Computer Aided Engineering Design

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Lecture - 22

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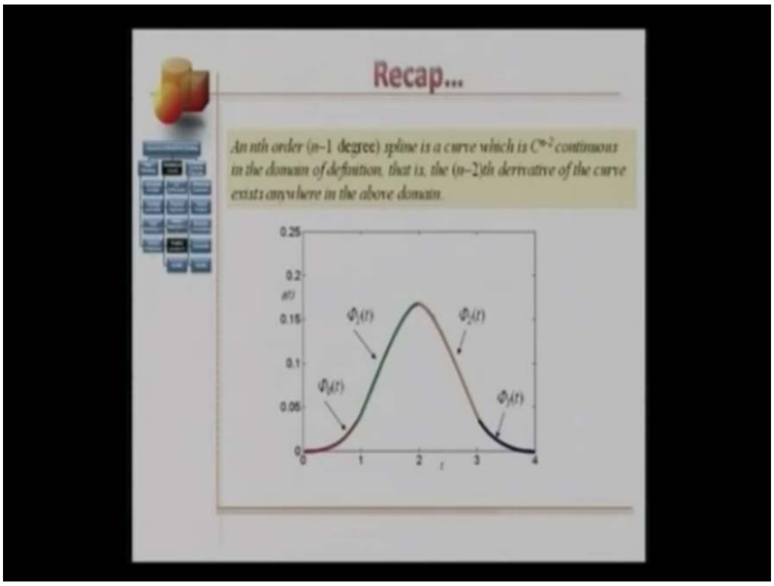

Hello and welcome, a little recap on what we have been doing in splines. First the definition, an nth order which n minus 1 degree curve as spline is C n minus 2 continuous everywhere in the domain definition, that is nth domain of t going from minus infinity plus infinity. Now, this definition is something that we will be keeping in mind throughout a discussion explains. In previous lecture, we had seen how construct belt shape B spline basic function to (( )).

You would agree with me that the mathematic process quite about involve an t d s, the both figure that is probably not a good idea to continue constructing basic functions using polynomial splines. And you are looking for alternatives, here with regard to the basic function. Let me revise a basic things, number 1 note this belt shape piece pie compasses basic function, is not asking the curve. It is compose of 4 segments. This is a cubic line, and that the reason wise compose of 4 individual segments. Each segment is a pure cubic palmer. These here are junction points, and these points' procession slope, and second derivative are constant, any curve for witch these conditions are not there will not be a spline. Let me replace any nth order or n minus 1 degree curve, which is not c n minus 2 continuous will not be a spline to many regular splines.

Let me, come up with the third stage a spine of degree n minus 1 has to be precisely see n minus 2 continuous. In t ranging from minus infinity to plus infinity. Today we will see how to construct d spine to this functions using the, Newton's divide differences. So let us start with the algebraic form of the, Newton's divide differences approach curve in that. We have seen in previous lecture this is a little revision for you so y of x j, x j, plus. 1 after x j plus a within square time process. Now this rotation something that. We have seen represents the divided represents can written as summation goes from 0 to k of y sub j plus r over w prime x sub j plus r. Can you tell which divided deference was this expression represent now, when k is equals to 0. We have y that equals to k that is 0 divided, when k is equal to1. We have y of x j x j plus 1 that is the first dividend.

So, this notation here represent the k h divide in this expressions on the right hand side. W x can be constructed as x minus x j times x minus x j plus 1 after x minus x j plus k and w frank. The first derivative of raise to 2 x, w prime. It is j plus r is simply this expression. To evaluated at x j plus r to revise to test again. Whether the expression. Whole Square for k is equal to 0. We have y of x j equals y of x j over w frank x j with simply y j in this case. We have w of x equals x minus x j so w frank is 1 for k equals 1. We have the first divide Refer with base j is of x j so y of x j x j plus 1 equals from this relation. Here y of j over w frank of x j plus y plus j plus 1 over w prime of x j plus 1 here w of x equals x minus x j times x minus x j plus 1. So w frank the first derivative is equals to x minus x j plus 1 plus x minus x j if you plug in x j. This expression goes out to be have x j minus x j plus 1. If you plug in x j plus 1. This expression is 0 and, we have x j plus 1 minus x j.

So y of x j plus x j plus 1 equals y j over x j minus x j plus 1 plus y j plus 1 over x j plus 1 minus x j. This is equal to y j plus 1 minus y j over x j plus 1 minus x j. This is the first divide difference of base x j are definition for k equals to, we have y of x j x j plus 1 x j plus 1 equals y j over w prime x j plus y j plus 1 over w frank x j plus 1 over w prime x j plus 1 y j plus 2 over w prime x j plus 1 from this relation.

Again the by the rules of divide differences. The second divide differences can be expressed using subtraction between 2 divide differences on this case from the left hand side of y plus x j plus 1 x j plus 2 minus y minus x j x j plus 1 over x j plus 2 minus x j. Here w x is equals to this sequence x minus x j times x minus x j plus 1 times x minus x j plus 2 although. I am not given you to generate the proof of this expression. I have mention the 4 that u might want. You prove this using mathematical induction for k k equals to 2. You can write however algebrately that. This expression which is by the definition of the divide differences will be identical to this expression here.

Let us, now continue to construct. These time basis function using. Newton's divide differences. We must be wondering. How is it these divide differences are related to these time basis function. You will probably see that. Now this lecture is number 22 b spine basis function. The construction using. Newton's divide difference approach .I said before will keep reminding us. That nth order spine. That is c n minus c 2 continuous. Coming back to divide differences, Let us consider a truncated power series function f of t equals t raise to m minus 1 sub plus. Now this is interesting function for values of t greater than 0 or even equal to 0 see plus m minus 1 is equal to t raise to n minus 1 and for values of t smaller than 0. That is the negative values of t this function of here is 0. You can think of the function getting truncated and t equals 0.

Let us see how the plot of this function 0 this is how the truncate or cuss function. Those as increase for values of t .Which are following are 0, this function of 0 for values of t greater than 0 this is accrue (( )) of degree. M minus 1 within that. You can see this polynomial it is getting truncate at t for 0.

Now let us look at certain properties of this truncated power. These functions look at the values of this function. The first derivative of this function and this second derivative of this function. After the m minus 2 derivatives of this function try to compute, the derivative from the left hand side and the derivative from the right hand side. So, for values of t greater than 0. Hence this is pure (( )) the first derivative will be m minus 1 times t raise to 1 minus 2. The second derivative will be m minus 1 times m minus 2 times 3 raise to m minus 3 and so on and so far. You know that and if you evaluate these derivatives a t equals to 0. You will find that all these derivatives will be 0 at equal 0.

So these are right hand side derivatives. That is the derivatives for t equals 0 plus, how about those for the left hand side will the function is 0 for negative values t so all these derivatives will also is 0. So you would see there is match in the derivatives. The first the second the third up to the m minus ((towards)) derivatives. The match the value of t slightly larger than 0 and for values of t is like this more than 0. You can say that all these derivatives exist and of course,, we have position continent aspect in other words. If you look at this function overall this function is m minus 2 continuous at this junction point. T equal 0 but, it will greater than 0 this function is infinitely differentiable and so the case for t smaller than 0 how about the m minus 1 derivative at t equals 0 plus so f m minus 1 t equal 0 plus is equal to m minus 1 factorial. Where is b right hand right derivative, while f m minus 1 at t equals to 0 minus is 0.

So you would see that the m minus 1 derivative is not unique. In other words this function here is not t m minus 1 continuous at the junction point. Let us look at this (( )) an nth order spines c n minus 2 continuous for this truncated ((power)) function satisfies of the definition of the spine. Yes so by the definition spine. This truncated power function f of t is to t raise to m minus 1 plus is a spine of order m so far further can we this play the truncated power function. A little more that is can we play with the junction point. Here may be s or can we reflect this function about the y axis again may be s.

Let us investigate now let us define another function f of t sub j colon t. This is equal t j minus t sub plus raise to m minus 1. Now this function which is (( )) of truncated power functions is equal to t j minus t raise to m minus 1 for values of t smaller than t j remember that. This number here it should be positive and this function equal to 0 for values of t greater than t. In other words at this number here is positive. We get a pure polynomial a degree m minus 1 and this number negative. We get 0. How would the plot of this function here like so. We have a junction point p j right here for values of t greater than j the function of 0 .The values of t smaller than t j. This function is a pure a line. We have changed the junction point and, we have also reflected this function about the vertical action now. Why the observation that. We made for the free rent truncated power function. Which this 1 can be similarly, say this same about this new function.

We observed that f t j colon t which is equal to t j minus t raise to minus 1 truncated at equals t j is continuous at t equals t j and that the derivatives up to n minus 2 all exist and there all unique. So we would conclude that t j minus t raise to m minus 1 plus is also a spine of order m a definition. Here again before we move further let me go to the board and try to generalize , this familiar 15;40. This generalization will help us introduce, Newton’s divided difference along with this Poincare r differences so this is, what I had mention before f of t j colon t and can written as t j minus t truncated m minus 1 and this is equal to t j minus t raise to m minus 1 for t smaller than equal to t j and this is equal to 0 for t greater than t j let me generalize. This expression let me introduce another parameter called s f s colon t equals s minus t truncated minus 1. Which is point to be equal to s minus t raise to m minus 1 if this number here is faster it would mean that if t smaller than s equal to smaller than s and then this number or this function is 0 if t is greater than s.

So now this is the 2 parameter function. We can treat either s as a variable and t as a parameter a vice versa will see, how we can use that. When we are discussing divided differences a little bit. This is where we introduce Newton's divided differences. We are going to be making a proposition about this proposition is base on following observation.

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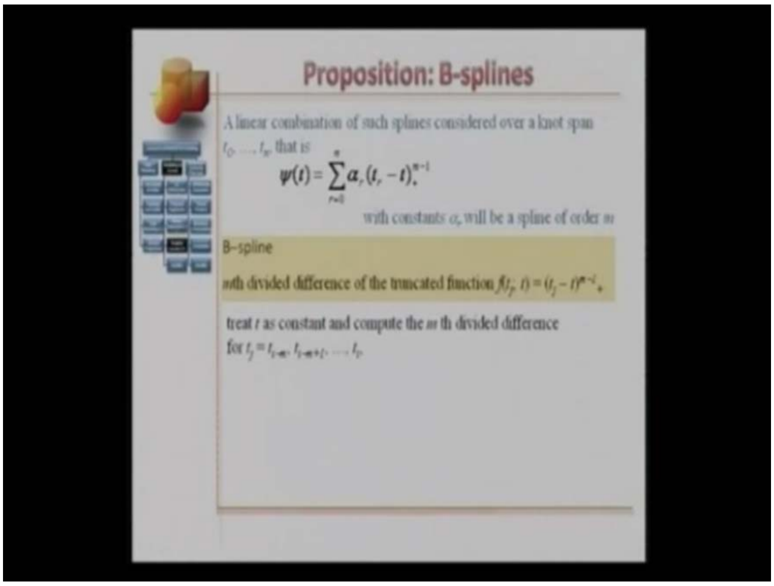

A linear combination of these b spines. That we had seen before consider over not's span t 0, t 1, t 2, until t n. That is sum function psi of t. Which is equal to summation r going from 0 to n of alpha r times t r minus t prime created raise to n minus 1 here alpha r are constants and the observation is that psi t will be spine of order m. would you agree (( )) We have seen before that each of these terms or whatever t r is a spine of order m.

So why not the linear combination of, these would be of spine. The observations seem good and base on this observation. We say the following a b spine. Basis function is the m divided difference of the truncated function, f t j colon t which is equal to t j minus t point crated raise to m minus 1. If you find get a little difficult to understand this replace t j by s going back and repeating myself a b spine basis function can be seen as the m divided difference of the truncated function f s colon t. This is equal to s minus t truncated raise to minus 1. It would be a little difficult for you to believe this for. I will try to show you how this is true the idea here, the treat t as constant or s parameter and compute the m divided difference for t j going from t sub i minus m t sub i minus m plus 1 t sub i minus m plus 2 until t i.

Let me explain, this procedure using the generalized truncated power functions say, if want to evaluate this function at s 0. I will say that f of s 0 colon t is equal to s 0 minus t truncated raise to m minus 1 with this information. Likewise f of s 1 colon t equals s 1 minus t truncated m minus 1. Now using the notation of the divided differences. I think consider, this expression to be the 0 divided difference with respect with this, likewise this will be 0 divided differences to respect respond being consistent in the notation. I. write this thing as f within squared parenthesis s 0 colon t 0 divided difference, with respect to s as 0, here likewise this can be written as f s 1 colon t. I using s 0 and s 1, how do I compute the first divided difference.

The first divided difference look like within square parentheses s 0 comma s 1 colon t, this is equal to f within square parentheses s 1 colon t minus f within square parentheses s 0 colon t over s 1 minus s 0. If I want to compute the first divided difference using s 1 and s 2. I will have to write this thing as f of s 1, s 2, and colon t this is equal to f s 2 colon t minus f s 1 colon t over s 2, minus s 1.

Now using these 2 how do, I compute the second divided difference. So the second divided difference will look like f within square parentheses and this is s 0, s 1, s 2, colon t and, this would be equal to f within square parentheses s 1, s 2, colon t minus f s 0, s 1, colon t over s 2, minus s 0. If we not to be very difficult. If you understand these divided differences for you to generalize and find. What f s 0, s 1, until s k colon t (( )) and this would be take divided difference and, if you are interested, this would be f of s 1, s 2, until s k colon t minus f of s 0, s 1, s k minus 1 colon t over s k minus s 0, you would be able to generalize this further note that you do not have to start which is 0 in start with s r for example.

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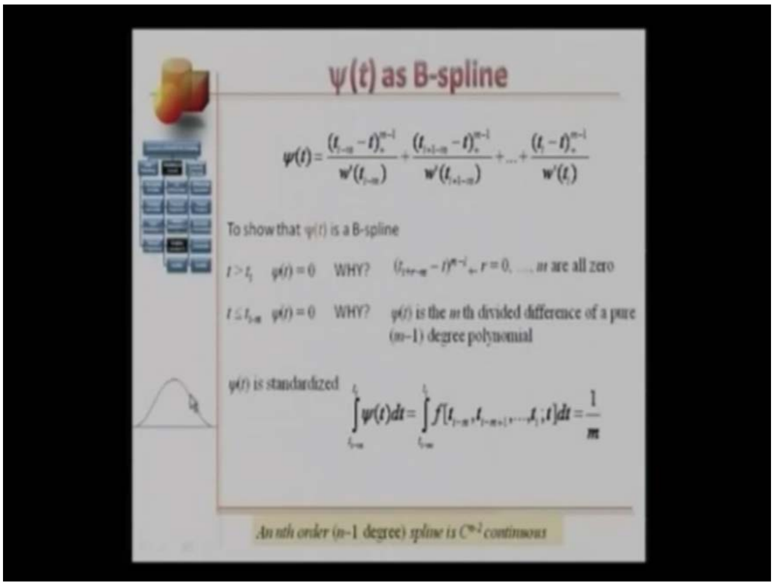

Now let us come back to our discussion on b spines basis functions So vm divided difference can be denoted as f within square parentheses t i minus m i minus m plus 1 until t i colon t equals summation are going from 0 to m of t i plus r minus m minus t truncated raise to m i minus 1 over w prime evaluated at t sub i plus r minus m and, we are saying as per our proposition, is that this expression is equal to psi t or b spine basis function. These kind of have to imagine this time right.

I will gradually show you. Why this is B- spine basis function. Let us (( )) expression on the left hand side all. We have d1 is, we have to place s is 5 t’s. How many not do you think. We are using m plus 1 knots or. We are using m not’s respond here. This expression is the m divided difference of the truncated (( )) function. You would have noticed that mind. This expression in middle here is the algebraic form of the m divided difference, you must be conversion with this expression of (( )) look at the numerator term in this summation and try to relate this entire expression with this expression. Alpha r times t r minus t truncated times 1. What do you absorb you would not each term by itself this is prime of order m 1 over w prime evaluated and t i minus m are the constants. Alpha r to in a sense this expression is a linear combination to be precise a waited in information of all clients of order m and therefore, this expression overall is B- spine of order m, because each term is a spine of order m here.

Let me expand this summation here and right side t as t i minus m minus t truncated raise to m minus 1 over w prime evaluated at ti minus m plus t i plus 1 minus m minus t truncated raise to m minus 1 over w prime at t r i minus m plus 1 until the last term. Which is t i minus t truncated raise to m minus 1 over the first derivative of w evaluated at t i. You will have m plus 1 term like these. I am repeating myself here but, this would help both of us understand that sense each term is a spine of order m since each term as m minus 2 unique derivatives the summation of all these term will have m minus 2 unique derivatives and by definition.

That would be a spine here w t is the product of t minus t i minus n times t minus t i minus 1 plus 1 until t minus t i and as you know w prime t is the first derivative with respect to t, if you are confuse with the notation as to y.We are using t i minus m t i minus m plus 1 and so on so 4th here and y t here .There is an easier way of resolving ,your confusion retain the knots as f i minus m s i minus m plus 1 until s i and then replace s by t cannot confuse do not worry about it.

Now y is psi of t of B- spine basis function .Do you remember this differentiate function. We will see here that psi of t essentially shapes like (( )) function. I had borrowed this equation from the previous slide psi of t equals t i minus m minus t truncated raise to m minus 1 over w prime t i minus m plus t i minus m plus 1 minus t truncated m minus 1 over w prime t i minus m plus 1. The last term in this summation is t i minus t truncated raise to m minus 1 over w prime t i.

We would show that psi of i is B- spine basis function. In a sense a t greater than t i, so in this figure here t i is this not here and t i minus 1 is this not here and the other knots are intermediate knots. So for t greater than t i psi of t is equal to 0. You are looking at this (( )) so the varies of t here can, we anywhere between t i and infinity. Why is that so why is it therefore, varies of t is greater than t i psi t is equal to 0 absorb this expression carefully. In a sense absorb these truncated signs carefully here. We assume that t i minus m t i minus m plus 1 until t i. They are all arranged in a sending all so for values of t greater than t i this expression. Here will be negative and this truncated aspect of the function will make this term 0 take a term on the left of this t i minus 1 minus t truncated raise to m minus 1.

Here again t will be greater than t i minus 1 and so that term will be 0. So if you go back left word. you would see that each term will have this number will in the parentheses as negative for t greater than t i, because of which this truncated sign will make the numerator in each thermal is summation 0. We can generalize this. We can say that t i plus r minus m minus t truncated rise to m minus 1 for r going from 0 to m are all 0. So we have figure that in this region psi of t is 0.

How about the region on the left that is how about for t equal to smaller than t i minus m psi of t will again be 0 and y is that. This would be a little difficult for as to understand in the first instants but, maybe we will able to digest this as, you move on lets go back to the same expression psi of t in terms of divided differences. What is this s is the m divided difference of truncated power function of degree. m minus 1 keeps that in mind. Let me repeat this for you. This is the m divided difference of tuckered power function of degree m minus 1.

Now values of t smaller than equal to t i minus m. What do you think will happen to these truncation signs. They will all drop because all these numbers with in the parentheses in the numerator. They will all the first it is, because of this that the truncation form will drop that the numerator will be a pure polynomial of degree n minus 1. I repeat the numerator will be pure polynomial of degree m minus 1 and, you have seen this before .What is the m divided difference of a pure polynomial degree m minus 1 so here goes. I have said psi of t is the m divided difference of a pure m minus 1degree polynomial. Then me ask this question again. What happens to the m divided difference of m minus 1 degree polynomial. The answer is this is going to be 0. We are seen this before and. I will discuss an example through, which you can see this again so, we are figure that psi is 0 for t greater than t I.

Which is this range here and psi is 0 for t smaller than or equal to t i minus m. Which is the range to the left here, how about for values of t in between t i minus m and t i and this here. We will see this psi of t is standardized that is the area under side t from t i minus m to t i. Which is integration psi t d t from t i minus m to t i. Which is equal to integration t i minus m to t ti the m divided difference of truncated (( )) of function a degree m minus 1 is equal to 1 over m. If would be little difficult for you to (( )) this time but, we will show you this time (( )) we will show that integration is equal to 1 over m using piano’s theorem.

That says that m minus 1 factorial time’s g with in square parentheses. t i minus m t i minus m plus 1 after t i colon t .Where g could be any function here and this expression is the m divide difference of g is equal to integration from t i minus m with t i of psi t times the m derivative of g times d t for those interested. They can look for the proof for the piano’s theorem will simply gives it to demonstrate standardization of psi t. If you choose g t as t raise to m which is a pure polynomial of degree m.

The m derivative of g is factorial m plug this information here. You also notice that. The m divided difference of t raise to m is 1. We can check this thing out for values of m equals 1, 2, 3, and so on and so far if you plug in this information here and this information here. We will have m minus 1 factorial equal’s m factorial integration. ti minus m to t i psi of d t .Which would mean that integration psi of t d t with limits t i minus m to t i is equal to m minus 1 factorial over m factorial. This is 1 over m.

To summarize we have seen that psi of t is 0 for t greater than t j psi of t is 0 for t is smaller than t j minus m in case t j happens to be the last knot and the area under psi t in between the limits t j minus m t j is equal to 1 over m all we to see is whether psi of t is non negative and this not span and you will see that and the next lecture.

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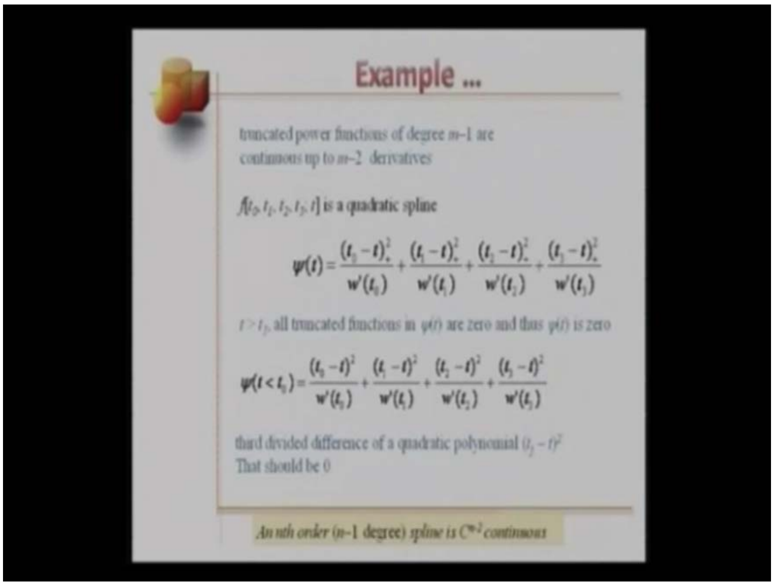

Let us work on a example show using plots, that psi of t. This is given by the third divide difference of truncated curve function. f with in square parentheses t 0, t 1, t 2, t 3, colon t .Which we would know is equal to summation r going from 0. Which 3 t r minus t truncated squared over w prime t r is a quadratic B- spine with w of t given as t minus t 0 times t minus t 1 until times t minus t 3 w prime t as the first derivative of w with t and assume that, all these knots t 0, t 1, t 2, and t3, are in the ascending order. We need to show that psi of t is a quadratic spine that psi of t is equal to 0for values of t not lined. The range t 0 and t 3. The fact that psi of t is non negative is .What we will see in the next lecture and that psi of g is standardize. That is integration from t 0 to t 3 psi t d t is equal to 1 over 3.

So, we know that individual Poincare curve functions of degree m minus 1 are continuous up to m minus 2 derivatives. Look at this definition here and using that. We can see that. The third divide difference of Poincare of function of degree. 2 is a quadratic in terms of the knots t 0, t 1, t 2, and t 3. This is how the algebraic expression of the Newton’s divide differences would be psi of t is equal to t 0 minus t Poincare. The whole squared over w prime t 0 plus t minus t truncated squared over w prime and t 1 plus t 2 minus t truncated squared over w prime and t 2 plus t 3 minus t truncated. The whole square over w prime of t 3 for t greater than t3 all truncated function in psi of t are 0 and thus psi of t 0.

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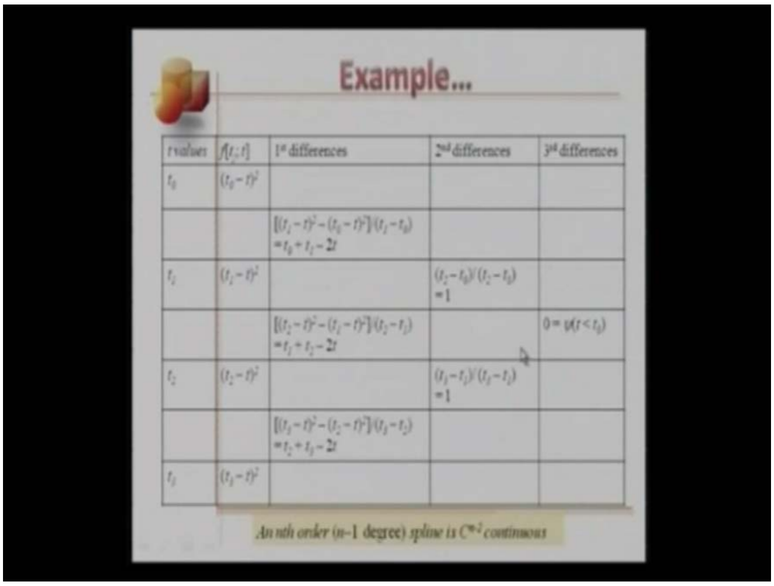

In a sense all these numbers within the parenthesis are negative psi for values of t smaller than t 0 and written as this expressions with all the truncated signs drop t 0 minus t squared over w prime t 0 plus t 1 minus t the whole squared over w prime t 1 plus t 2 minus t. The whole squared over w prime t 2 plus t t3 minus t the whole squared over w prime t t3. Now this is a pure quadratic polynomial the third divided difference of a quadratic polynomial t j minus t the whole squared should be 0 and we will see that

Now will compute the, Newton's divide difference in the tabular form. We have the first column. Where we arranged the t values. We had the 0 divide differences f of t j colon t. The first differences, the second differences, and the third differences. We arrange the knots t 0, t 1, t 2, and t 3. In the first column, you know this. We arrange the values in the second column t 0 minus t squared t 1 minus t squared t 2 minus t squared and t 3 minus t squared.

Here t is smaller than t 0 therefore, we have drop all these Poincare (( )) The first difference using this information here and this information here can written as t 1 minus t the whole squared this number minus t 0 minus t the whole squared over t 1 minus t 0 and this is equal to t 0 plus t 1 minus 2 t. Likewise using, the second row and the third row. We compute the first divide difference as t 1 plus t 2 minus 2 t in a similar passion using. The third row and 4th row the first divide difference would be t 2 plus t 3 minus 2 t for the second differences .We will use this information and this information .So t 1 plus t 2 minus 2 t minus t 0 minus t 1 plus 2 t over t 2 minus t 0 which will be equal to t 2 minus t 0 over t 2 minus 20 and this would be equal to 1.

Likewise using this row here and this row here the second difference would be t 3 minus t 1 over t 3 minus t 1. This is also equal to 1. Now remember this. When we are discussing the piano's theorem t m divide difference of t raise to m would be 1This is something very similar here and the third divide difference of course will be 1 minus 1 over t 3 minus t 0 which is 0 and this would be for all values of t smaller and t 0.