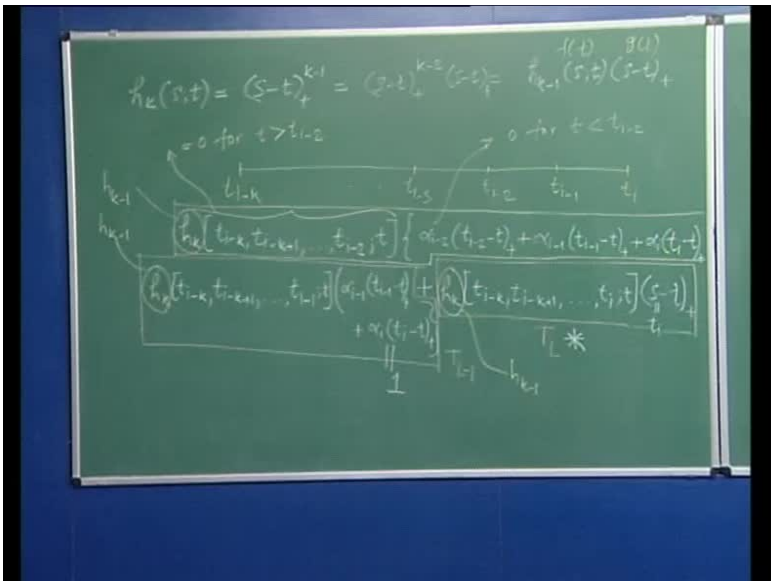

I just realize just I have made a little mistake here, this thing should be h k minus 1 likewise, this is h k minus 1 and so is this. You might what to make corrections in your notes obviously, this is so because here I have h k minus 1 and I have been computing the divided differences of h k minus 1 all along. Now, coming back here, what would you say this is, this is the k th divided difference of truncated for function of order k or degree k minus 1. So, by definition this should be a spline.

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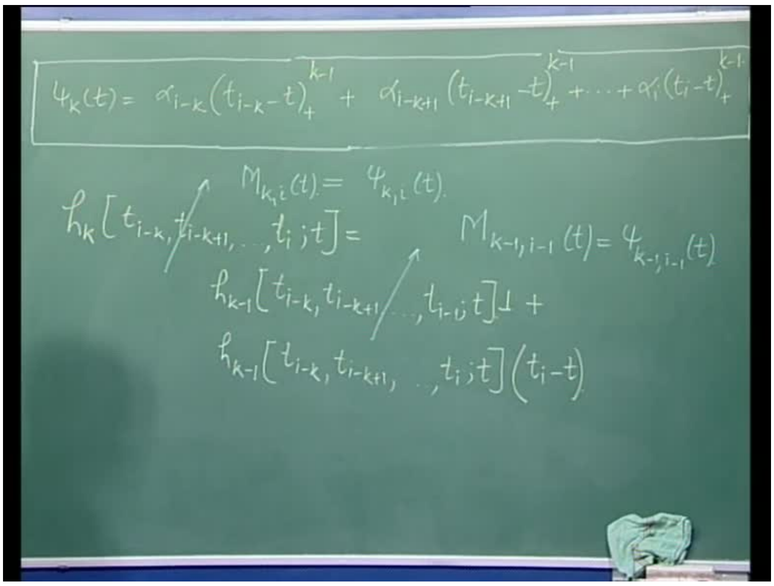

This should be M k, the last part is i of t also, this is psi k i t. We have considered both conventions or both notations, your free to use either psi or M, k is the order, i is the index of the last knot. What do have to say about this, look at these knot here. So, this for correspond to the k minus 1 divided difference or h k minus 1. What is h k minus 1? H k minus 1 is S minus t truncated k minus 2, this is the truncated part function of order k minus 1 by the same argument that we have used here, this correspond to the k minus 1 divided difference of order k minus 1 truncated part function.

So, this would be a spline a b-spline basis function what is the order, the order is k minus one of course, and what would be the last knot, the last knot would be i minus 1, in terms of psi this is psi k minus 1 i minus 1 t. How about this now, we will have to be quite careful here, these knots convey that this expression here is the k th divided difference of order k minus 1 truncated for function. It may be a little difficult for us to think about this as a spline as if now, we will have to convert this expression suitably, how do we do that.

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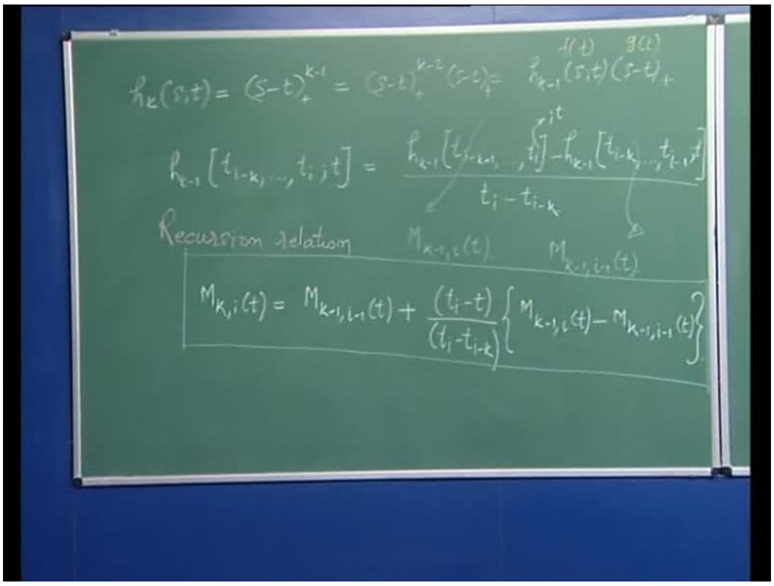

So, I can write h k minus 1 t i minus k t I colon t as I will have to be a little careful here h k minus 1 t i minus k plus 1 t i minus h k minus 1 t i minus k t i minus 1 colon t. I will have a colon t here over t i minus t i minus k. What have I done, I have written that k th divided difference in terms up to k minus 1 divided difference. So, this is a k minus 1e divide difference of h k minus 1 h k minus 1 is the order of this truncated part of function. Likewise, this also the k minus 1 divided difference of S minus t truncated k minus 2.

Now, by definition these two are splines as well, what will this be? First the order then the knot so, this M k minus 1 i of t how about this expression, again first the order then the last knot this is M k minus 1 i minus 1 t. I am now ready to combine the entire information into a recursion relation for b-spline basis functions. So, I start with M k i t equals M k minus 1 i minus 1 t plus t i minus t over t i minus t i minus k times M k minus 1 i t minus m k minus 1 i minus 1 t. What do we observer here? We observe that a b spline basis functions of order k can be written in terms of two b spline basis functions each of order k minus 1, the first b-spline basis function is M k minus 1 i minus 1 and the second b-spline basis function is M k minus 1 i, this is you recursion relation.

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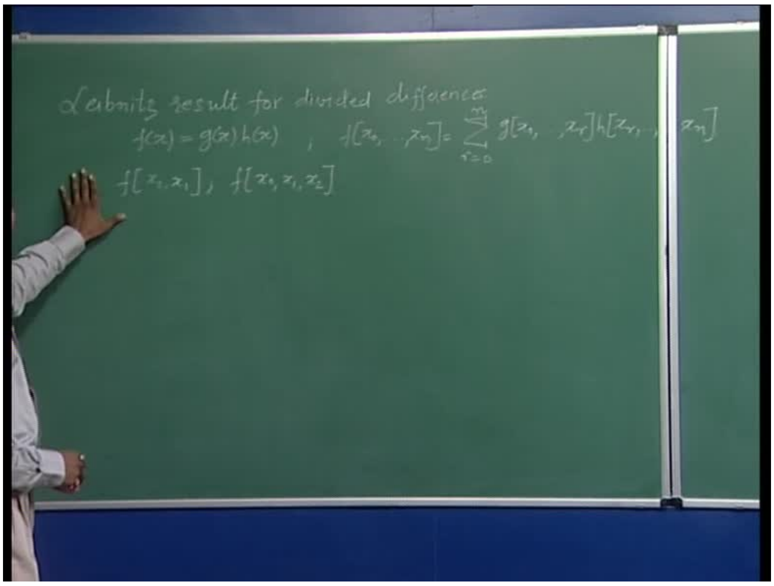

I will try to give you a general proof of the Leibnitz result for the divided differences. So, you would know, what Leibnitz result for divided differences is given a function f of x and if that function can be written as a product of two functions, g of x and h of x the nth divided difference f square brackets open x 0, x 1 up to x n square brackets closed. So, the nth divided difference of f is given by summation index r going from 0 to n th, r th divided difference of g, that is g within square parenthesis x 0, x 1 up to x r times h within square parenthesis x r, x r plus 1 up to x n. In the previous lecture, I had shown this result to be true for a few cases namely, for the first divided difference f x 0, x 1 and for the second divided difference x 0, x 1 and x 2 in f. Today I will try to give you a general proof of this statement.

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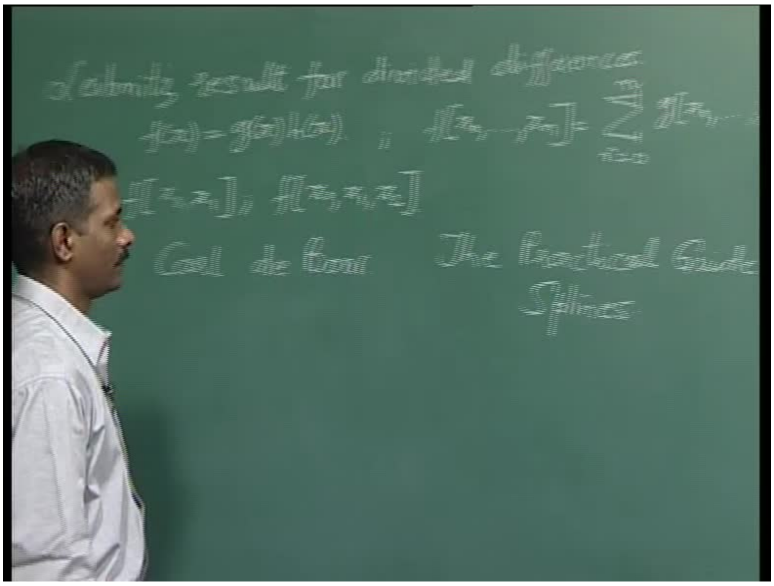

This proof is from Carl De Boor, the book is called the practical guide to splines.

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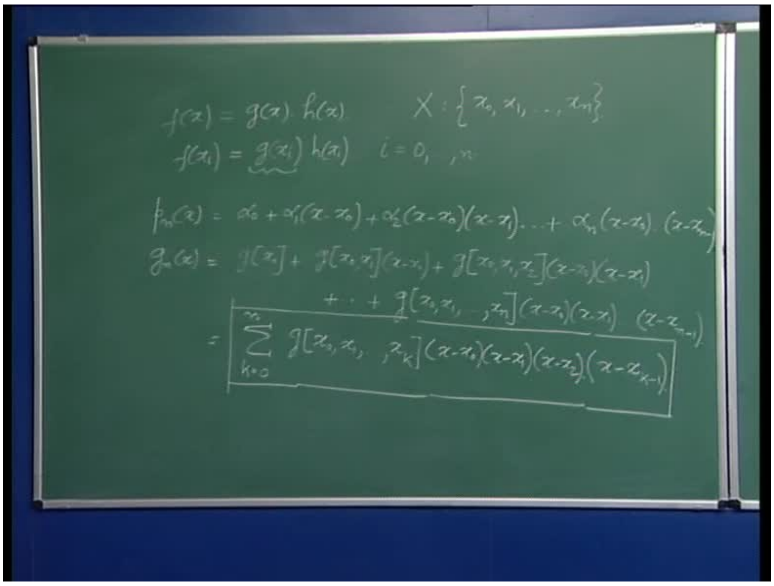

So, let us here started with the proof, we have a function f of x written as a product of two functions g of x and h of x and then we have a set capital X of x values x 0, x 1 up to x n. We assume that we know, f x i which is equal to g x i times h x i for all i going from 0 to n. Let us start using some tricks, what this statement would mean is that there would be n plus 1 values of g of x i available to us, using that let us try to construct a degree n polynomial to represent g x.

Recall the Newton’s interpolation formula, a polynomial of degree of n in x is given by alpha 0 plus alpha 1 x minus x 0 plus alpha 2 x minus x 0 x minus x 1 plus alpha n x minus x 0 x minus x n minus 1, this is a degree n in x. And of course, you would that alpha 0, alpha 1, alpha 2 up to alpha n are all divided differences. Now, let us express this result to represent g x as a polynomial of degree n. So, g n x equals I am going to be writing the divided differences directly, as per the notations that we know.

So, g sub n of x equals g square x 0 plus g squared x 0 x 1 times x minus x 0 plus g square brackets x 0 x 1 x 2 times x minus x 0 x minus x 1 plus some more terms, and then the last term will be g x 0 x 1 x n square parenthesis closed, x minus x 0 x minus x 1 and then the last term here will be x minus x sub n minus 1. In short I would be able to write the expression for g n x as, summation index k going from 0 to n g within square brackets x 0 x 1 x sub k square brackets closed x minus x 0 x minus x 1 x minus x 2 and the last term will be x minus x sub k minus 1 will leave this result. So, will try to block it as a plan now, will do something very similar for h of x as well we have n plus 1 values of h of x i available to us, it should be possible for us therefore, to construct a polynomial of degree n in h as well, but will use a slightly different trick instead of going from x sub 0 to x sub n will reverse the order.

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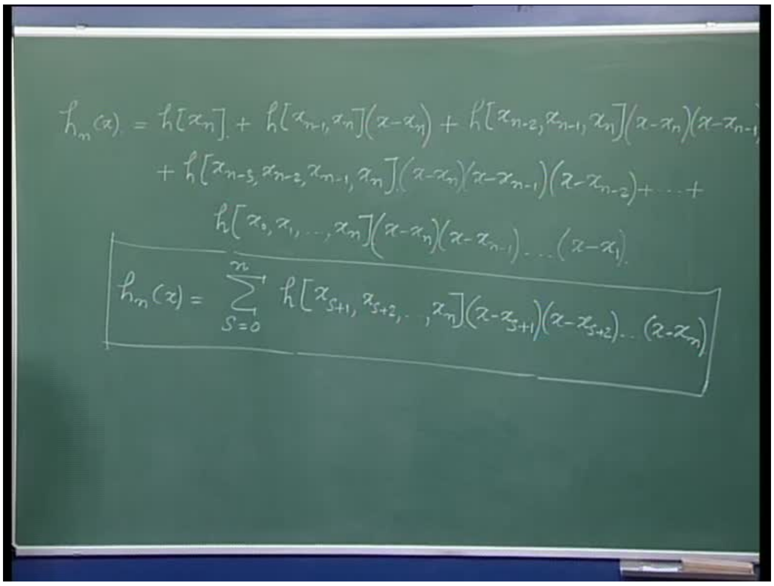

So, h sub n of x this is the nth degree polynomial representing the function h of x, this is equal to h x n. We would want to be careful because we are working with the reversed order now, h of x n is the zero th divide difference that is the value of h at x sub n plus the first divide difference involving x n minus 1, and x n times x minus x n. Remember again that we are working in the reverse order, plus the third will be h x n minus 2 x n minus 1 x n times x minus x n x minus x n minus 1 plus.

Let us write as the forth term as well, h x n minus 3 x n minus 2 x n minus 1 x n times x minus x n x minus x n minus 1 x minus x n minus 2 plus there will be some more terms, plus try to figure out for the last term here would be. Again will have the n th divided difference and that would involve all values from x 0 to x n, this is x 0, x 1 up to x n times again, we are working in the reverse order x minus x n x minus x n minus 1, and the last term here would be x minus x 1.

Once again, we computed these divided differences in case of g from left to write in x i and to compute the nth degree polynomial to represent h of x, we went the other way around. In short I can write h n equals summation index s going from 0 to n h square bracket open x s plus 1 x s plus 2 up to x n times x minus x s plus 1 x minus x s plus 2 up to x minus x n, this is an important result for us so will block it.

Note the difference in these two expressions, the divide difference here involves x 0 x 1 up to x k. The divide difference here involves x s plus 1 x s plus 2 up to x n, the minimum value of x is 0. So, the starting value is x 1 here the starting value here is x 0 with regard to these terms, we have here x minus x 0 x minus x 1 up to the last term as x minus x k minus 1, the maximum value of k is n so, the last term here will be x minus x n minus 1 for k equals n. However, here the last term is slightly different, here we have x minus x n, the starting term of course, is x minus x 1 for s equal 0. So, there are minor differences in the two expressions, let us not lose our perspective.

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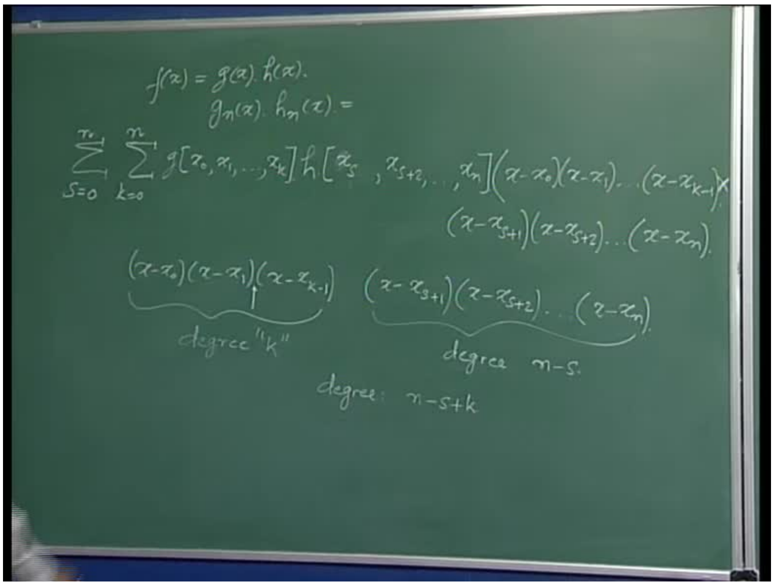

We call that f x was equal to g x times h x, let us now multiply the polynomial approximation of g, with the polynomial approximation of h. So, g n of x times h n of x will be equal to summation copy this expression here, k equal 0 to n g x 0 x 1 up to x k copy this expression here. So, I am rearranging terms h x s plus 1 x s plus 2 up to x n times the expression, involving the x terms x minus x 0 x minus x 1 up to x minus x k minus 1 from here and again the expression involving the x terms from here, times x minus x s plus 1 x minus x s plus 2 x minus x n.

Let me double check for got this right x 0 to x k x k x s plus 1 to x n plus x minus x 0 x minus x 1 up to x minus x k minus 1, we are x minus x s plus 1 x minus x s plus 2 up to x minus x n we are. And of course, here will have a double summation, index s would also go from 0 to n. Let us worry about these coefficients a little later, what is of concern to us is this expression, let me copy this expression down x minus x 0 x minus x 1 x minus x k minus 1 here will have some more terms, and then will have x minus x s plus 1 x minus x s plus 2 up to terms, last term x minus x n. There is a minor typo here this should be x sub s here.

Now, coming back to this expression, what you think the degree of this part is, this is linear for quadratic for x 1. So, if I multiply of these terms, I will get degree k in x how about these terms if I set s equal 0 then I will have x minus x 1 x minus x 2 up to x minus x n and that will have the degree n in x. So, this expression will have degree n minus s so, the overall degree for this entire expression is n minus s plus k.

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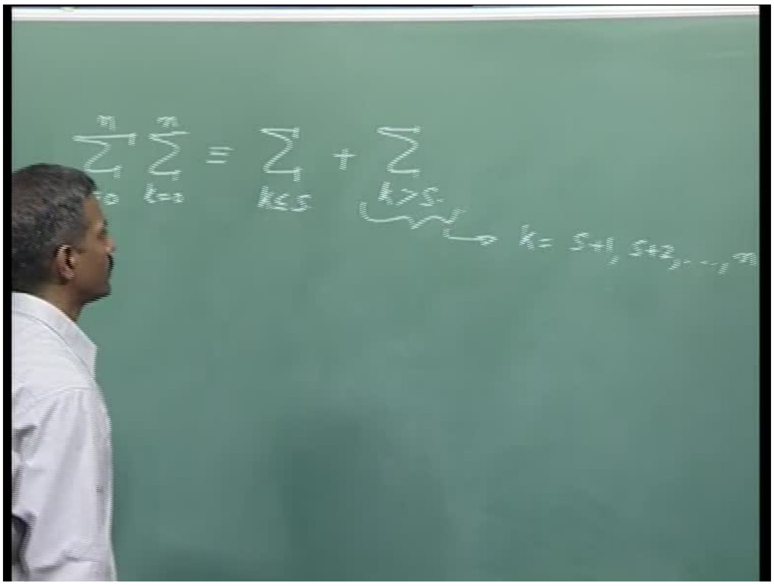

Now, let us come back, this here is a single term and this double summation. Now, let us try to express this double summation as two parts, s going from 0 to n, k going from 0 to n. Let us try to express these as the summation of all terms for the case, when k is smaller than or equal to s and then summation of all terms for cases when k is greater than s.

Let us try to investigate, what happens to this particular expression for these two cases let us take this case first k greater than s. So, here k can assume values s plus 1 s plus 2 and so on so for up to n. Remember the maximum value of k is n. Now, concentrate on this part here for k equals s plus 1, we have x minus x s for k equal x plus 2, we have x minus x s and then we have x minus x s plus 1. So, if you notice for all values of k greater than s, we will have terms involving x minus x 0 x minus x 1 x minus x 2 up to x minus x n. What is the consequence?

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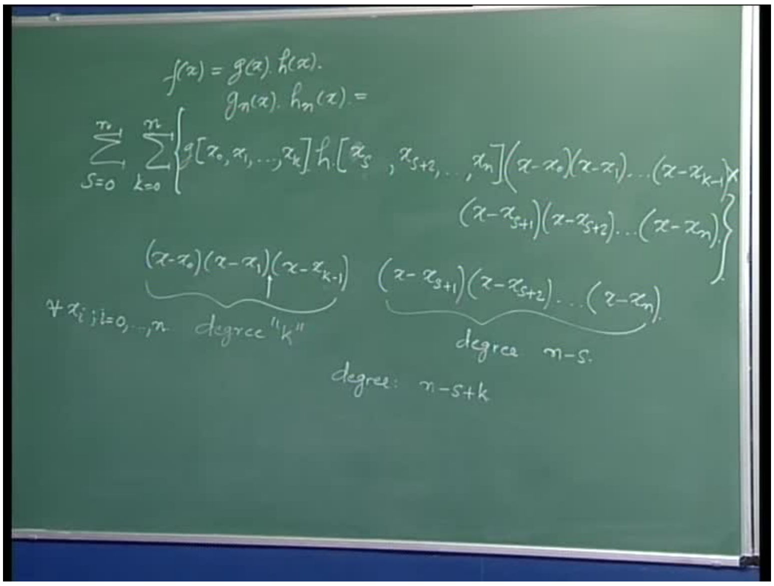

The consequence is that this entire expression will vanish for all x i, i going from 0 to n once again for values of k greater than S, this expression here will involve terms like x minus x 0 x minus x 1 x minus x 2 all the way up to x minus x n, there will be some square terms and so on. So, for and this entire expression will become 0 for all values of x i for i going from 0 to n. So, if you think about it this part of the summation does not bother us, it is this part that need to vary about us.

The first thing we need to do now is vary about the degree of the terms, emanating from this summation for values of k smaller than or equal to S. Let us come back you would know the degree of the entire expression which is here is n minus s plus k. The maximum value of k can be S for this case which would mean that the maximum degree of this expression here or of this expression here is n if you set k equals s. So, let us get a perspective back, let me a wild all the confusion and let me replace or summarize, whatever we have done so for.

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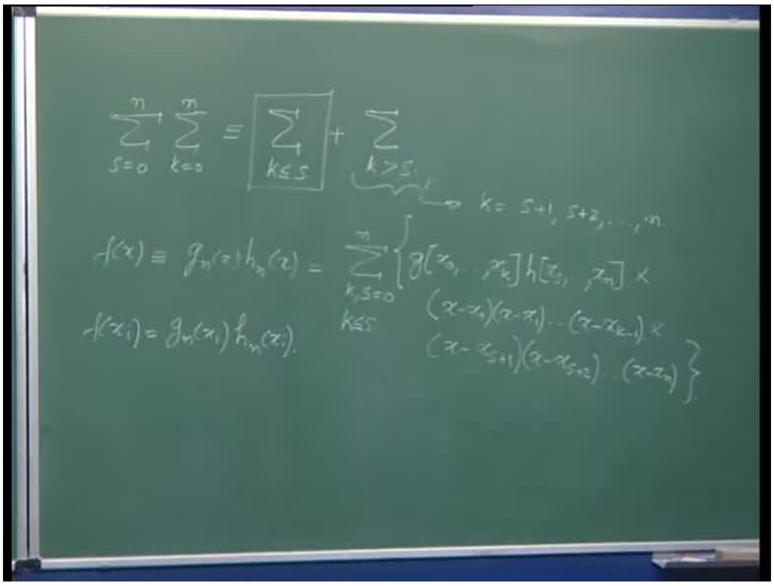

So, f of x was being represented as the product of g n x h n x now will be written as summation k n S both going from 0 to n, but k has to be smaller than or equal to S and then we have terms involving the divided differences g x 0 to x k h x s to x n times, the terms involving x s x minus x 0 x minus x 1 up to x minus x k minus 1 times x minus x s plus 1 x minus x s plus 2 up to x minus x minus x n.

Now, since this part in the summation does not contribute to this result, what can implies is that it has to be this part or rather this expression here on the right hand side, that would contribute to f x i equals g n x i times h n x i. And we have also seen that the maximum degree in x here will be for k equals s and that would be n. In other words, this statement has to satisfy this equation for values of i going from 0 to n, think about this way, like we approximated g and h using the corresponding n th degree polynomials, we can do the same for f.

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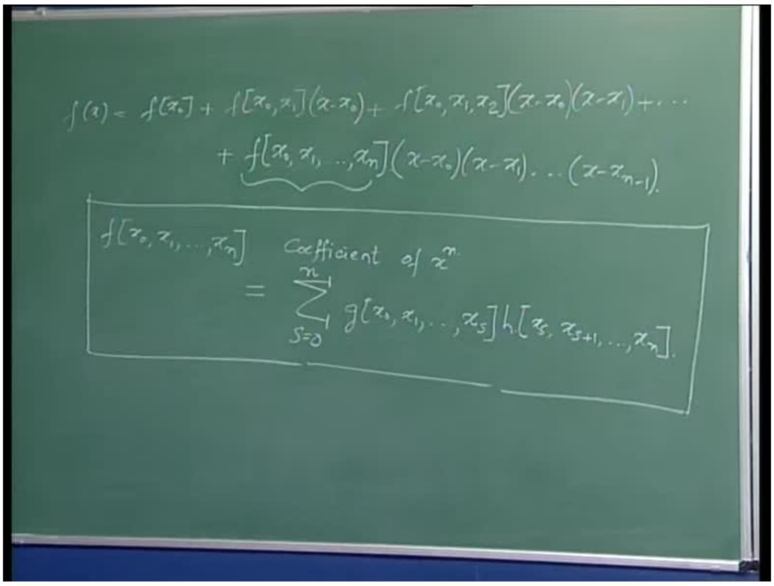

So, using Newton's polynomial we can write the nth degree approximation for f and that would f of x equals f x 0 plus f x 0 x 1 x minus x 0 plus f x 0 x 1 x 2 x minus x 0 x minus x 1 plus. And the last term of course, is the nth divide difference in f, and then x minus x 0 x minus x 1 up to x minus x n minus 1. Now, let us to compare the coefficients, try to figure what would be the coefficient of x raise to n clearly, the nth divided difference in f coefficient of x is raise to n is call the leading coefficient.

Now, let us try to come back here, this again pretty much looks like Newton polynomial in f in some manner, let us try to extract all the coefficients corresponding to x raise to n from here, remember that will get the degree n for this expression only for values of k equals s, and also remember that this is a double summation. So, we set k equals s then this coefficient becomes g x 0 x 1 x s h x s x s plus 1 up to x n.

Now, this expression correspond to only one term here, how about in-coperating summation in s, this is a double summation sign the constraint here is only on values of k. So, being co-operate summation in s, s would go from 0 to n; and so this expression would correspond to the coefficient of x raise to n, and for the Newton's polynomials to be identical from this result, and from this result these coefficients have to be identical. And if you notice, this is the Leibnitz result for divided differences.