Computer Aided Engineering Design
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Lecture - 21
B-spline Segments and Curves

Hello, I am welcome let us continue our discussion on B-spline segments and curves.

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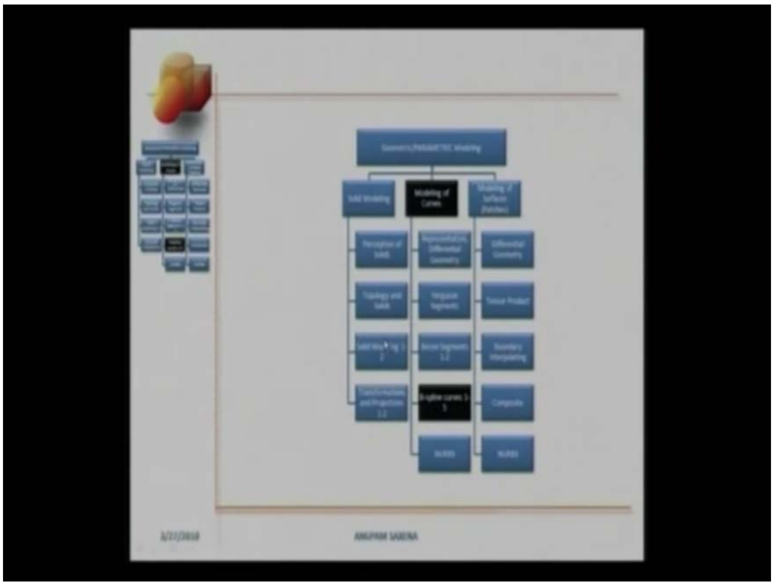

In particular we are trying to design B-spline basis function.

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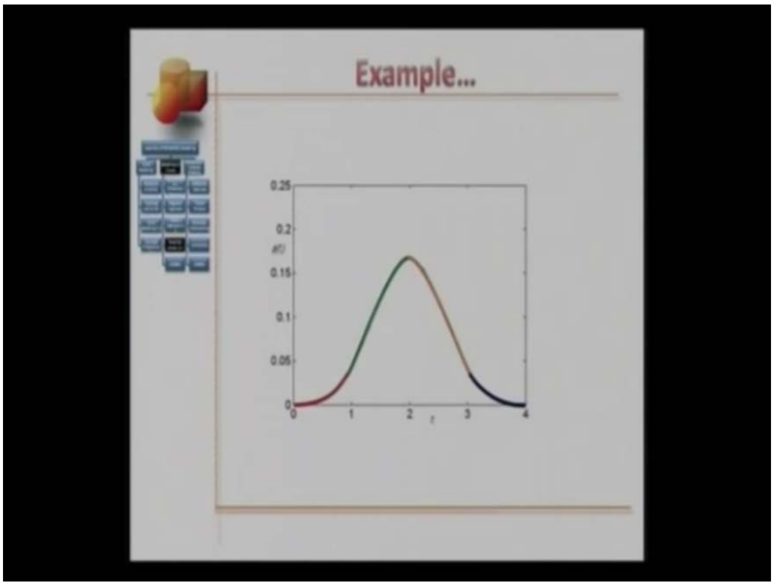

Let us look at the example; That we had discuss in the previous lecture, we computed a composite b spline continuous cubic, b spline basis function using polynomial splines. this is how this cubic function will look like, we call that we had used a cube conditions the position slope and second derivative at this point 0. The position slope and the second derivative at this point are 0 and this b spline basis function is non zero over this knot span, this is knot span has 1, 2, 3, 4, 5 knots. After recall that we have had called this basis function as a fundamental line. Because the 4 knots fonts were essential to provide the minimum support for this base function, notice that this basis function was compose of 4 individual pure cubic segments.

What we have to say about the contrary conditions are these 3 junction points. You are right the position slope and the secondary derivative are continuous here continuous here and the 3 are continuous here, which would make this basic function a c 2 continuous basic function throughout this span of t in the sense for t going from minus infinity to plus infinity this will be c 2 continuous through up in the previous 2 lectures we had also seen that it was very inconvenient to ask to compute the b spline basis functions using (( )) and because of that we are going to be seeking up alternative methods of computing basis function.

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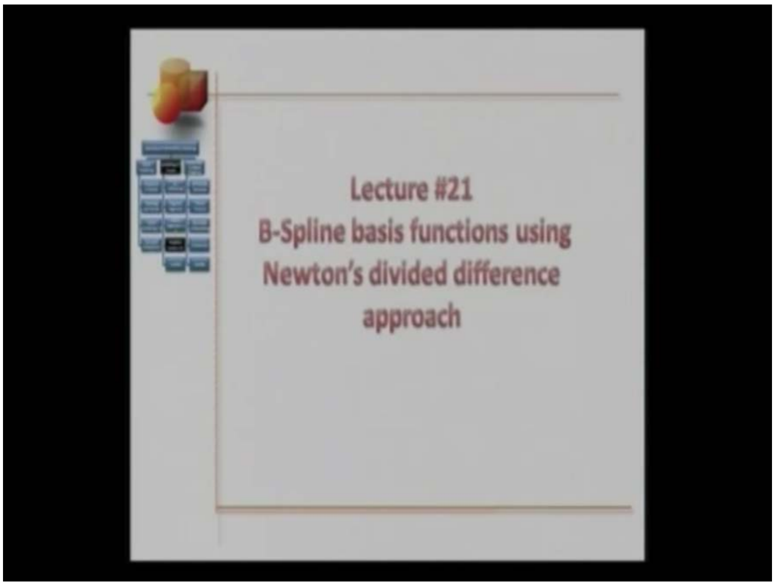

Let us look at one of them today's lecture is about computing b spline basis functions using Newton's divided difference approach.

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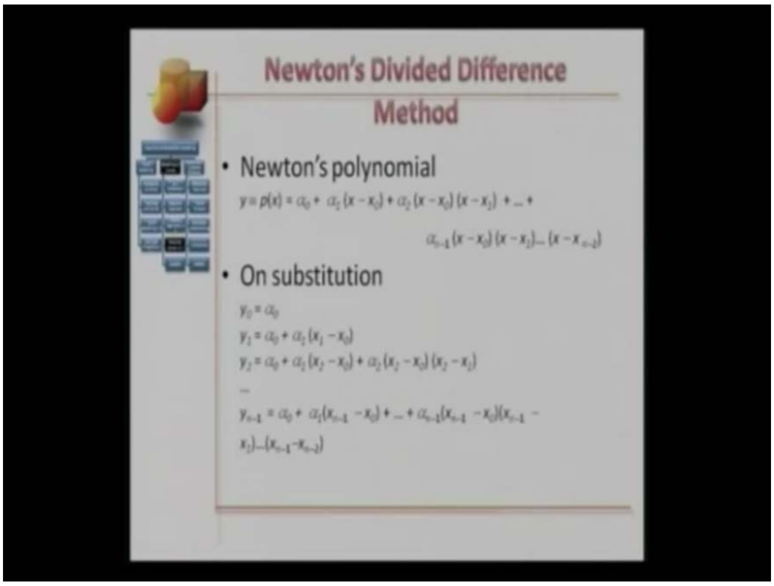

This is lecture NO 21 first a little we capitalist on the Newton's divided difference approach remember that we had discussed this method briefly when talking about curve interpolation. So let us say we start with Newton's polynomial I hope you remember what that is just in case if you do not here is a recap the polynomial y is written as p of x which is alpha 0 times alpha 1 times x minus x 0 plus alpha 2 times x minus x 0 times x minus x 1 and so on so after the last term alpha sub n minus 1 times x minus x 0 times x minus x 1 after times x minus x minus 2.

So given x 0 y 0 x 0 y 1 x 2 y 2 after x n minus 1 y n minus 1 we should be able to compute Newton's polynomial alpha 0 alpha 1 alpha 2 are unknown co efficient are this time also recall have we computed these unknown constants in the substitute y as y 0 for x as a x 0 all these terms vanish only alpha sub 0 remains which is equal to y 0, second step a b substitute for x 1 y 1 we have y 1 equals alpha 0 plus alpha 1 times x 1 minus x 0 all these terms starting from the third come over here there vanish we already know alpha 0 we would be able to compute what alpha 1 is likewise using the third data point x 2 y 2 we have y 2 equals alpha 0 plus alpha 1 times x 2 minus x 0 plus alpha 2 times x 2 minus x 0 times x 0 minus x 1 all the other terms will vanish we have alpha 0 from here we have alpha 1 from here we can compute what after towards and so on

This way we would be able to compute all these unknown co efficient after alpha and minus 1 and you tell me about the degree of this polynomial we should be able to figure now this is the linear term this is the quadratic term and this would be the term corresponding to degree you would know at better would you say this term correspond to degree n minus 1 in x anyhow.

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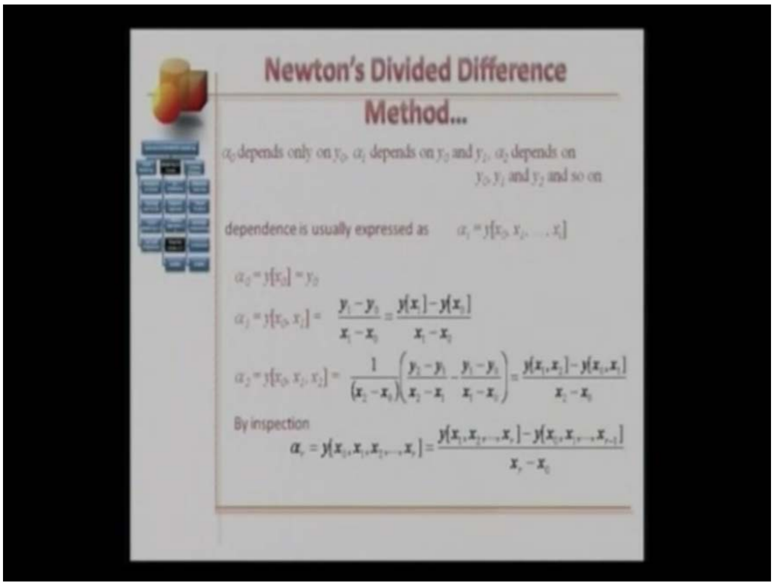

Let us move forward notice what alpha 0 would be alpha 0 would depend only on y sub 0 alpha 1 would depend on the first 2 data points in particular y 0 and y 1 alpha 2 would depend on the first 3 data points y 0 y 1 and y 2 and so on using this observation let me introduce sum rotations in general we can express this dependence as follows alpha I is equal to y with in squared brackets now this is important y within square brackets the function of x 0 x 1 after x I this try to go back here alpha 0 that is for I equal 0 would depend only on x 0 for I equals 1 alpha 1 will depend on x 0 and x 1 for I equals 2 alpha 2 will have relation corresponding to x 0 x 1 and x 2 and so on.

So for we are using squared parenthesis to express the inter relation between the co efficient and the data points in the Newton’s divided difference ((fallond)) so using this notation we have alpha 0 which is equal to y within squared parenthesis this x 0 this is equal to y 0 alpha 1 is equal to y with in squared parenthesis x 0 x 1 you would know 1 that is which is y 1 minus y 0 over x 1 minus x 0 now how can you express y 1 in terms of this notation right here y within square parenthesis x 1 minus y within square parenthesis x 0 over x 1 minus x 0 so far so good how about alpha 2 from this notation we have alpha 2 is equal to y within square parenthesis x 0 x 1 and x 2 and this is equal to 1 over x 2 minus x 0 times y 2 minus y 1 over x 2 minus x 1 minus y 1 minus y 0 over x 1 minus x 0 you would know that this is the second valid difference and these 2 are the first valid differences.

Now, try to introduce this notation in this expression right here y 2 minus y 1 over x 2 minus x 1 can be expressed as y within squared parentheses x 1 and x 2 try to relate this expression with this point and raise the index by 1 y 1 minus y 0 over x 1 minus x 0 as the same as y within square parentheses x 0 x 1 its right here and we borrow 1 over x 2 minus x 0 from here can you think off generalizing this by observation or by inspection it is possible in general alpha r can be written as y within square parenthesis x 0 x 1 x 2 after x r and this is equal to y within square parentheses x 1 x 2 after x r minus y within square parentheses x 0 x 1 after x r minus 1 over x r minus x 0.

Let us see if it is true put r equals 2 here alpha 2 equals y of x 0 x 1 x 2 if r is equals to 2 here will have y of x 1 and x 2 which is right here and then this would be followed by minus y by x 0 and x 1 which is right here when r is equal to 2 will need only x 1 and the denominator here will have x of 2 and x of 0 there is a nice way I remember this what is important are the (( )) this index this index right here this index right here and these 2 indices now this index is 1 plus the first index that appears this index is the last index for this 1 is the first index that appears here and for this last index if you look at this index I will have to subtract 1 for this is x of r minus 1 and then x of r which is the final x here minus x of 0 which is the initial x now this is the story when I have data points starting from 0 1 at the (( )) I should be able to compute the divided difference corresponding to any set of intermediate data point us.

How about for y within square parenthesis x sub s x sub s pula 1 x sub s plus 2 after x r remember what I told you right here this index for corresponds to 1 plus the first index so the first index is s this point here will be s plus 1 this index will corresponds to the last index here it is the first index in the center and for this 1 I will have to subtract 1 from the last index over x r minus x s which is x r minus x s here with a little practice you will be able to remember this.

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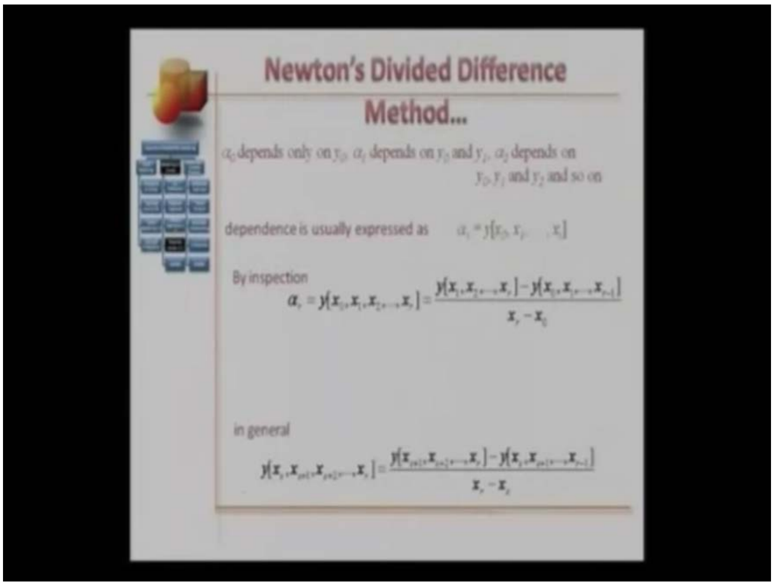

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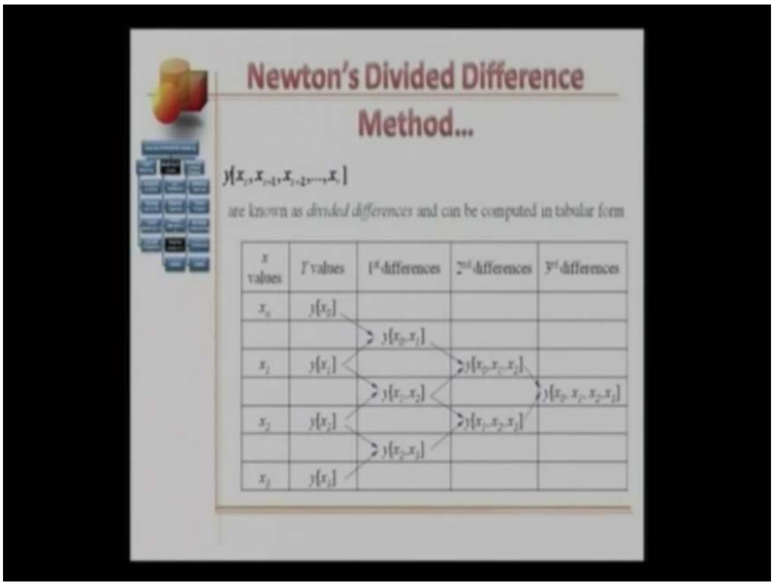

Now this is the tabular form that we have seen in few examples: Here if you recall I had worked a very simple example on board for ten into Newton's divided difference matrix and a similar approach was used when computing intermediate decasegive points in case of be the segments let us revisit this method for first y within square parenthesis x s x s plus 1 x s plus 2 after x r are known as divided difference.

They are the co efficient and as I said the can computed in tabular form quite easily will have to arrange the data column y the first column will corresponds to all the x values x

0 x 1 x 2 until x and minus 1 or x n or whatever the second column will corresponds to all the y values the third column will have the first divided differences, so what would make the second column here the y values they would be this 0 divided differences the fourth column here with corresponds to this second divalent differences and so on so for the third divide differences for the fifth column any how lets arrange first all the x values x 0 x 1 x 2 and x 3 in this example; We are working with 4 data points correspondingly will have y within square parenthesis x 0 which is simply y 0 y within square parentheses x 1 which is simply y 1 this is y 2 and likewise this is y 3 how would you compute the first divided differences you need to use only this information.

This would be y of x 0 and x 1 which is y 1 minus y 0 over x 1 minus x 0 likewise the divided difference corresponding to this information here will be y within square parentheses x 1 and x 2 which is y 2 minus y 1 over x 2 minus x 1 about the third 1 can you tell me what this is you guessed it right of x 2 x 3 and this is y 3 minus y 2 over x 3 minus x 2 what is next how do we compute this second derivatives for that we would need to use this information here let me to view the cube for the first 1 the second divide difference will be y of x 1 and x 2 minus y of x 0 x 1 remember the notation we had discussed previously over x 2 minus x 0 what have a done here I have consider the limits of this larger interval and that would be equal to y within square parentheses x 0 x 1 and x 2 how about the second divide function here.

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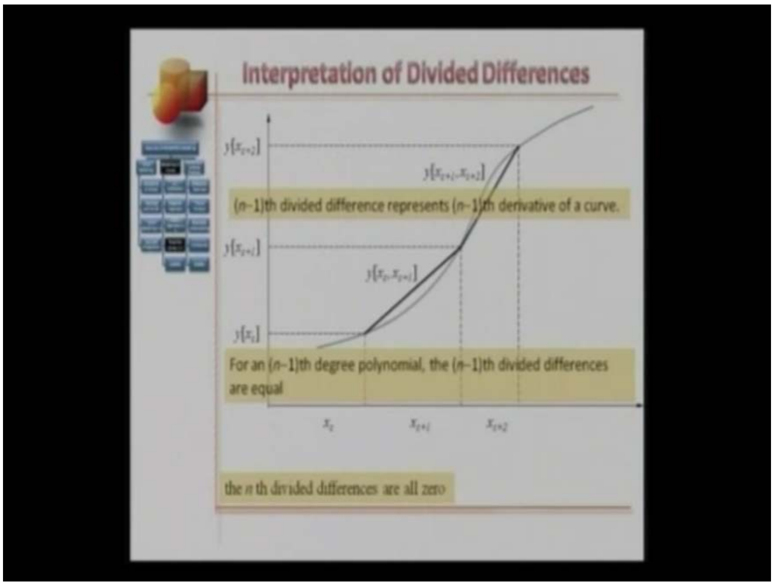

This would be y of x 2 x 3 minus y of x 1 x 2 over x 3 minus x 1. Once again I have skipped x 2 here and consider the limits of this larger interval and this will be denoted as per our convention by y of x 1 x 2 and x 3 and finally, the third divide difference for this example; will be this number right here minus this number right here over a larger entry this number minus this number and can you know tell what this notation of this divided principle this will be y with in square parentheses this x 0 x 1 x 2 and x 3 it might be looking like little complicated for you to compute divide differences in this manner.

Will work in an example; in this lecture and hopefully that would help you consolidate the concept. so far so good with regard to the mathematics behind the divide difference for physically what would the divide differences maid considered a general (( )) let us say this is x s and this is y of x s which is y x, let say another point x s plus 1 correspondingly a value is y x plus 1 y of x s plus 1 third point x s plus 2 y x plus 2 joint the first 2 points here this line will represent the slope in the decrease sense of course, and the slope will be equal to y x plus 1 minus y s over x s plus 1 minus x s.

Which is the first divided difference using this 2 points so the first divided difference will represent the local slope again as you would see denoted by y of x s x s plus 1. likewise we can do very similar for the second and the third data points the slope here given by y x plus 2 minus y x plus 1 over x s plus 2 minus x s plus 1 and that is represented using a convention by y of x s plus 1 x s plus 2.

So these are all local slopes represented by the first divided difference how about the second divided deference the second divide difference will corresponds to y of x s plus 1 x s plus 2 minus y of x s x s plus 1 divided by x s plus 2 minus x s this would locally represent the second derivative of the curve in general the n minus one divide difference would represent the n minus 1 derivative of the curve remember the term that I am using here it represent it is not equal to the n minus 1 derivative of the curve now this is something very interesting. For an n minus 1 degree polynomial the n minus 1 divided difference are all equal and correspondingly the n the divided difference are all 0.

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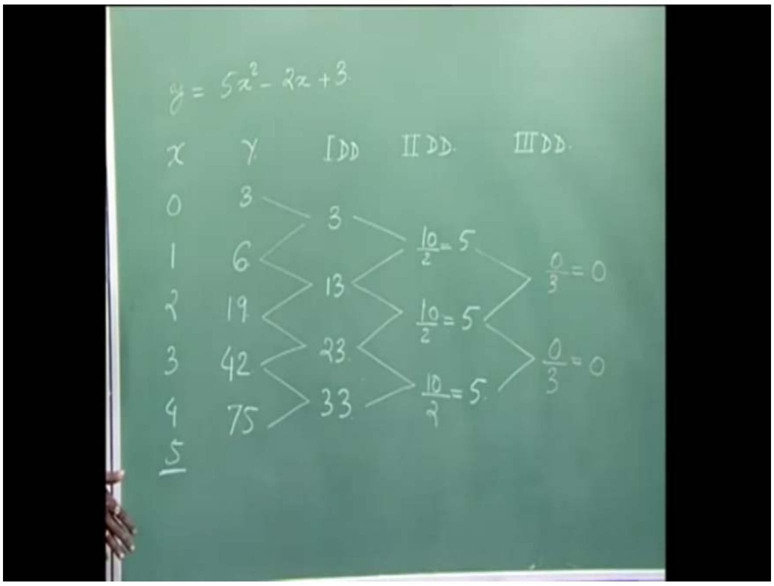

Let us me use this example; and explain a few things on the board for you. Let me make an example for you let me consider of quadratic polynomial y equals let say 5 x squared minus 2 x plus 3 let me generate some data points and arrange them in the tabular form so, will have the x values will have the y values let me take ((genic)) form values to make our lives simpler 0 1 2 3 4 5 for x equals 0 y equal 3 for x equals 1 will have 5 minus 2 which is 3 plus 3.which is 6 for x equals 2 will have 5 times 4 20 minus 4 16 plus 3 19 for x equals 3 this is 9 times 5 45 minus 6 which makes a 39 plus 3 42, looks like I am going to be testing my mathematics for x equals 4 16 times 5 it is 8 80 minus 8 is 72 plus 3 75, let us stop here let me consider this if I need to be late now let me arrange the column for the first derivative I call it 1 d d how would I talk computing those divide differences I use information corresponding to first and the second row and this will be 6 minus 3 over 1 minus 0 which is 1

So, 6 minus 3 this is 3 likewise 19 minus 6 is 13 over 1, 42 minus 19. So let me compute 2 minus 20 with this 22 and 1 less which would make it 23, 23 over 1 its 23 likewise 75 minus 42 would be 33 over 1 so this is 33 let me now compute the second divided difference I call it 2 d d now for this I am going to be doing 13 minus 3 which is 10 which is 10 over 2 minus 0 which is 2 this is equal to 5 how about the second divide difference corresponding to this information so this will be 23 minus 13 which is again 10 over 3 minus 1, which is 2 this is again equal to 5, for this 1 here this is 33 minus 23 which is 10, over 4 minus 22 again equal to 5.

Recall; What I had previously imagine the second divided difference of a degree 2 pure polynomial are all equal 55 and 5 what we have to say about the third divide differences I call them 3 d d since this 5 minus 5 0 over all have to be (( )) here enlarging the interval so this would be 3 minus 0 which is 3 this is 0 likewise 5 minus 5 again 0 over 4 minus 1, 3 which is 0 every time I compute this sub sequent divide difference I will have to enlarge interval here by 1 keep that in mind what did I tell you before as well the third divide difference corresponding to a degree 2 pure polynomial are all 0 in general the n plus 1 divide differences corresponding to a degree n pure polynomial will all be 0. We can tested out for as many cases we want I find lot comfortable to compute the Newton’s divide differences this way we must be wondering how the Newton’s divide differences are related to b spline basis functions how we are going to construct b spline basis functions using this divide differences I call it to later but…

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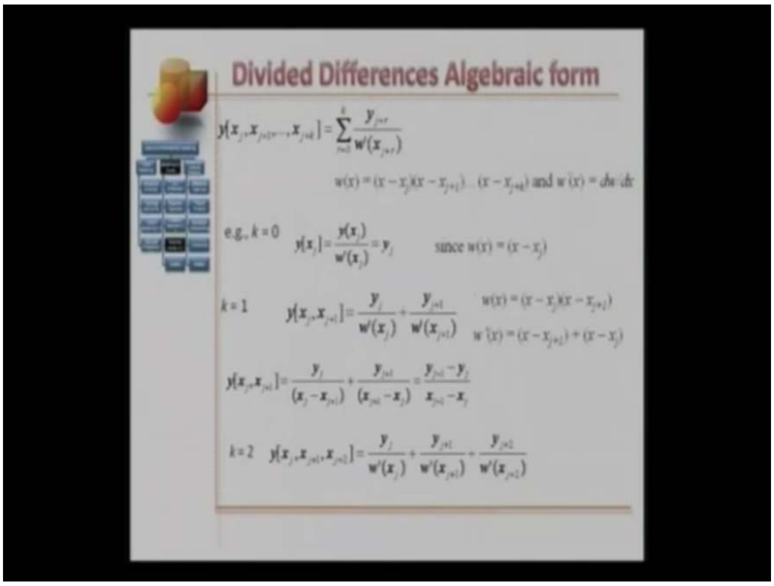

The First let me also give you an algebraic form of the valid (( )) So why within square parenthesis x sub j x sub j plus 1 until x sub j plus k is written as summation an index are going from 0 to k of y sub j plus r over w prime this probably not from here w prime that is equals derivative of w I will talk about that now y w prime evaluated at x j plus r. So consider of function w of x equals x minus x j times x minus x j plus 1 until x minus x j plus k and w prime x is the first derivative of w with respect to x to many indices and it might not very care to you in first go once again try to remember these terms so here index r goes from 0 to k, k corresponds to last index j plus k so this is k here y of j plus r r is going from 0 to k that would mean that we are considering the yes corresponding to x j x j plus 1 until x j plus k over w how you would want to construct that w is these terms here x minus x j times x minus x j plus 1 and so on so fourth after x minus x j plus k in a sense all unit do or remember is subtract all these terms x j x j plus 1 respectively from x and much apply these terms to construct differentiate of u once to this grater x and evaluate w prime x at x j plus r.

Let us me give you few examples; That will help you understand and remember this a little bit for example, when k is equal to 0 we are talking about y of x j alone so y of x j is equal to y of x r is 0 here y of x j over w prime of x j What is w prime of x j? is the first term here x minus x j differentiate this once to get 1 so this is y of x j over 1 which is simply y j how about for k equals 1 we have y within square parentheses x j x j plus 1 right here in numerators for these 2 terms will be y j and y j plus 1 y j and y j plus 1 the denominators will be w prime evaluated as x j and w prime evaluated at x j plus 1 right here how do you construct w w will be x minus x j times x minus x j plus 1 differentiate that once to get w prime x equals x minus x j plus 1 plus x minus x j so w prime of x j is x j minus x j plus 1 and w prime at x j plus 1 is x j plus 1 minus x j.

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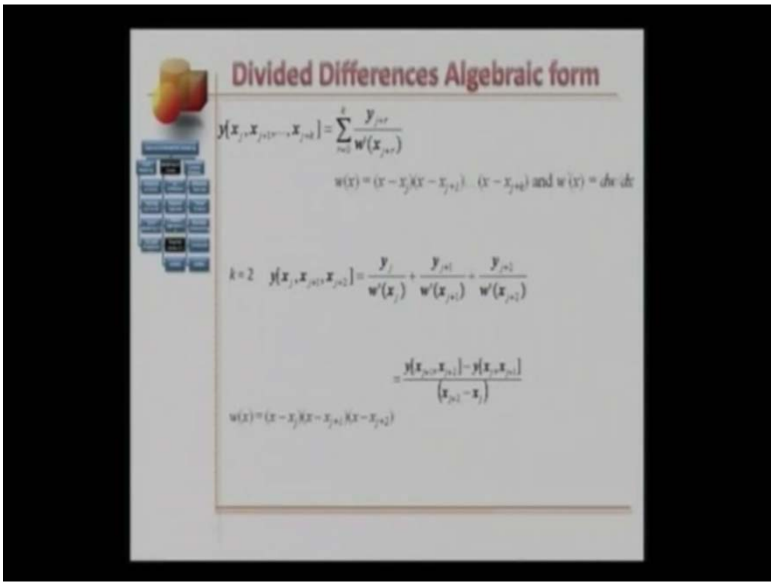

This term goes on substitute for those term here so y of x j x j plus 1 equals y j over x j minus x j plus 1 plus y j plus 1 over x j plus 1 minus x j so this term here is a negative of this term so this would make this term of left hand side equals y sub j plus 1 minus y j over x j plus 1 minus x j and this expression is very familiar to you by now of k 2 how many term we had here 3 so y of x j x j plus 1 x j plus 2 within squared parentheses this is denoting the second divide difference here this would be equal to y j over w prime at x j plus y j plus 1 over w prime at x j plus 1 plus y j plus 2 over w prime and at x j plus 2 are about w in this case w will have 3 terms in x.

If you think about this the second divide difference corresponding to the data points j j plus 1 j plus 2 can be written as y of x j plus 1 x j plus 2 minus y of x j x j plus 1 over x j plus 2 minus x j coming back w here w is constructed using these 3 terms corresponding to x j right here x j plus 1 right here and x j plus 2 right here you know how to differentiate this once evaluate the result at x j x j plus 1 and x j plus 2 and substitute those results back once I have told you is only the algebraic expression corresponding to this divide principle y of x j x j plus 1 until x j plus k I had not yet derive this expression here you might want to consider this as an assignment.

And I will give you a few as to how to derive this expression I would suggest that you used mathematical induction for derivation you probably would had studied this technique in your first year or second year of undergraduate mathematic start with y and x j may be try to convince to yourself with the 2 derivative differences already done this exercise here assume that this expression is true for say y equals x j x j plus 1 after x j plus k and using this result itself try to compute the next divide difference, which is y of x j x j plus 1 until x j plus k plus 1 and try to show that using this result itself you are able to accurately compute the subsequent divided (( )) this is mathematical induction (( ))

Why do not have work in example; (( )) this will help you consolidate the algebraic form of the divided differences it is important for you to understand this because we have needing these concepts later on when we discuss b spline basis function. Let me make up an example; So say I am going to use 4 data points 0 data points let us say 0 1 the data points is that say 2, 3 the second data point is that say 3, 5 and the fourth data point here is that say 5 minus 2, I have these data points correspondingly I would be able to compute until the third divide difference this try to compute directly using the algebraic form.

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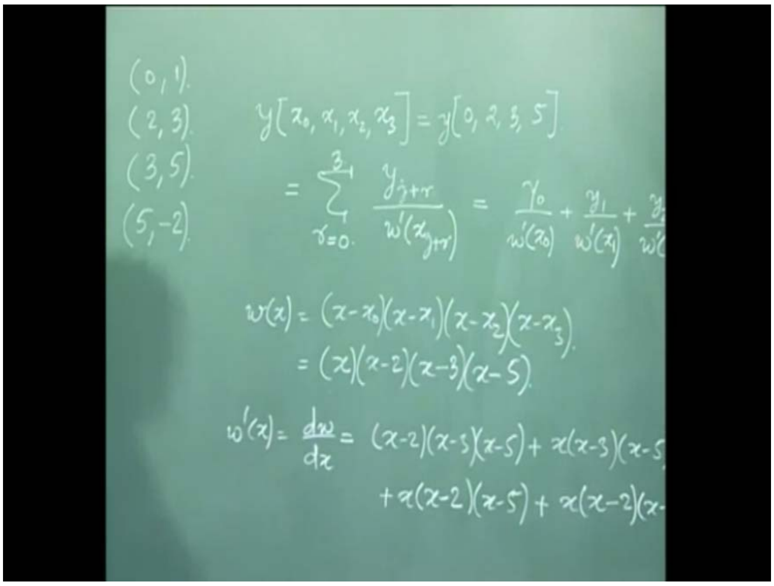

So y of x 0 x 1 x 2 and x 3 y of 0, 2, 3 ,5 is equal to summation the index r going from 0 2,3 y sub j plus r over w prime x j plus r, let me write this thing in elaborate form so that we are all understand now j here a 0 so will have y 0 over w prime evaluated as x 0 plus y 1 over w prime and x 1 plus y 2 over w prime and x 2 plus y 3 over w prime and x 3 now we know what these values of y 0, y 1, y 2, and y 3, which are 1, 3, 5 and minus 2 respective this try to first construct w and then evaluate these denominator.

So w here is equal to x minus x 0, x minus x 1, x minus x 2, and x minus x 3, we know what x 0 x 1 x 2 and x 3 are let us plug in those values directly this is x x minus 2, x minus 3, x minus 5, now w priming is equals total w of total x we can differentiate this expressions as follows x minus 2, x minus 3, x minus 5, plus x x minus 3, x minus 5, plus x, x minus 2 ,x minus 5, plus x x minus 2, x minus 3, essentially differentiate this expression considering the first derivative of each of this terms attar time.

Retain the 3 terms differentiate these terms to get this retain the first third and the fourth term differentiate this term to get the second term here and so on so you would know this keep this expression ready in your note we will be needing this.

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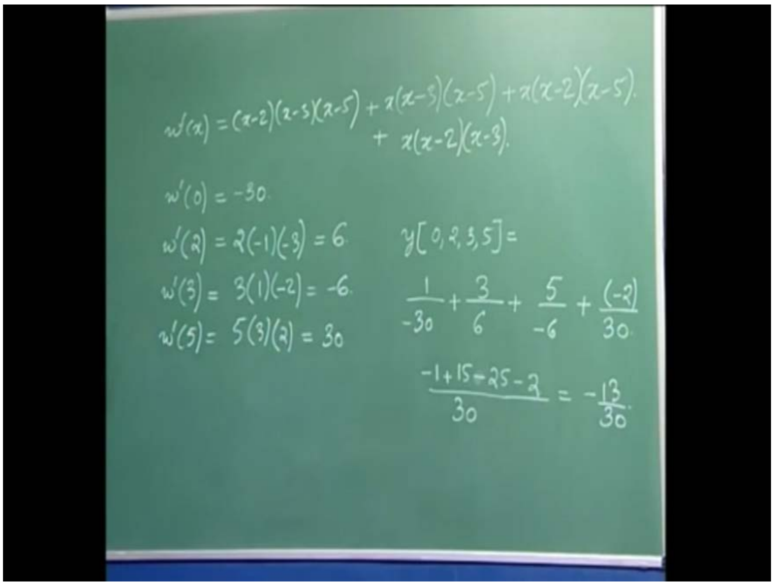

Ok, So what we have done is I have copied this expression here now ((yes)) w prime at 0, so the second third and the fourth terms will all be 0 is this first term. Which will be non 0, this is equal to 2, times 3, times 5, with a negative sign minus 30, about w prime at x equals 2, which corresponds to this term here, now here you can notice this term will vanish this 1 and this 1 these are all vanish what am I left with this is equal to 2 times 1 minus 3 which is minus 1 and 2 minus 5 which is minus 3 this is equal to 6, w prime corresponding to x 2 which is three is equal to this term this term and this term.

They are all go and only the third term remain this will correspond to 3 times 3 minus 2 is 1 and 3 minus 5 is minus 2 this is minus 6 and finally, w prime at 5 is equal to the terms corresponds to the last term here this this and this they are all vanish 5 times 5 minus 2 is 3 times 5 minus 3 is 2 this is equal to 30 and so y of 0 2 3 5 is equal to y 0 which is 1 here over w prime at 0 which is minus 30 plus y 1 which is 3 here over w prime 2, which is 6, plus y 2, which is 5, over w prime 3, which is minus 6, plus by 3, which is minus 2, over w prime at 5, which is 30, you can work hard the math and simplify this to get a number and try to compare what we get with a number that you would get when you are using the tabular form to compute the Newton’s divide differences.

Let me try to simplify this and see if I can go anywhere so if I take the least common factor which is 30, I get minus 1, plus 15, plus 25, I think I will have minus sign here minus 2, this is equal to minus 10, minus 11, minus 13, minus 13, over 30.

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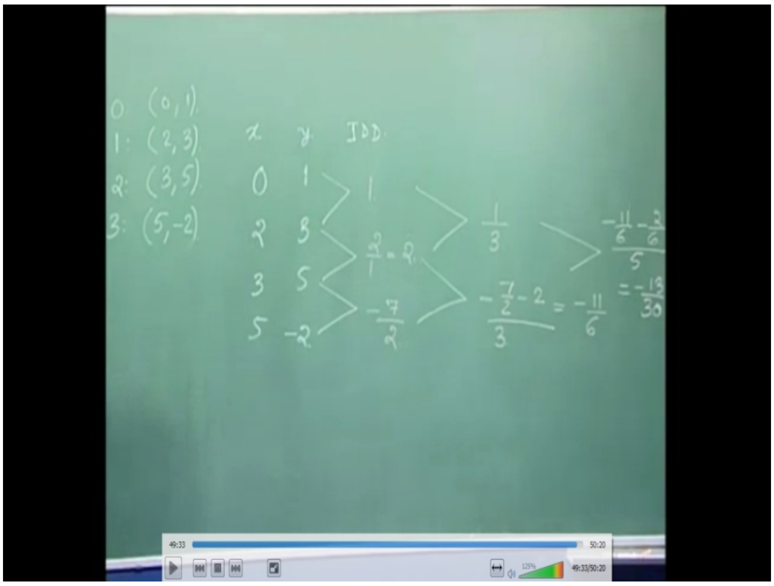

Now let me use the tabular form, to see if I get to the same number arrange all x's in a single column 0, 2, 3, and 5, arrange all yes in the next column 1, 3, 5, minus 2, and then so these are the ax's these are the x, and then start computing the divide differences so the first divide difference combined these 2, 3 minus 1, over 2 minus 0, which is 1, combined these 2, 5, minus 3, is to 3, minus 2, as 1, so this is 2 over 1 is equal to 2, minus 2, minus 5, which is minus 7, over 5, minus 3, is 2, coming to the next point 2, minus 1, is 1, over now I will have to enlarge being interval here 3, minus 0, which is 3, likewise combining these 2, minus 7, over 2, minus 2, over 5, minus 2, which is 3, I can simplify this as minus 11, over 6, and the final divide difference is minus 11, over 6, minus 1 over 3.I am gonna multiply and divide by 2, here and this would be minus 2, over 6 ,over this entire (( )) which is 5 and this would be equal to minus 13, over 13, so you just solve that we called the same number it is really matter .If we use the tabular form or the algebraic form they would gave you the same result .