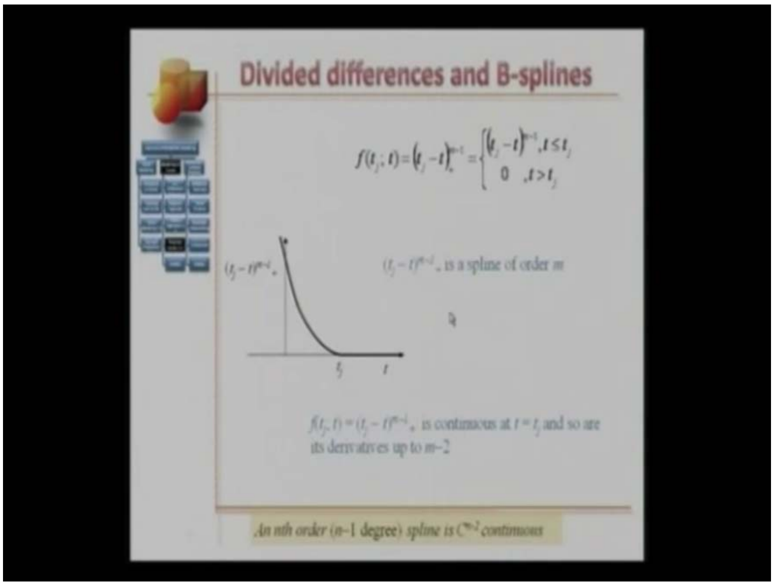
Computer Aided Engineering Design
Prof. Anupam Saxena
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 23

Good morning and welcome, we continue today with our discussion on Newton's divided difference, and how to use that to derive 'B-Spline basis function' are quick recap on what we did previously, divided differences and B-Splines (( )).

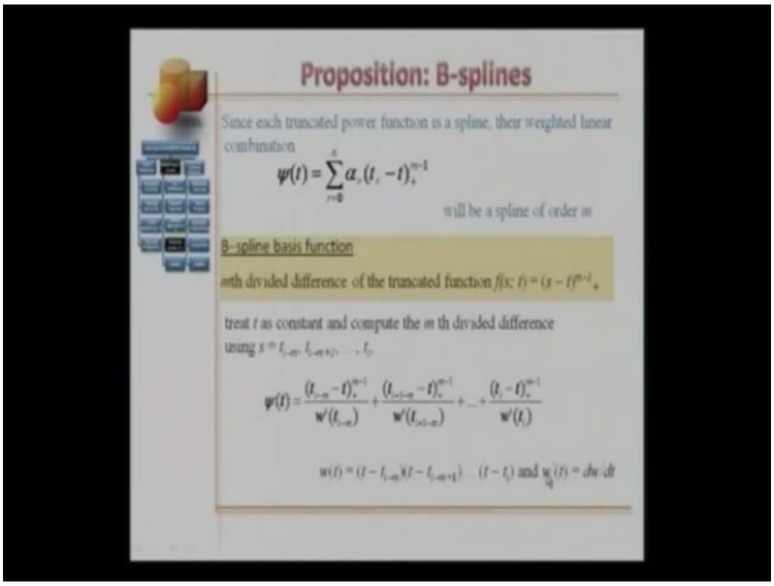
(Refer Slide Time: 00:39)



So, previously we studied truncated power function f of t sub j colon t is equal to t j minus t truncated raise to m minus 1. This function is pure polynominal t j minus t raise to m minus 1, for values of t smaller than t j, that is for this value here positive and this function get truncated implying that the value of this function is 0 for values t greater than t j.

This is how this function looks like, for values of t smaller than t j, the function pure polynomial of degree m minus 1 and for value of t greater than t j, this function is truncated, we have a o value. So, f t j colon t vectors t j minus t truncated raise to m minus 1. This continuous r t equal t j write here and so all its derives up to m minus 2, recall with definition of (( )), an nth order which is an n minus 1 degree curve is Spline f curve is c m minus 2 continuous, everywhere in the domain by this observation it truncated of function is Spline and it is Spline of order m.

(Refer Slide Time: 02:32)

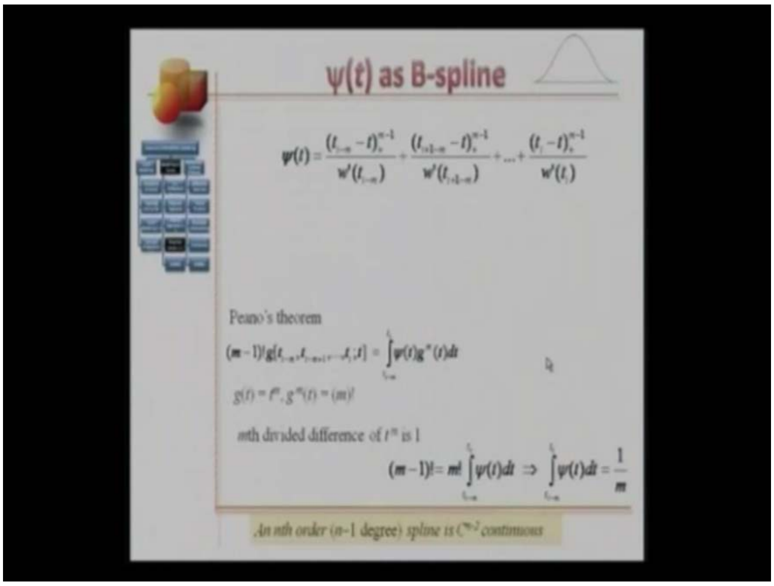


We have also made proposition is regarded to B-Splines basis function, lets revise that again since each truncated power function as a Spline it is natural their weighted linear combination which is summation r going from 0 to n r for r t r minus t truncated raise to m minus 1 the called this function psi of t is also Spline of order m and so we have meant that B- Spline basis function is mth divided difference of the truncated function f s colon t which is equal to s minus t truncated raise to m minus 1.

The twist here is to hold t is constant and compute the divided differences with respect to s and introduce not in colon and it was this way of computing B- Splines basis functions that we thought would be work while explaining will do so and today's lecture and few more as I mentioned before we might from to treat as constant and compute the n th divided difference using s as t i minus m t i minus m plus 1 after t i.

Using the algebraic form this divided difference the look like psi of t is equal to t i minus m minus t truncated raise to m minus 1 over w frame at t i minus m plus t i minus m plus 1 minus t truncated raise m minus 1 over w frame t i plus 1 minus m and the last term in summation would be t i minus t truncated raise to m minus 1 over w frame evaluated t i. The w is the product of t minus t i minus m t minus t i minus m plus 1 and so on after t minus t i and w frame is the first derivative of w with respect to t.

(Refer Slide Time: 05:29)



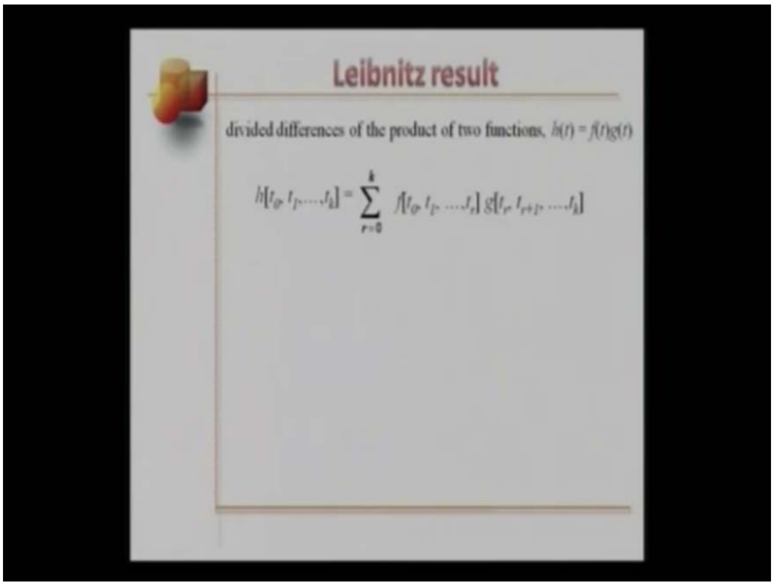
We tell in to this expression further and be showed that psi of t is 0 of values of t greater and t i and over values of t smaller and t i minus n. So this not on the right this t i and this not on the left is t i minus m. So for t co-oral ant t i psi t is o and that is because t i plus r minus m minus t truncated rise to m minus 1 for r going from 0 to m are all 0.

This is because each of this terms the become negative and this cos of the truncated psi here. All this terms the individually become 0 for values of t smaller than equal to t i minus n again psi t is 0, this is because psi t nth divided difference of pure m minus 1 degree polynomial. We are seen a few examples and previous lecture and we have shown this to be true also psi of t is standardized.

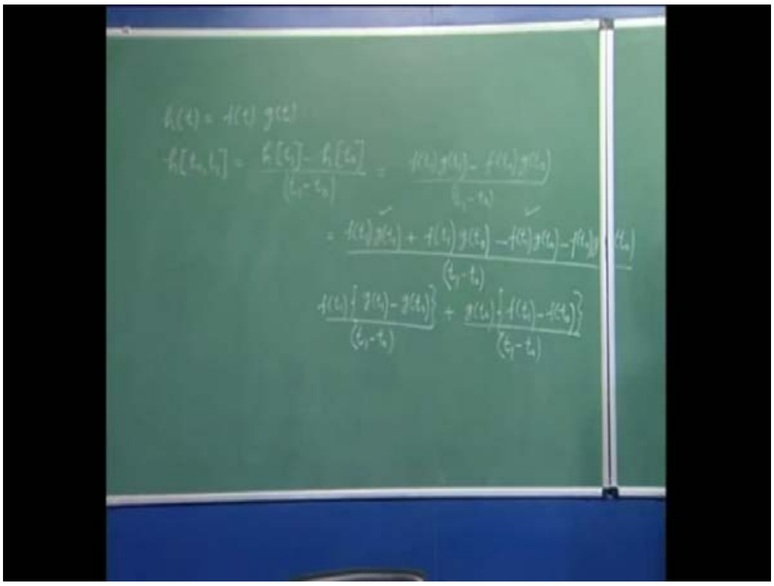
We showed this using the Peano’s theorem that is factorial m minus 1 times the nth divided difference of and constant g is equal to integration from t i minus m to t i psi of t times the nth derivative of g times g t we chose g t is equal to t raise to m so nth derivative of t g is factorial m and nth divided difference t raise to m is 1 that is this information in to yellow statement.

We can see that m minus 1 factorial this equal to m factorial times integration from t i minus m to t i of psi d t, which implies integration from t i minus m to t i of psi d t equal 1 over n and this is the standardization condition for B- Spline functions. All the need to do is to show that psi t is non negative of values of t in between t i minus 1 and t i ensure that little later continuing with the discussion on B- Spline basis function and today will try to derive the recursion relation polynomial. A lot of books directly start with recursion relation will try to prove recursion relation or B- Spline basis function.

(Refer Slide Time: 09:14)



(Refer Slide Time: 10:22)



We start with Leibnitz result, how to you compute the divided differences of the product of the two function, say h t equals f t times g t, Leibnitz result says that a kth divided difference, that h within prentices t o t 1 up t k is equal to summation r going from 0 to k, the r divided differences of f which is f within prentices t 0 t 1 up to t r times this divided difference here of the function g represented by g within prentices t r plus 1 up to t k all are prove this result for now all those this result is true for a generic case.

I will show that this result holes for the first few cases so h of t equal to f of t time's g of t. Just look at what f t 0 t 1 by definition is defined as h of t 1 minus h of t 0 over t 1 minus t 0, this is equal to f of t 1 g of t 1 minus f of t 0 g of t 0 over t 1 minus t 0.I will add and subtract 1 term here. I can write this thing as f of t 1 g of t 1 plus f of t 1 g of t 0 minus i subtract the same term here, f of t 1 g of t 0 and I will both term of this here minus f of t 1 g of t 0 over t 1 minus t 0.

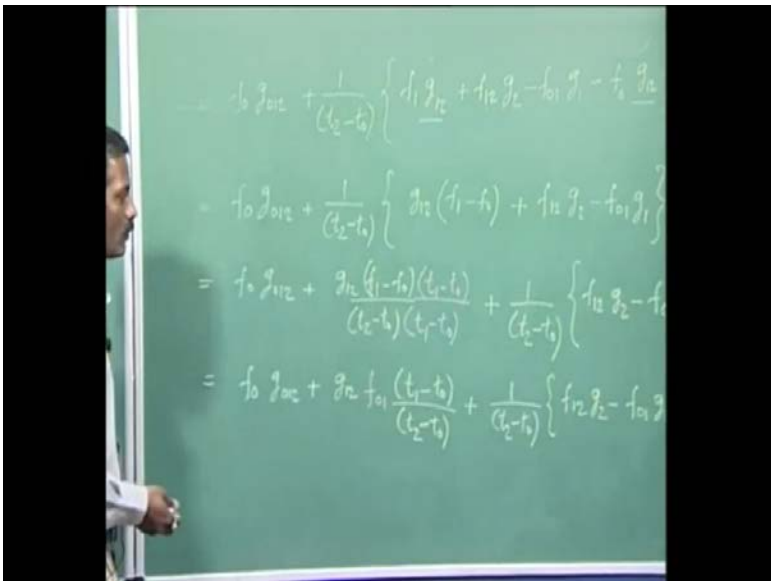
Now what I will do is I will take f of t 1 common in between this term. So f of t 1, g t 1 minus g t 0 over t 1 minus t 0. I will take care of this term and this term and here i have g of t 0 is common this would be plus g of f t 0, what I am left with is f of t 1 minus f of t 0 write here over t 1 minus t 0. Notice what g t 1 minus g t 0 over t 1 minus t 0, likewise what is f of t 1 minus f of t 0 over t 1 minus t 0.

We have f of t 1 this is the first divided difference in g should be g t 0 t 1 plus here, we have g t 0, here we have a first divided difference in f. And this is what the Leibnitz result products for the first divided difference of a function h t that can be represented as a product f of t g t. Let us see if the Leibnitz result gives the correct production for the second divided differences. So h t equals f t, g t and we looking at what h t 0 t 1 t 2 else. To make my algebra a little center I will interfused a short hand, so what I will do so represents the second divided differences h t 0 t 1 t 2 as h sub 0 1 2.

Now by definition h t 0 t 1 t 2 or h 0 1 2 is equal to h t 1 t 2 minus h t 0 t 1 over t 2 minus t 0 using the short hand rotation, this is h 1 2 minus h 0 1 over t 2 minus t 0. The Leibnitz result gives aqua rate prediction for the first divided differences. So, I can substitute for h 1 2 in terms of the divided differences in f and g. So, h 0 1 2 equals now, what is h 1 2, h 1 2 will be f 1 g 1 2 plus f 1 2 g 2 minus what is h 0 1, this would be f 0 g 0 1 minus f of 0 1 g 1 over t 2 minus t 0. Can I simplify this further maybe I can, I have to add and subtract a few terms.

Let us say I add plus f of 0 g 1 2, now what I intend to is I intend to extract the second dived difference in g and that is the reason and have add f 0 g 1 2. I need to subtract the same term f 0 g 1 2. Let us see what happens. If I combined this 2 terms, what do you have I have f of 0 g 1 2 minus g 0 1 over t 1 minus t 0 plus 1 over t 2 minus t 0 times whatever is left, f 1 g 1 2 plus f 2 f 1 2 g 2 this term here minus f 0 1 g 1 minus f 0 g 1 2. What is this term here this is the second divided difference in g if you notice.

(Refer Slide Time: 19:34)

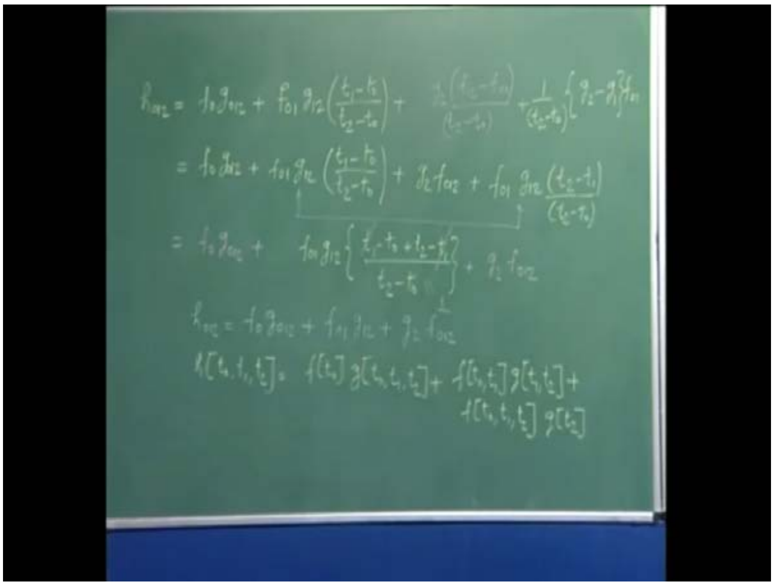


So this is equal to f of 0 g 0 1 2 plus 1 over t 2 minus t 0 this term here, let me rewrite this for you f 1 g 1 2 plus f 1 2 g 2 minus f 0 1 g 1 minus f 0 g 1 2, what is next. Now let us see if I can further simplify this expression, this figure if anything is common out here. I see there is 1 term g 1 2 here the first divided differences of g and g 1 2 here, maybe I can take this common, I can write this thing is equal to f 0 g 0 1 2 plus 1 over t 2 minus t 0, g 1 2 f 1 minus f 0. I have taken care of this term write here and this term write here, but I am left is plus f 1 2 g 2 minus f of 0 1 g 1. Now if I notice this expression I can see the first dived difference in f getting predicated.

So this expression becomes f 0 g 0 1 2 plus g 1 2 f 1 minus f 0 over t 2 minus t 0 and I multiplied and dived t 1 minus t 0 this is t 1 minus t 0 over t 1 minus t 0. What my left 1 over t 2 minus t 0 f 1 2 t 2 minus f of 0 1 g 1 this is f 0 g 0 1 2 plus g 1 2 f 1 minus f 0 over t 1 minus t 0 is the first divided difference in f. I can write this term as f 0 1 times whatever is left, t 1 minus t 0 over t 2 minus t 0 plus 1 over t 2 minus t 0, f 1 2 g 2 minus f 0 1 g 1.

What do we have we have f 0 g 0 1 2 plus some constant times f 0 1 g 1 2 and what I will looking for we are looking for f 0 1 2 that is the second divided difference in f how can be extract this second divided differences in f f 0 1 2 i would need to add and subtract something else here what i have do is i add g 2 f 0 1 and subtract g 2 f 0 1 let us we what happen this 2 terms are within this prentices.

(Refer Slide Time: 24:51)

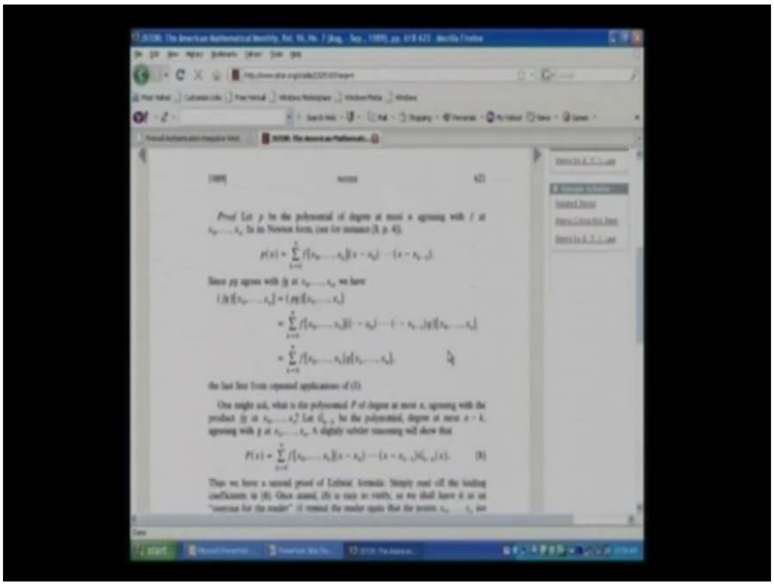


So h 0 1 2 equals f 0 g 0 1 2 plus f 0 1 g 1 2 t 1 minus t 0 over t 2 minus t 0 plus many combined this term and this term. What would this give me g 2 f 1 2 minus f 0 1 over t 1 minus t 0 plus 1 over t 2 minus t 0. Whatever is left, so I have consider this term here and this term here and what I am left is g 2 minus g 1 f 0 1 if u notice f 0 1 is common in between this 2 terms. I should be able to simplify the matters a lot from here. This is f 0 g 1 2 there will be 0 here so f 0 g 0 1 2 plus f 0 1 g 1 t 1 minus t 0 over t 2 minus t 0 plus what is this, this is g 2 the second divided difference in f f 0 1 2 plus. I can simplify this further this is f I can multiplied and dived by g 2 minus g 1 and I do that and I simplify this little further this would be g 1 2 times t 2 minus t 1 over t 2 minus t 0.

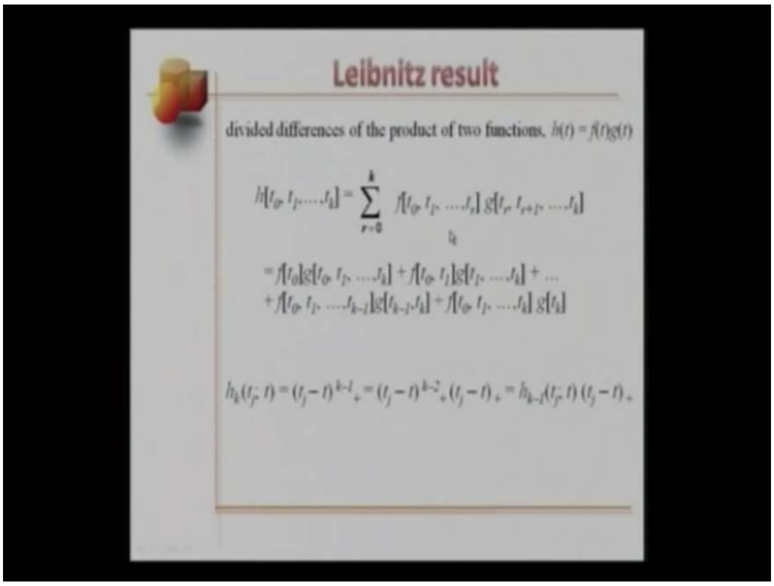
Now concentrate on this 2 expression f 0 1 g 1 2 is common, so this is f 0 1 g 1 2 t 2 minus t 0 is common denominator. We have here t 1 minus t 0 plus t 2 minus t 1, whatever I left with and left with f 0 g 0 1 2 plus and left with g 2 f 0 1 2 if i simplify this factor of further, t 1 canceled out t 2 minus t 0 canceled out with t 2 minus t 0 here. So this term is equal to 1 so finely h 0 1 2 equal f 0 g 0 1 2 plus f 0 1 g 1 2 times plus g 2 f 0 1 2.

So, if I let go of short hand notation and if I expand this using the convectional that we have flowed for the divided differences, this is h t 0 t 1 t 2 this is equal f t 0 f t 0 t 1 t 2 plus f t 0 t 1 g t 1 t 2 plus f t 0 t 1 t 2 which is the term here times g t 2. This is what is preciously predicated by the Leibnitz result for the second divided differences. All though I will quite elaborate and my algebra and showing the Leibnitz result to be 2 for the first and the second divided differences in h I am not sure if that would be the best way prove the general result.

(Refer Slide Time: 31:16)



(Refer Slide Time: 31:54)

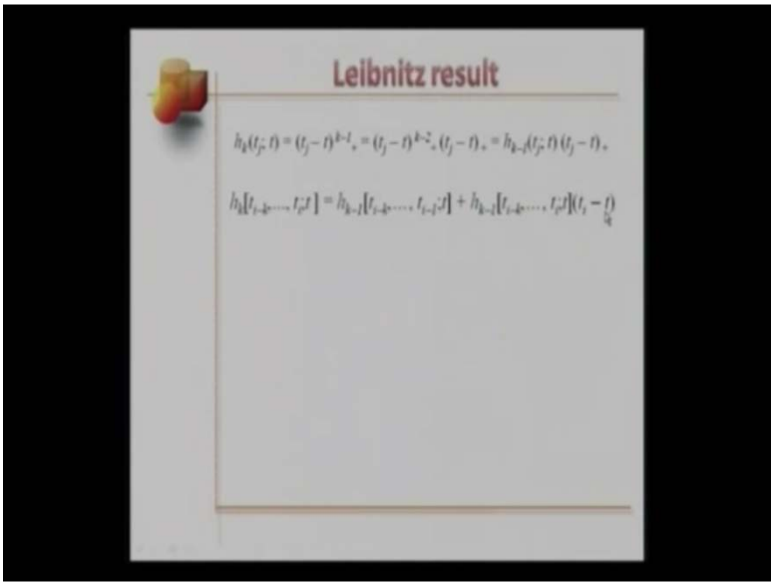


You can look at much element prove by searching to the net for example, I went through Google website and I found this paper a remark on divided difference by e t y ly. This paper presents this much element prove of Leibnitz result.

You might want to take a look; however here our intention is different we are going to be using this result to derive the recursion relation for B- Spline basis function. Any how the general result h within prentices t 0 t 1 after t k to expand this summation here. I get f of t 0 t 1 up to t k plus f of t 0 t 1 times g of t 1 up to t k plus of few more terms and the last term would be the kth divided difference in f and the 0th divided difference in g evaluated of t sub k. Now to be able to use this result and derive the recursion relation to B- Spline basis functions.

All we need to do as use h of k of t sub j colon t, which are truncated functions of degree k minus 1 t j minus t truncated raise to k minus 1 is equal to t j minus t truncated k minus 2 times t j minus t truncated. So what have d1 here is i expressed h of k which are k minus 1 degree from truncated function into a multiplication of to functions 1 is truncated functions of degree k minus 2 and the other 1 is a truncated function of degree 1. I write this thing as h sub k minus 1 t j colon t times t j minus t truncated.

(Refer Slide Time: 35:13)

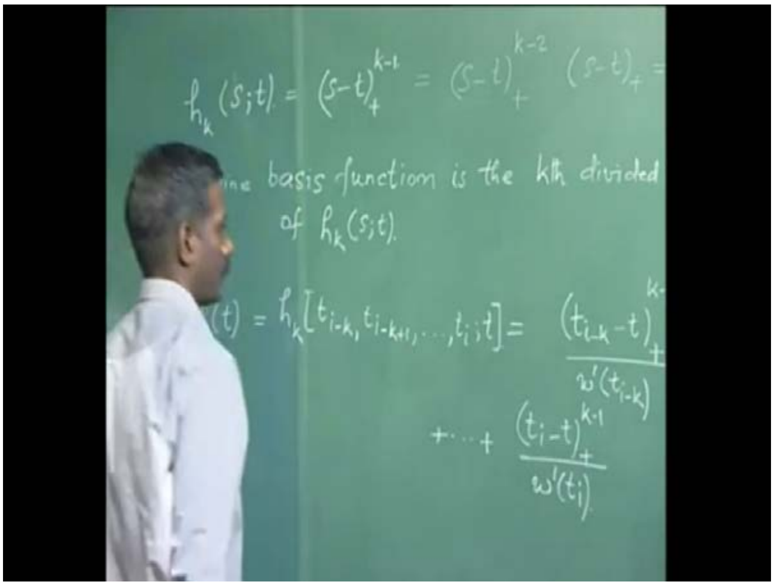


Try to relate this h with this h here k is subscript introduces simply to be note the order of the truncated function so the degree k minus n the order k which is represented by the subscript by h k t j colon t equals h k minus 1 t j colon t times t j minus t truncated. So h k minus 1 t j colon t you can think of that function to be f of t and t j minus t truncated we can think of this function to g of t and apply the Leibnitz result.

And we will see how to get recursion relation from here. So h of k t j colon t is equal to h of k minus 1 of t j colon t times t j minus t truncated. This try to apply Leibnitz result, remember what is said about B- Spline basis function in terms of the divided differences of B- Spline basis function would be the kth divide difference of this truncated functions degree k minus 1.

Let us try to compute kth divided differences of h k t j colon t again. I am going to be using this same x convention that have been using before. I denote the last knot by t i and the first knot by t i minus k so the kth divide difference of h of k is represented by t i minus k t i minus k plus 1 and so on up to t i colon t and this be equal to h k minus 1 within prentices t i minus k t i minus k plus 1 up to t i minus 1 colon t plus h of k minus 1 t i minus k plus 1 up to t i colon t times t i minus t which is a regular linear polynomial.

(Refer Slide Time: 37:16)



I have describe few step here let me show u those step on the board, so let start with of truncated function of order k and let me represents using 2 parameters s and t this is equal to s minus t truncated raise to k minus 1. Now B-Spline basis function is the kth divided differences of h k s colon t, we have made this position before and we have shown that this is indeed true. Let me go back little bit and let me rewrite this thing as s minus t truncated k minus 2 times s minus t truncated. So, I can write h sub k s colon t in terms of the product of the 2 functions, this is equal h k minus 1 s colon t times s minus t truncated by the notation we use here.

So this represents the k minus 1th order truncated of function. Now let us concentrate on this position little bit B- Spline bases function is the kth divided difference of h sub k s colon t truncated power function of order k, how to you write the kth divided difference yellow k. We can use the algebraic form psi t and I am going to be introducing a subscript here to represents the order of B- Spline basis function.

This is equal to h k t i minus k t i minus k plus 1 up to t i colon t and using the algebraic form this would be equal to t i minus k minus t truncated k minus 1 over w frame evaluated at t i minus k plus t i minus k plus 1 minus t truncated raise to k minus 1 over t i minus plus 1 up to t i minus t truncated raise to k minus 1 over w frame t i, let me make matter is algebraic center here and not very much interest here of this denominator B-Spline, so let me replace the denominator by non constant.

So psi k t equal alpha i minus k t i minus k minus t truncated k minus 1 plus alpha i minus k plus 1 t i minus k plus 1 minus t truncated k minus 1 plus alpha i t i minus t truncated raise to k minus 1, this is what I am interested alpha i minus k alpha i minus k plus 1 all these constant over here are respectively 1 over w frames.

Let us block this result as I going to be using it a little later that is go back to this expression here, h of k s colon t equals h k minus 1 s colon t times s minus t truncated and using the Leibnitz result, let us try to compute the kth divided difference to get of B-Spline basis function on the left hand side how to write this so using Leibnitz result the kth divided difference of h sub k i can write thing as t i minus k t i minus k plus 1 up to t i colon t this is equal to and I can use Leibnitz result to directly on the right hand side.

Let me call this f of t and let me call this g of t. So in terms of f and g i have f of t i minus k, g t i minus k t i minus k plus 1, t i plus f t i minus k, t i minus k plus 1 times g i minus k plus 1 t i plus many term and last term will be f t i minus k, t i minus k plus 1, t i g t i. Now let us try to relate these divided differences with the truncated power function.

Here f of t i minus k is the 0th divided differences of h k minus 1 g of t i minus k t i minus k plus 1 up to t i is the kth divided difference of s minus t truncated. How was this second term f of t i minus k t i minus k plus 1 as the first divided difference of h k minus

1 s colon t, g of t i k minus 1 up to t i is the k minus 1th divided difference of s minus t truncated and so on.

Until we get last term here f of t i minus k t i minus k plus 1 up to t i is the kth divide differences of h k minus 1 s colon t and g t i is simply the value of s minus t truncated and s equals t i and this result not all term is important to us, infect is so happen that only that last but, 1 term and the last term will be non 0 and the all the other terms will be 0 and this time I will agree with you that this is looking way to complicated by if u (( )) there with me I think I would able to simplify by this expression a lot better.